GRASSMANN PHASE SPACE THEORY

for the

BEC/BCS CROSSOVER

in

COLD FERMIONIC ATOMIC GASES

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Introduction

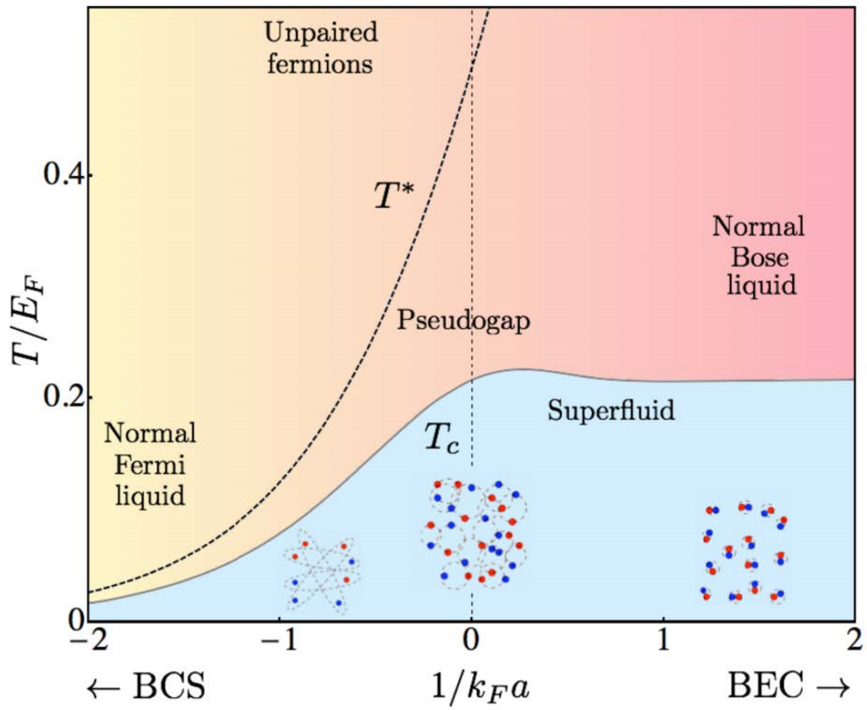
♦ Non-interacting untrapped fermionic atoms at zero temperature form a Fermi gas - each energy state only being occupied by atoms with different spins due to Pauli principle.

♦ Energy states filled up to Fermi energy E_F - associated wave number k_F proportional to inverse of average atom separation.

♦ Weakly interacting fermions - interaction changed via Feshbach resonance methods -> BEC/BCS crossover.

♦ For case of spin ½ atoms - possibility of two atoms with opposite spins and momenta pairing up and behaving as a single entity (Cooper pair).

♦ Two extreme possibilities - depends on whether two-body scattering length a is positive or negative.



♦ For $1/(k_F a) << 0$ superfluid BCS behaviour occurs - loosely bound Cooper pairs of atoms with opposite spins and momenta.

♦ For $1/(k_F a) >> 0$ a BEC forms based on tightly bound pairs of atoms with opposite spins and momenta - constitute a spin 0 bosonic molecule.

♦ Positions of fermionic atoms in Cooper pairs described by quantum correlation functions (QCF) - measured using Bragg spectroscopy.

♦ General behaviour described using BCS-Leggett theory (Leggett 80a, Ketterle 08a), (a mean field theory).

♦ Strong interaction regime (where scattering length $|a| >> 1/k_F$) presents challenge for developing theories of interacting ultracold fermionic atomic gases. Pair size here is of order interparticle separation $1/k_F$ (Ketterle 08a).

♦ Grassmann phase space theory (GPST) enables c-number equations to be derived for evolution (over time or temperature) of QCF describing position probabilities for atoms of opposite spin in (a) a single Cooper pair or (b) two Cooper pairs.

♦ GPST applicable to simple fermion systems (few modes, few fermions) - such as Fermi-Hubbard model, coherence in a single Cooper pair.

♦ Aim is see if GPST numerically applicable to study crossover, such as in unitary regime - compare to mean field theories.

♦ Qn: Are only fermion modes near Fermi surface important ?

References

Leggett80a - A. J. Leggett, In *Modern Trends in the Theory of Condensed Matter,* (eds) A.Pekalski *et. al.,*Springer-Verlag, Berlin 1980) p14.

Ketterle08a - W. Ketterle and M. W. Zwierlein, in *Ultracold Fermi Gases, Proc. International School of Physics, Varenna 2006,* (eds) M. Inguscio *et. al.*, (IOS Press, Amsterdam 2008).

Fermion Theory

♦ Field Theory and Diagrammatic methods - Which Feynman diagrams are important ?

♦ Quantum Monte Carlo methods - Fermion sign problem ?

♦ Variational approaches - BEC-Leggett theory. State assumes pairs of opposite spins all with same single pair wavefunction ?

♦ Phase space methods - Quantum state for m modes represented by Distribution Function for Phase Space Variables

★ Gaussian projector - PSV are $2m \times 2m$ c-number covariance matrices

★ Coherent state projector - PSV are $2m$ Grassmann variables

Grassmann Algebra and Calculus

- Grassmann variables satisfy all the usual rules of algebra except for multiplication and complex conjugation.
- Anti-commutation rules apply for multiplication. Boson oprs commute.

$$\{h_i, h_j\} = 0 \qquad \{h_i, \widehat{c}_j\} = \{h_i, \widehat{c}_j^{\dagger}\} = 0$$

- Square and higher powers of Grassmann variables are zero $h_i^2 = h_i^3 = .. = 0$. Grassmann variables have no inverse.
- Grassmann functions are of form - $f_0, f_i,$ etc are c-numbers.

$$f(h_1,h_2,..,h_n)=f_0+\sum_i f_i h_i+\sum_{i<j} f_{ij} h_i h_j+..+f_{123..n} h_1 h_2..h_n$$

♦ Grassmann calculus based on rules to differentiate or integrate 1 and h.

♦ Left and right differentiation of Grassmann functions

$$\frac{\overrightarrow{\partial}}{\partial h_i} h_j = h_j \frac{\overleftarrow{\partial}}{\partial h_i} = \delta_{ij}$$

♦ Left and right integration of Grassmann functions

$$\int dh_i 1 = 0 \qquad \int dh_i h_j = \delta_{ij} \qquad \int 1\, dh_i = 0 \qquad \int h_j\, dh_i = -\delta_{ij}$$

♦ Proceed by moving the Grassmann variable being differentiated or integrated either to the left or right of all other Grassmann variables and then apply the above rules.

Grassmann Phase Space Theory for Fields

- Time Evolution

$$\frac{\partial}{\partial t}\widehat{\rho} = \frac{-i}{\hbar}[\widehat{H}_f, \widehat{\rho}]$$

- Hamiltonian - Interacting Fermions ($g = 4\pi a_{s(eff)}\hbar^2/m$)

$$\hat{H}_f = \int dr \left(\sum_{\alpha} \frac{\hbar^2}{2m} \nabla\hat{\Psi}_{\alpha}(r)^{\dagger} \bullet \nabla\hat{\Psi}_{\alpha}(r) + \sum_{\alpha} \hat{\Psi}_{\alpha}(r)^{\dagger} V_{\alpha} \hat{\Psi}_{\alpha}(r) \right.$$

$$\left. + \frac{g}{2} \sum_{\alpha} \hat{\Psi}_{\alpha}(r)^{\dagger} \hat{\Psi}_{-\alpha}(r)^{\dagger} \hat{\Psi}_{-\alpha}(r) \hat{\Psi}_{\alpha}(r) \right)$$

- Field Operators, Grassmann Field Functions - $\alpha = \{up, down\}$

$$\widehat{\psi}_{\alpha}(r) = \sum_{i=1,..,m} \widehat{c}_{\alpha i} \; \phi_{\alpha i}(r) \qquad \hat{\Psi}_{\alpha}(r)^{\dagger} = \sum_{i=1,..,m} \widehat{c}^{\dagger}_{\alpha i} \; \phi^{*}_{\alpha i}(r)$$

$$\psi_{\alpha}(r) = \sum_{i=1,..,m} g_{\alpha i} \; \phi_{\alpha i}(r) \qquad \psi^{+}_{\alpha}(r) = \sum_{i=1,..,m} g^{+}_{\alpha i} \; \phi^{*}_{\alpha i}(r)$$

- Fermion Bargmann Coherent States $([\widehat{c}_{\alpha i}, \widehat{c}^{\dagger}_{\beta j}]_{+} = \delta_{\alpha i, \beta j})$

$$|\psi(r)\rangle_B = \exp(\prod_{\alpha} \sum_{i} \widehat{c}^{\dagger}_{\alpha i} \; g_{\alpha i}) \; |0\rangle = \prod_{\alpha i} (|0_{\alpha i}\rangle - g_{\alpha i} |1_{\alpha i}\rangle)$$

$$|\psi^{+}(r)^{*}\rangle_B = \exp(\prod_{\alpha} \sum_{i} \widehat{c}^{\dagger}_{\alpha i} \; (g^{+}_{\alpha i})^{*}) \; |0\rangle = \prod_{\alpha i} (|0_{\alpha i}\rangle - (g^{+}_{\alpha i})^{*} |1_{\alpha i}\rangle)$$

- Eigenvalue Equations

$$\widehat{\psi}_{\alpha}(r) \, |\psi(r)\rangle_B = \psi_{\alpha}(r) \, |\psi(r)\rangle_B \qquad {}_B\langle \psi^{+}(r)^{*} | \, \hat{\Psi}_{\alpha}(r)^{\dagger} =_B \langle \psi^{+}(r)^{*} | \, \psi^{+}_{\alpha}(r)$$

♦ B Distribution Functional

$$\widehat{\rho} = \int \prod_{\alpha} d\psi_{\alpha}^{+} d\psi_{\alpha} B[\underline{\psi}(r)] \widehat{\Lambda}[\underline{\psi}(r)] \quad \widehat{\Lambda}[\underline{\psi}(r)] = |\psi(r)\rangle_B \langle \psi^{+}(r)^{*}|_B$$

$$\underline{\psi}(r) = \{\psi_a(r), \psi_a^{+}(r)\} \equiv \{g_{\alpha i}, g_{\alpha i}^{+}\} = \underline{g},\ B[\underline{\psi}(r)] \equiv B(\underline{g}),$$

$$\prod_{\alpha} d\psi_{\alpha}^{+} d\psi_{\alpha} \equiv \prod_{\alpha} dg_{\alpha 1}^{+} .. dg_{\alpha n}^{+} dg_{\alpha n} .. dg_{\alpha 1}$$

♦ Quantum Correlation Functions for Position Prob Density

$$\Lambda(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p) = \hat{\Psi}_{\alpha_1}(r_1)^{\dagger} \cdots \hat{\Psi}_{\alpha_p}(r_p)^{\dagger} \lfloor 0 \rangle\langle 0| \hat{\Psi}_{\alpha_p}(r_p) \cdots \hat{\Psi}_{\alpha_1}(r_1)$$

$$P(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p) = Tr(\widehat{\rho}\, \Lambda(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p))$$

$$= \int\int d\psi^{+} d\psi\, \psi_{\alpha_p}(r_p) \cdots \psi_{\alpha_1}(r_1) B[\psi(r), \psi^{+}(r)] \psi_{\alpha_1}^{+}(r_1) \cdots$$

♦ Functional Fokker-Planck Equations - Correspondence Rules

$$\hat{\rho} \Rightarrow \hat{\Psi}_{\alpha}(r)\,\hat{\rho} \quad B[\psi] \Rightarrow \psi_{\alpha}(r)\,B[\psi] \qquad \hat{\rho} \Rightarrow \hat{\rho}\,\hat{\Psi}_{\alpha}(r) \quad B[\psi] \Rightarrow B[\psi]\,\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha}^{+}(r)}$$

$$\hat{\rho} \Rightarrow \hat{\Psi}_{\alpha}^{\dagger}(r)\,\hat{\rho} \quad B[\psi] \Rightarrow \frac{\overrightarrow{\delta}}{\delta\psi_{\alpha}(r)}\,B[\psi] \qquad \hat{\rho} \Rightarrow \hat{\rho}\,\hat{\Psi}_{\alpha}^{\dagger}(r) \quad B[\psi] \Rightarrow B[\psi]\,\psi_{\alpha}^{+}(r)$$

♦ Functional Fokker-Planck Equations - Drift $A_{\alpha A}$ and Diffusion $D_{\alpha A\,\beta B}$

$$\frac{\partial}{\partial t}B[\psi] = -\sum_{\alpha A}\int dr\,(A_{\alpha A}[\psi(r),r]\,B[\psi])\,\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha A}(r)}$$

$$+\frac{1}{2}\sum_{\alpha A,\beta B}\iint ds\,dr\,(D_{\alpha A\,\beta B}[\psi(s),s;\psi(r),r]\,B[\psi])\,\frac{\overleftarrow{\delta}}{\delta\psi_{\beta B}(r)}\,\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha A}(s)}$$

- Ito Stochastic Field Equations

$$\delta\widetilde{\psi}_{\alpha A}(r,t) \equiv \widetilde{\psi}_{\alpha A}(r,t+\delta t) - \widetilde{\psi}_{\alpha A}(r,t)$$
$$= A_{\alpha A}[\widetilde{\psi}(r,t),r]\,\delta t + \sum_{a} B_{a}^{\alpha A}[\widetilde{\psi}(r,t),r]\;\delta\omega_a(t_+)$$

- Stochastic Wiener Increments - C-Numbers

$$\overline{\delta\omega_a(t_+)} = 0 \qquad \overline{\delta\omega_a(t_+)\delta\omega_b(t_+)} = \delta_{ab}\;\delta t$$

$$\overline{\delta\omega_a(t_+)\delta\omega_b(t_+)\delta\omega_c(t_+)} = 0$$

$$\overline{\delta\omega_a(t_+)\delta\omega_b(t_+)\delta\omega_c(t_+)\delta\omega_d(t_+)} = \overline{\delta\omega_a(t_+)\delta\omega_b(t_+)}\;\overline{\delta\omega_c(t_+)\delta\omega_d(t_+)}$$
$$+ \overline{\delta\omega_a(t_+)\delta\omega_c(t_+)}\;\overline{\delta\omega_b(t_+)\delta\omega_d(t_+)}$$
$$+ \overline{\delta\omega_a(t_+)\delta\omega_d(t_+)}\;\overline{\delta\omega_b(t_+)\delta\omega_c(t_+)}$$

- Takagi Factorisation

$$D_{\alpha A\,\beta B}[\widetilde{\psi}(s,t),s;\widetilde{\psi}(r,t),r] = (BB^T)_{\alpha A\,\beta B} = \sum_a B_a^{\alpha A}[\widetilde{\psi}(s,t),s]\,B_a^{\beta B}[\widetilde{\psi}(r,t),r]$$

- Quantum Correlation Functions - Position Probability Density

$$P(\alpha_1 r_1,\alpha_2 r_2\cdots\alpha_p r_p) = \overline{\widetilde{\psi}_{\alpha_p}(r_p)\cdots\widetilde{\psi}_{\alpha_1}(r_1)\;\widetilde{\psi}^{+}_{\alpha_1}(r_1)\cdots\widetilde{\psi}^{+}_{\alpha_p}(r_p)}$$

Phase space average replaced by stochastic average.

- Temperature Evolution - $\beta = 1/k_B T$

$$\widehat{\sigma} = \exp(-\beta\widehat{H}_{f,}) \qquad \widehat{\rho} = \widehat{\sigma}/Z \qquad Z = Tr\;\widehat{\sigma}$$

$$\frac{\partial}{\partial\beta}\widehat{\sigma} = \frac{-1}{2}[\widehat{H}_f,\widehat{\sigma}]_+$$

Analogous B distribution functional, FFPE, Ito SDE.

Stochastic Momentum Fields

- Momentum Fields - Box normalisation $V = L^3$

$$\tilde{\psi}_{\alpha}(s) = \frac{1}{\sqrt{V}} \sum_{k} \exp(ik \bullet s)\, \tilde{\phi}_{\alpha}(k) \quad \tilde{\psi}_{\alpha}^{+}(s) = \frac{1}{\sqrt{V}} \sum_{k} \exp(-ik \bullet s)\, \tilde{\phi}_{\alpha}^{+}(k)$$

$k \equiv \{k_x(\frac{2\pi}{L}), k_y(\frac{2\pi}{L}), k_z(\frac{2\pi}{L})\}$ - where k_x, k_y, k_z are all integers.

- QCF via stochastic momentum fields - One Cooper pair

$$\begin{aligned} X(d\,r_1, u\,r_2; t) &= \overline{\widetilde{\psi}_d(r_1,t)\,\widetilde{\psi}_u(r_2,t)\;\widetilde{\psi}_u^{+}(r_2,t)\,\widetilde{\psi}_d^{+}(r_1,t)} \\ &= \frac{1}{(\sqrt{V})^4} \sum_{k_1,k_2,k_3,k_4} \exp i\{(k_1 - k_4) \bullet r_1\}\;\; \exp i\{(k_2 - k_3) \bullet r_2\} \\ &\quad \times \overline{\widetilde{\phi}_d(k_1,t)\,\widetilde{\phi}_u(k_2,t)\;\widetilde{\phi}_u^{+}(k_3,t)\,\widetilde{\phi}_d^{+}(k_4,t)} \end{aligned}$$

♦ QCF via stochastic momentum fields - Two Cooper pairs

$$X(dr_1, dr_2, ur_3, ur_4; t)$$

$$= \overline{\widetilde{\psi}_d(r_1,t)\,\widetilde{\psi}_d(r_2,t)\,\widetilde{\psi}_u(r_3,t)\,\widetilde{\psi}_u(r_4,t)\;\;\widetilde{\psi}_u^+(r_4,t)\,\widetilde{\psi}_u^+(r_3,t)\,\widetilde{\psi}_d^+(r_2,t)\,\widetilde{\psi}_d^+(r_1,t)}$$

$$= \frac{1}{(\sqrt{V}\,)^8} \sum_{k_1,k_2,k_3,k_4} \sum_{k_5,k_6,k_7,k_8} \exp i\{(k_1 - k_8)\bullet r_1\}\;\; \exp i\{(k_2 - k_7)\bullet r_2\}$$

$$\times \exp i\{(k_3 - k_6)\bullet r_3\}\;\; \exp i\{(k_4 - k_5)\bullet r_4\}$$

$$\times \overline{\widetilde{\phi}_d(k_1,t)\,\widetilde{\phi}_d(k_2,t)\,\widetilde{\phi}_u(k_3,t)\,\widetilde{\phi}_u(k_4,t)\;\;\widetilde{\phi}_u^+(k_5,t)\,\widetilde{\phi}_u^+(k_6,t)\,\widetilde{\phi}_d^+(k_7,t)\,\widetilde{\phi}_d^+(k_8,t)}$$

Ito SDE - Momentum Fields

- Time Evolution - Sum over l, q implied

$$\widetilde{\phi}_u(k, t+\delta t) = F_{u,u}(k,l;q,\delta t)\widetilde{\phi}_u(l,t) + F_{u,d}(k,l;q,\delta t)\widetilde{\phi}_d(l,t)$$

$$\widetilde{\phi}_d(k, t+\delta t) = F_{d,u}(k,l;q,\delta t)\widetilde{\phi}_u(l,t) + F_{d,d}(k,l;q,\delta t)\widetilde{\phi}_d(l,t)$$

$$\widetilde{\phi}^+_u(k, t+\delta t) = F^+_{u,u}(k,l;q,\delta t)\widetilde{\phi}^+_u(l,t) + F^+_{u,d}(k,l;q,\delta t)\widetilde{\phi}^+_d(l,t)$$

$$\widetilde{\phi}^+_d(k, t+\delta t) = F^+_{d,u}(k,l;q,\delta t)\widetilde{\phi}^+_u(l,t) + F^+_{d,d}(k,l;q,\delta t)\widetilde{\phi}^+_d(l,t)$$

- The F quantities are - $\lambda = \sqrt{-ig/2\hbar V}$

$$F(k,l;q,\delta t) = \begin{bmatrix} \delta_{q,0}\ \delta_{k,l}\left(1 + \frac{i}{\hbar}\frac{\hbar^2 k^2}{2m}\delta t\right) & \lambda\ \delta_{(k-q),l}\ (\delta\widetilde{\omega}^q_{u,d} + i\delta\widetilde{\omega}^q_{d,u}) \\ \lambda\ \delta_{(k+q),l}\ (\delta\widetilde{\omega}^q_{u,d} - i\delta\widetilde{\omega}^q_{d,u}) & \delta_{q,0}\ \delta_{k,l}\left(1 + \frac{i}{\hbar}\frac{\hbar^2 k^2}{2m}\delta t\right) \end{bmatrix}$$

and

$$F^{+}(k,l;q,\delta t) = \left[\begin{array}{|c|c|} \hline \delta_{q,0}\ \delta_{k,l}\left(1 - \frac{i}{\hbar}\frac{\hbar^2 k^2}{2m}\delta t\right) & \lambda\ \delta_{(k+q),l}\ (\delta\widetilde{\omega}^{q}_{u+,d+} + i\delta\widetilde{\omega}^{q}_{d+, \\ \hline \lambda\ \delta_{(k-q),l}\ (-\delta\widetilde{\omega}^{q}_{u+,d+} + i\delta\widetilde{\omega}^{q}_{d+,u+}) & \delta_{q,0}\ \delta_{k,l}\left(1 - \frac{i}{\hbar}\frac{\hbar^2 k^2}{2m}\delta t\right) \\ \hline \end{array} \right.$$

QCF - Time Evolution

- Subsitute for stochastic momentum fields, no-correlation thm -> average over products of F, F^{+} and of $\widetilde{\phi}_{\alpha}(l), \widetilde{\phi}^{+}_{\beta}(m)$ factorises, averages over F, F^{+} products determined analyticly.
- QCF for one Cooper pair satisfy

$$\frac{\partial}{\partial t} \overline{\widetilde{\phi}_{d}(k_1,t)\widetilde{\phi}_{u}(k_2,t)\ \widetilde{\phi}^{+}_{u}(k_3,t)\widetilde{\phi}^{+}_{d}(k_4,t)}$$

$$= \sum_{l_1 l_2 l_3 l_4} \left\{ \begin{array}{l} \delta_{k_1,l_1}\,\delta_{k_2,l_2}\,\delta_{k_3,l_3}\,\delta_{k_4,l_4}\left(\frac{i}{\hbar}\left\{\frac{\hbar^2 k_1^2}{2m}+\frac{\hbar^2 k_2^2}{2m}-\frac{\hbar^2 k_3^2}{2m}-\frac{\hbar^2 k_4^2}{2m}\right\}\right) \\ +2\lambda^2\left\{\delta_{k_1,l_1}\,\delta_{k_2,l_2}\,\delta_{(k_3+k_4),(l_3+l_4)}\right\} -2\lambda^2\left\{\delta_{(k_1+k_2),(l_1+l_2)}\,\delta_{k_3,l_3}\,\delta_{k_4,l_4}\right\} \end{array} \right.$$

$$\times \overline{\widetilde{\phi}_{d}(l_1,t)\widetilde{\phi}_{u}(l_2,t)\ \widetilde{\phi}^{+}_{u}(l_3,t)\widetilde{\phi}^{+}_{d}(l_4,t)}$$

♦ QCF for two Cooper pairs

$$\frac{\partial}{\partial t}\left[\begin{array}{c}\widetilde{\phi}_d(k_1,t)\widetilde{\phi}_d(k_2,t)\ \widetilde{\phi}_u(k_3,t)\widetilde{\phi}_u(k_4,t)\\ \times\ \widetilde{\phi}_u^+(k_5,t)\widetilde{\phi}_u^+(k_6,t)\widetilde{\phi}_d^+(k_7,t)\widetilde{\phi}_d^+(k_8,t)\end{array}\right]_{StochAver}$$

is given by

$$\sum_{l_1}\sum_{l_2}\sum_{l_3}\sum_{l_4}\sum_{l_5}\sum_{l_6}\sum_{l_7}\sum_{l_8} M((k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8;l_1,l_2,l_3,l_4,l_5,l_6,l_7,l_8)$$

$$\times\left[\begin{array}{c}\widetilde{\phi}_d(l_1,t)\widetilde{\phi}_d(l_2,t)\ \widetilde{\phi}_u(l_3,t)\widetilde{\phi}_u(l_4,t)\\ \times\ \widetilde{\phi}_u^+(l_5,t)\widetilde{\phi}_u^+(l_6,t)\widetilde{\phi}_d^+(l_7,t)\widetilde{\phi}_d^+(l_8,t)\end{array}\right]_{StochAver}$$

$M((k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8; l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8)$ is given by

$$
\left[
\begin{array}{c}
(\delta_{k_1,l_1}\,\delta_{k_2,l_2}\,\delta_{k_3,l_3}\,\delta_{k_4,l_4}\,\delta_{k_5,l_5}\,\delta_{k_6,l_6}\,\delta_{k_7,l_7}\,\delta_{k_8,l_8}) \\
\times\left(\frac{i}{\hbar}\left\{\frac{\hbar^2 k_1^2}{2m}+\frac{\hbar^2 k_2^2}{2m}+\frac{\hbar^2 k_3^2}{2m}+\frac{\hbar^2 k_4^2}{2m}-\frac{\hbar^2 k_5^2}{2m}-\frac{\hbar^2 k_6^2}{2m}-\frac{\hbar^2 k_7^2}{2m}-\frac{\hbar^2 k_8^2}{2m}\right\}\right) \\
+(-2\lambda^2)\,\delta_{k_1,l_1}\delta_{k_2,l_2}\delta_{k_3,l_3}\delta_{k_4,l_4} \\
\times\left(\begin{array}{c}
\delta_{k_6,l_5}\delta_{k_8,l_8}\;\delta_{(k_5+k_7),(l_6+l_7)}+\delta_{k_6,l_5}\delta_{k_7,l_7}\;\delta_{(k_5+k_8),(l_6+l_8)} \\
+\delta_{k_5,l_6}\delta_{k_8,l_8}\;\delta_{(k_6+k_7),(l_5+l_7)}+\delta_{k_5,l_6}\delta_{k_7,l_7}\;\delta_{(k_6+k_8),(l_5+l_8)}
\end{array}\right) \\
+(2\lambda^2)\,\delta_{k_5,l_5}\,\delta_{k_6,l_6}\,\delta_{k_7,l_7}\,\delta_{k_8,l_8} \\
\times\left(\begin{array}{c}
\delta_{k_1,l_1}\delta_{k_3,l_4}\;\delta_{(k_2+k_4),(l_2+l_3)}+\delta_{k_1,l_1}\delta_{k_4,l_3}\;\delta_{(k_2+k_3),(l_2+l_4)} \\
+\delta_{k_2,l_2}\delta_{k_3,l_4}\;\delta_{(k_1+k_4),(l_1+l_3)}+\delta_{k_2,l_2}\delta_{k_4,l_3}\;\delta_{(k_1+k_3),(l_1+l_4)}
\end{array}\right)
\end{array}
\right]
$$

♦ Initial Condition - Assume all fermions initially non-interacting and initial state for zero temperature. All plane wave modes up to Fermi surface occupied by one spin down atom and one spin up atom. Evolution after interaction turned on.

$$\overline{\widetilde{\phi}_d(k_1)\,\widetilde{\phi}_u(k_2)\;\widetilde{\phi}_u^+(k_3)\,\widetilde{\phi}_d^+(k_4)}$$

$$= Tr\,(\widehat{\Phi}_d(k_1)\,\widehat{\Phi}_u(k_2)\,\hat{\rho}\,\widehat{\Phi}_u(k_3)^\dagger\,\widehat{\Phi}_d(k_4)^\dagger)$$

$$= \delta_{k_1,k_4}\;\delta_{k_2,k_3} \qquad k_1,k_2,k_3,k_4 \quad inside\;\; FS$$

$$= 0 \qquad otherwise$$

Solution of QCF Evolution Equations

♦ One Cooper pair time evolution - H is real, symmetric

$$\frac{\partial}{\partial t} X(d\,k_1, u\,k_2, u^+ k_3, d^+ k_4)$$

$$= \sum_{l_1 l_2 l_3 l_4} iH(k_1, k_2, k_3, k_4; l_1, l_2, l_3, l_4)\, X(d\,l_1, u\,l_2, u^+ l_3, d^+ l_4)$$

with $X(d\,l_1, u\,l_2, u^+ l_3, d^+ l_4) = \overline{\widetilde{\phi}_d(l_1,t)\,\widetilde{\phi}_u(l_2,t)\;\widetilde{\phi}_u^+(l_3,t)\,\widetilde{\phi}_d^+(l_4,t)}$.

♦ Solved using normalised, orthogonal column eigenvectors x_i of H and real eigenvalues λ_i

$$H\, x_i = \lambda_i x_i \qquad x_i^T\, x_j = \delta_{ij}$$

♦ Solution

$$X(t) = \sum_i c_i(0)\ \exp(i\lambda_i t)\ x_i$$

$$c_i(0) = x_i^T\ X(0)$$

Initial condition specified by coefficients $c_i(0)$ - determined from initial column vector $X(0)$.

♦ Method of solution requires first calculating matrix H and initial column vector $X(0)$, then determining eigenvectors and eigenvalues of H. Alternative is a step-by-step development of QCF.

Summary

♦ Grassmann Phase Space Theory (GSPT) applied to BEC/BCS crossover in cold fermionic atomic gases.

♦ Determined evolution eqns (time or temperature) of QCF specifying:

★ positions of fermionic atoms in a single Cooper pair -> size of Cooper pairs - expect smooth change in crossover.

★ positions of fermionic atoms in two Cooper pairs -> correlation between two Cooper pairs

- expected significant in strong interaction unitary regime where size of Cooper pair comparable to pair separation.

♦ QCF given in terms of stochastic average of products of Grassmann stochastic momentum fields.

- ♦ Using no-correlation theorem, GPST shows stochastic average of products of Grassmann stochastic momentum fields at later time (or lower temperature) is related linearly to stochastic average of products of Grassmann stochastic momentum fields at earlier time (or higher temperature).
- ♦ Matrix elements involved in linear relations are all c-numbers.
- ♦ Initial conditions include zero temperature non-interacting fermionic gas to study Cooper pair creation and correlations when interaction is switched on via Feshbach resonance methods.
- ♦ Behaviour for unitary regime of particular interest.
- ♦ Numerics still to be done - Quantum Monte-Carlo methods ?