Space-Time Inside a Star

Sam Drake and Yang Shi
Flinders University: Mathematical Physics

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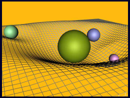
Problem Statement

How do black holes form?



* Energy (in all its forms) curves space-time.

* The curvature of space-time is the effect of gravity on energy (in all its forms)

Einstein's field equations for a perfect fluid in conjunction with the tensor give the O-V equation

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$$
$$T^{\alpha\beta} = (\rho + P)u^{\alpha}u^{\beta} + Pg^{\alpha\beta}$$
$$G^{\alpha\beta}{}_{;\beta} = 8\pi T^{\alpha\beta}{}_{;\beta} = 0$$

For static spherically symmetric space-times in hydro-static equilibrium these equations lead to the Oppenheimer-Volkoff Equation

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{[\rho + P][m + 4\pi r^3 P]}{r[r - 2m]}$$

The pressure gradient stops gravitational collapse as long as r-2m(r) > 0

A large but finite density can require an infinite pressure gradient to stop collapse.

What are the equations of state for stable stars and which ones lead to the formation black holes?

$$P(\rho) = ?$$

Background

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On Massive Neutron Cores

J. R. OPPENHEIMER AND G. M. VOLKOFF
Department of Physics, University of California, Berkeley, California

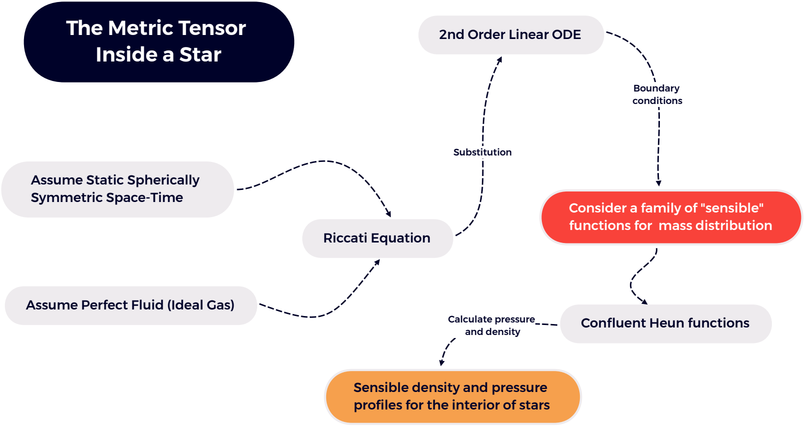
(Received January 3, 1939)

It has been suggested that, when the pressure within stellar matter becomes high enough, a new phase consisting of neutrons will be formed. In this paper we study the gravitational equilibrium of masses of neutrons, using the equation of state for a cold Fermi gas, and general relativity. For masses under $\frac{1}{3}\odot$ only one equilibrium solution exists, which is approximately described by the nonrelativistic Fermi equation of state and Newtonian gravitational theory. For masses $\frac{1}{3}\odot < m < \frac{3}{4}\odot$ two solutions exist, one stable and quasi-Newtonian, one more condensed, and unstable. For masses greater than $\frac{3}{4}\odot$ there are no static equilibrium solutions. These results are qualitatively confirmed by comparison with suitably chosen special cases of the analytic solutions recently discovered by Tolman. A discussion of the probable effect of deviations from the Fermi equation of state suggests that actual stellar matter after the exhaustion of thermonuclear sources of energy will, if massive enough, contract indefinitely, although more and more slowly, never reaching true equilibrium.

I. INTRODUCTION

FOR the application of the methods commonly used in attacking the problem of stellar structure[1] the distribution of energy sources and their dependence on the physical conditions within the star must be known. Since at the time of Eddington's original studies not much was known about the physical processes responsible for the generation of energy within a star, various mathematically convenient assumptions were made in regard to the energy sources, and investigation would afford some insight into the more general situation where the generation of energy is taken into account. Although such a model gives a good description of a white dwarf star in which most of the material is supposed to be in a degenerate state with a zero point energy high compared to thermal energies of even 10^7 degrees, and such that the pressure is determined essentially by the density only and not by the temperature, still it would fail completely to describe a normal main sequence star, in which

Solution Flow Diagram



- Starting from a the metric of a static, spherically symmetric spacetime:

$$\mathrm{d}\,s^2 = -e^{2\Phi(r)}\,\mathrm{d}\,t^2 + e^{2\Lambda(r)}\,\mathrm{d}\,r^2 + r^2\,\mathrm{d}\,\theta^2 + r^2\sin^2\theta\,\mathrm{d}\,\phi^2$$

we obtain the following expression for the density ρ and pressure P:

$$\rho(r) = e^{-2\Lambda(r)}\left(\frac{2\Lambda'(r)}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2},$$

$$P(r) = \frac{e^{-2\Lambda(r)}\left(-e^{2\Lambda(r)} + 2r\Phi'(r) + 1\right)}{r^2},$$

- where $\Phi(r)$ satisfies an 2nd-order ODE involving $\Lambda(r)$:

$$\Phi''(r) = \frac{r\Lambda'(r) - e^{2\Lambda(r)} + 1}{r^2} + \frac{\left(r^2\Lambda'(r) + r\right)\Phi'(r)}{r^2} - \Phi'(r)^2, \quad {}' = \frac{d}{dr}.$$

Riccati to a 2nd-order linear ODE

$$\Phi'' = A(x) + B(x)\Phi' - \Phi'^2,$$

- Let $\Phi' = y$, we have Riccati eqn:

$$y' = A(x) + B(x)y - y^2,$$

- $\Phi' = y = \frac{w'}{w}$ we obtain a linear 2nd-order ODE:

$$w'' = A(x)w + B(x)w'$$

with

$$A(x) = \frac{x\Lambda'(x) - e^{2\Lambda(x)} + 1}{x^2}$$
$$B(x) = \frac{x^2\Lambda'(x) + x}{x^2},$$

2nd-order linear ODE for different $\Lambda(x)$'s

An 2nd-order linear ODE:

$$w'' = A(x)w + B(x)w'$$

with

$$A(x) = \frac{x\Lambda'(x) - e^{2\Lambda(x)} + 1}{x^2}$$

$$B(x) = \frac{x^2\Lambda'(x) + x}{x^2},$$

Different choices of $\Lambda(x)$'s:

$$e^{2\Lambda(x)} = a = \text{const},$$

$$e^{2\Lambda(x)} = (ax + b)\,,$$

$$e^{2\Lambda(x)} = \left(\frac{a - bx}{1 - dx}\right),$$

$$e^{2\Lambda(x)} = (a(b - x)(c - x))\,,$$

$$e^{2\Lambda(x)} = \left(a(b - x)^2\right),$$

a, b, c and d are constants.

Boundary conditions $\Lambda(x)$, $\rho(x)$ and $P_(x)$:

$$e^{2\Lambda(1)} = \frac{1}{1 - \mu}, \quad \rho(1) = 0, \quad P(1) = 0.$$

Where $\mu = 2M/R$, M is the mass of the star and R is its radius.

Linear ODE and its singularities

Definition

Given a second-order linear ODE

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

Let $n \geq 0$, the *order of singularity* at $x = a$, be the minimal integer such that both limits

$$\lim_{x \to a} (x-a)^n p(x), \quad \text{and} \quad \lim_{x \to a} (x-a)^{2n} q(x)$$

are finite.

- $n = 0$, ordinary point
- $n = 1$, a regular singular point
- $n > 1$, irregular singular with *rank* $m = n - 1$

Behaviour of $x = \infty$ using $z = 1/x$ and checked at $z = 0$.

Special functions satisfying 2nd-order ODEs. Heun

- Singularities at $0, 1, a, \infty$ of order (1,1,1,1).
- Coalescence cascade of Heun functions:
  $(1, 1, 1, 1) \to (1, 1, 2) \to (2, 2)$...

Confluent Heun equation has singularities at $0, 1, \infty$ of order $(1, 1, 2)$:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha z - q}{z(z-1)} w = 0,$$

general solution $w(z) = c_1 H_1 + c_2 H_2$.

$e^{2\Lambda(x)} = (ax+b)$, linear case. Confluent Heun.

$$u'' = A(t)u + B(t)u'$$

with

$$A(t) = \left(\frac{\left(\mu^2 - 2\mu\right)\mu^4}{(\mu-1)^2 t^2} + \frac{\mu^4}{2(t-1)} - \frac{\left(\mu^2 - 2\mu - 1\right)\mu^4}{2(\mu-1)^2 t} \right)$$

$$B(t) = \left(\frac{\mu^2}{2(t-1)} + \frac{\mu^2}{t} \right),$$

where $' = \frac{d}{dt}$ and μ is a parameter. Asymptotic expansions at $x = 0$:

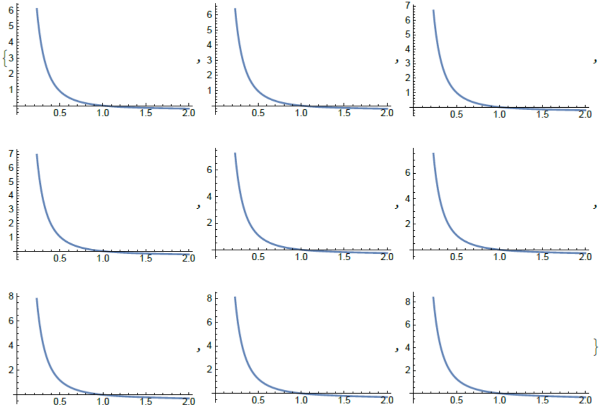
$$w(x) \sim c_2 x^{1-\frac{\sqrt{2\mu^2-4\mu+1}}{(1-\mu)}} \left(1 - \frac{\left(\frac{\mu^3-2\mu^2-\mu}{2(\mu-1)^2} + \frac{1}{2}\mu \left(1 - \frac{\sqrt{2\mu^2-4\mu+1}}{(1-\mu)} \right. \right.}{\frac{2\mu-\mu^2}{(\mu-1)^2} + \left(1 - \frac{\sqrt{2\mu^2-4\mu+1}}{(1-\mu)} \right) \left(2 - \frac{\sqrt{2\mu^2-4\mu+1}}{(1-\mu)} \right)} \right.$$

$$+ c_1 x^{1+\frac{\sqrt{2\mu^2-4\mu+1}}{(1-\mu)}} \left(1 - \frac{\left(\frac{\mu^3-2\mu^2-\mu}{2(\mu-1)^2} + \frac{1}{2}\mu \left(\frac{\sqrt{2\mu^2-4\mu+1}}{(1-\mu)} + 1 \right) \right.}{\cdots} \right.$$

Density is plotted for various values of μ's

In[∘]:=
```
Plot[ρ[x] /. {r → #}, {x, 0, 2}] & /@ {11/50, 23/100, 6/25, 1/4, 13/50, 27/100, 7/25, 29/100, 3/10}
```

Out[∘]=



Pressure is plotted for various values of μ's

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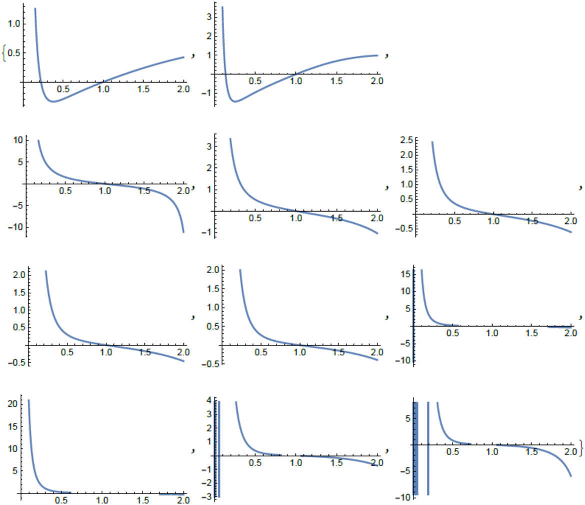
Plot[Pr[x] /. {r → #}, {x, 0, 2}] & /@
{11/50, 23/100, 6/25, 1/4, 13/50, 27/100, 7/25, 29/100, 3/10, 4/10, 5/10}
```



Summary and Future Work

- We have found some exactly solutions to Einstein's field equations that seem to correspond to realistic density and pressure profiles for stars.
- Further analysis of the properties of these solutions is required.
- A more general class of solutions exist and should be investigated.

Special fns	Singularity	$e^{2\Lambda(x)}$	Physics
Conf. Heun	$(1, 1, 2)$	linear	Schwarzschild
Heun	$(1, 1, 1, 1)$	?	-
?	$(1, 1, 1, 2)$	rational/quadratic	-
Painlevé	non-linear	?	-

- Coalescence cascade of Painlevé functions:

$$\begin{array}{ccccccc} \mathrm{P_{VI}} & & \mathrm{P_V} & & \mathrm{P_{III}} & & \\ \text{Gauss} & \to & \text{Kummer} & \to & \text{Bessel} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{P_{IV}} & & \mathrm{P_{II}} & & \mathrm{P_I} \\ & & \text{Weber} & \to & \text{Airy} & \to & \times \end{array}$$

- $\mathrm{P_{II}}$ equation: $\frac{d^2y}{dz^2} = 2y^3 + zy + \alpha.$