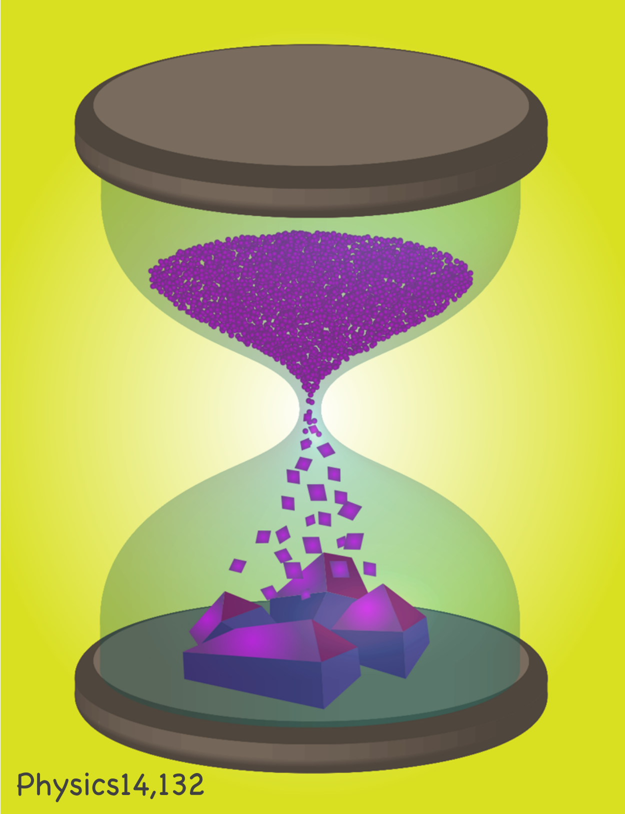
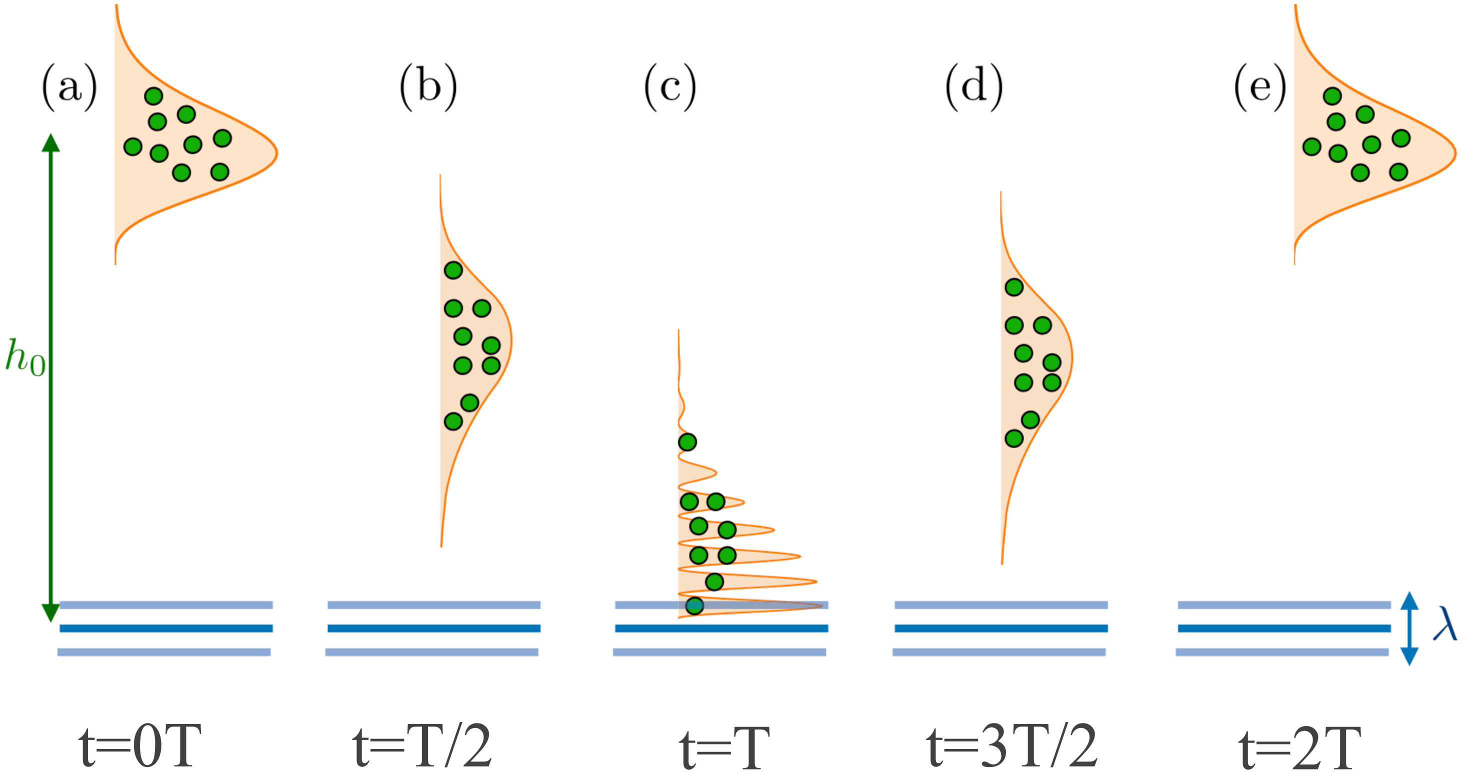
Discrete symmetry-breaking and time crystals (DTC) in continuous systems under periodic driving



Physics14,132



Jia Wang [a], Krzysztof Sacha [b], Peter Hannaford [c], Bryan J. Dalton [a]

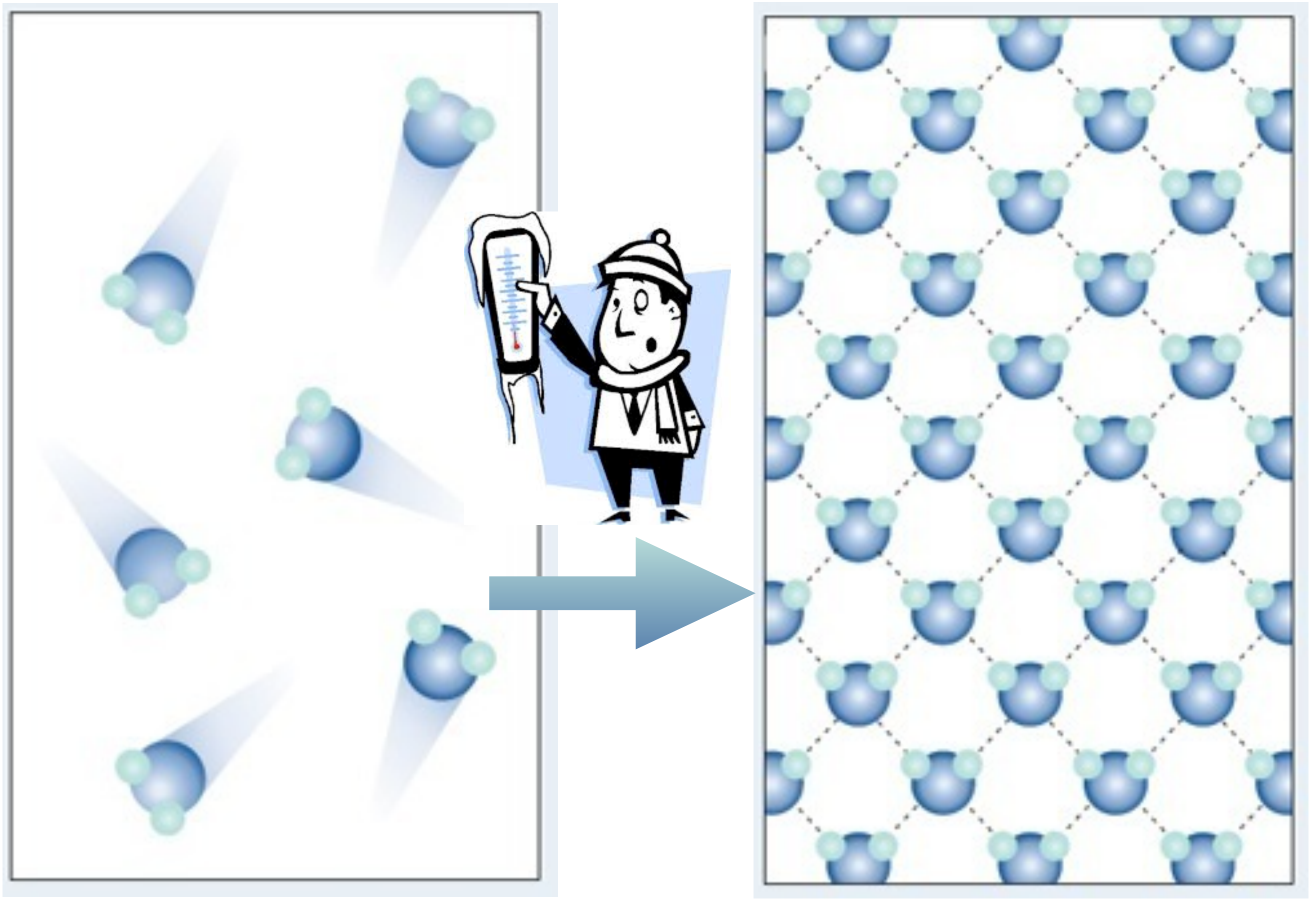
[a] Centre for Quantum Technology Theory, Swinburne University of Technology
[b] Instytut Fizyki Teoretycznej, Uniwersytet Jagielloński
[c] Optical Sciences Centre, Swinburne University of Technology

[1] New J. Phys. **23** 063012 (2021)
[2] PRA **104**, 053327 (2021)

AIP Congress, 16/12/2022

Crystallisation & Symmetry Breaking

Space crystallisation: Spontaneous breaking of space translational symmetry



Spontaneous symmetry breaking:

the Hamiltonian of the system respects a symmetry:

$$P^{\dagger}HP = H$$

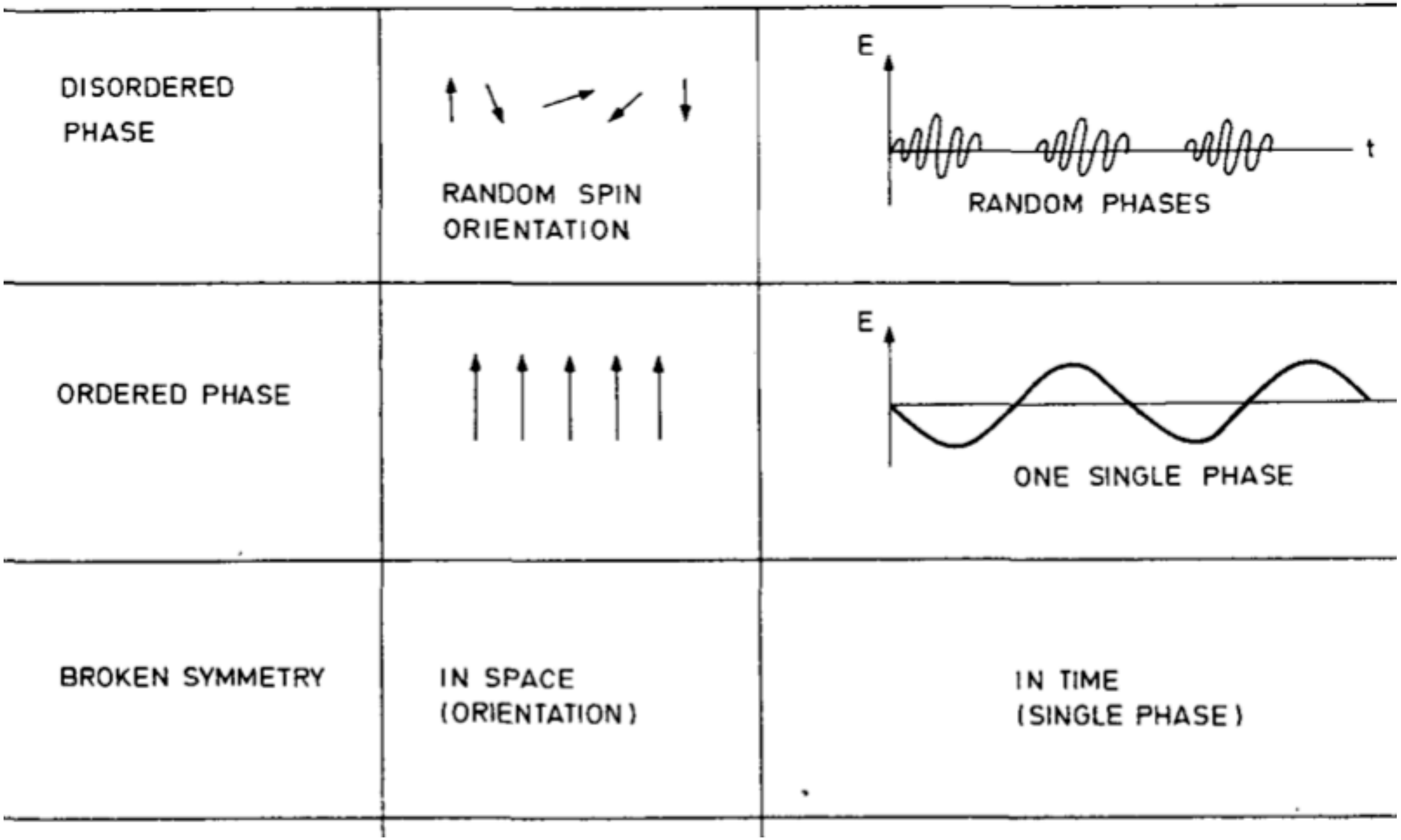
But the macroscopic equilibrium state of the system is non-invariant under the symmetry transformation

$$P\,|\psi_0\rangle \neq |\psi_0\rangle$$

(up to a global phase)

Phases of matter, such as crystals, magnets, and conventional superconductors, as well as simple phase transitions can be described by spontaneous symmetry breaking.—Wiki

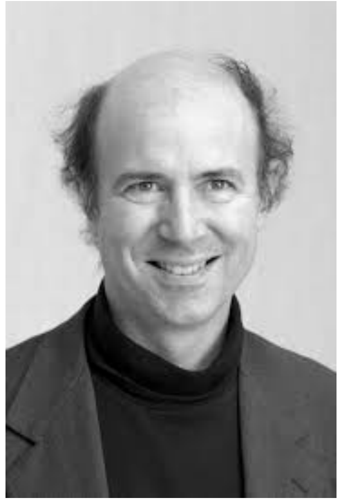
Magnetisation, Laser & Symmetry Breaking

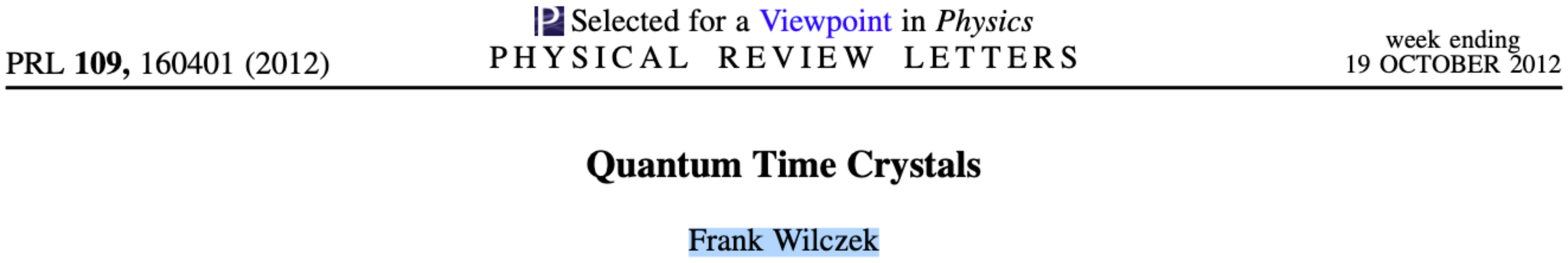


	FERROMAGNET Close quantum systems Equilibrium	LASER Open quantum system Non-equilibrium
DISORDERED PHASE	RANDOM SPIN ORIENTATION	RANDOM PHASES
ORDERED PHASE		ONE SINGLE PHASE
BROKEN SYMMETRY	IN SPACE (ORIENTATION)	IN TIME (SINGLE PHASE)

Laser Theory, H. Haken, 1973

What is a time crystal?





Selected for a Viewpoint in *Physics*

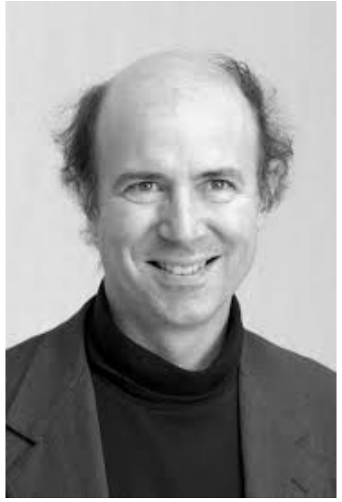
PRL **109,** 160401 (2012) PHYSICAL REVIEW LETTERS week ending 19 OCTOBER 2012

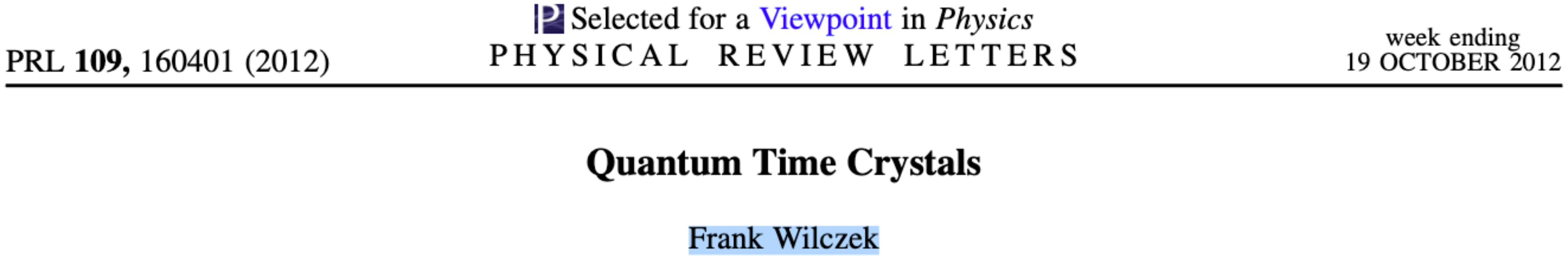
Quantum Time Crystals

Frank Wilczek

Some subtleties and apparent difficulties associated with the notion of spontaneous breaking of time-translation symmetry (TTS) in quantum mechanics are identified and resolved.

What is a time crystal?





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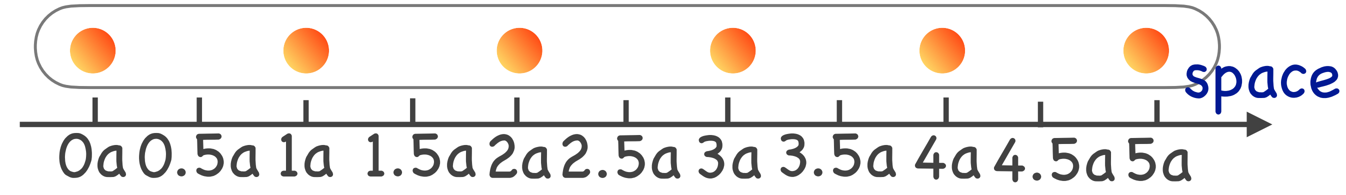

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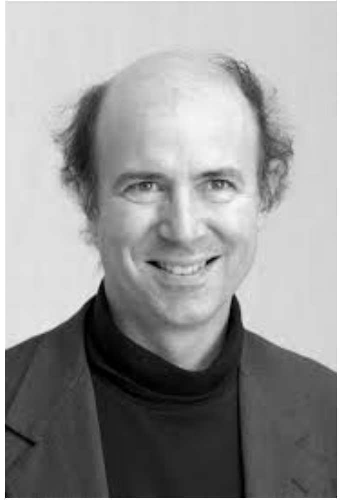

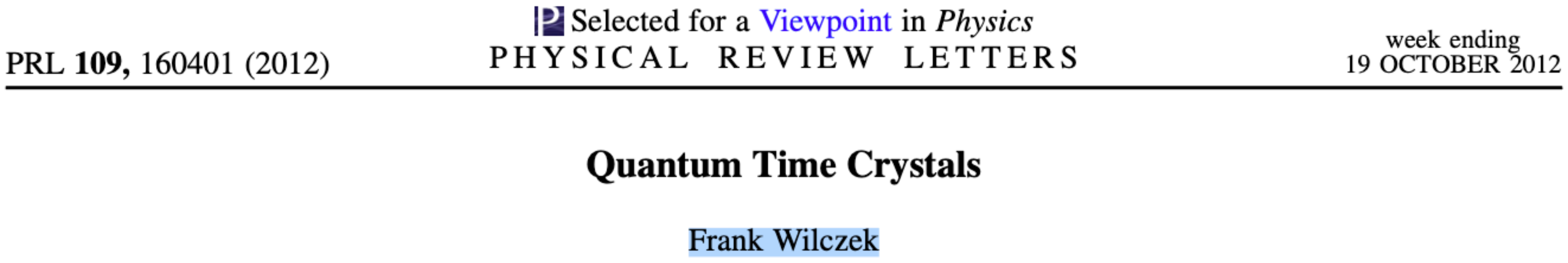

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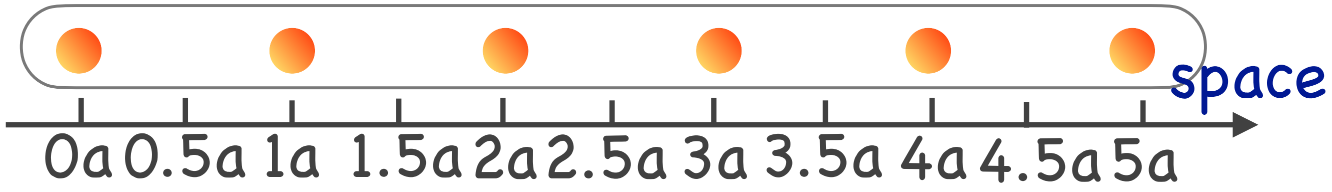
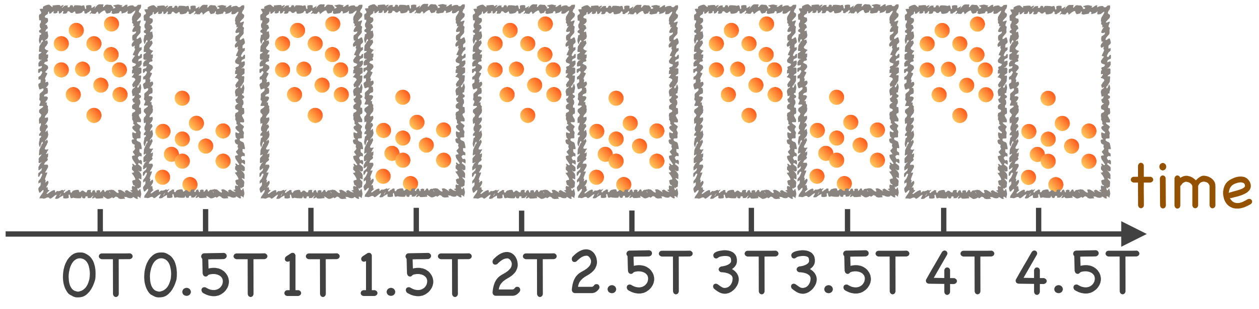


Spontaneous breaking of time translational symmetry: Time crystal $H(t) = H(t')$



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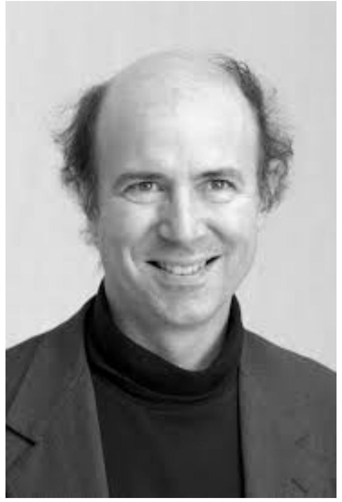

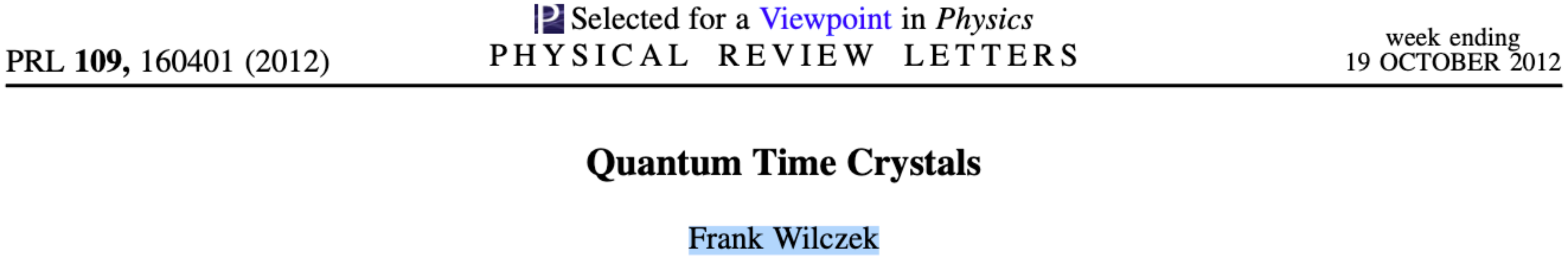

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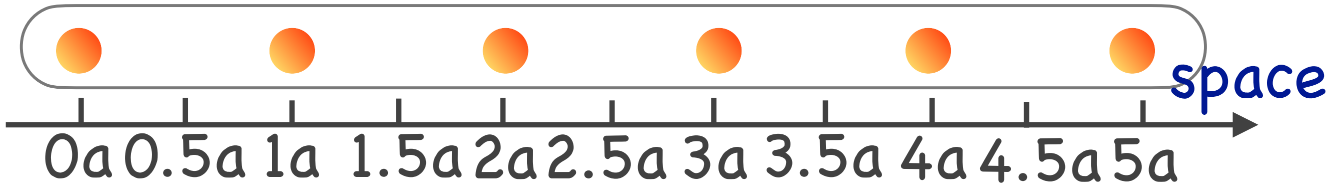
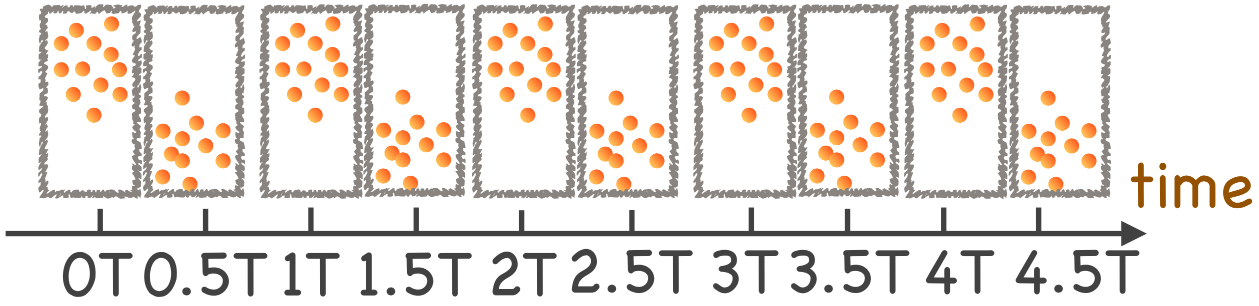
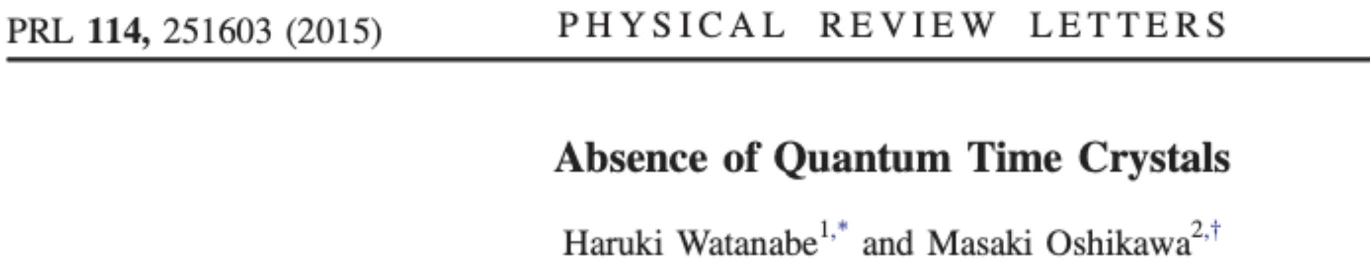


Spontaneous breaking of time translational symmetry: Time crystal

$$H(t) = H(t')$$





PRL **114,** 251603 (2015) PHYSICAL REVIEW LETTERS

Absence of Quantum Time Crystals

Haruki Watanabe[1,*] and Masaki Oshikawa[2,†]

… prove a no-go theorem that rules out the possibility of time crystals defined as such, in the ground state or in the canonical ensemble of a general Hamiltonian, which consists of not-too-long-range interactions.

Can a time crystal exist in a closed quantum system?

This no-go theorem **does not** rule out:

(1) long-range interacting system

(2) non-equilibrium systems (Floquet)

Discrete Time Crystal

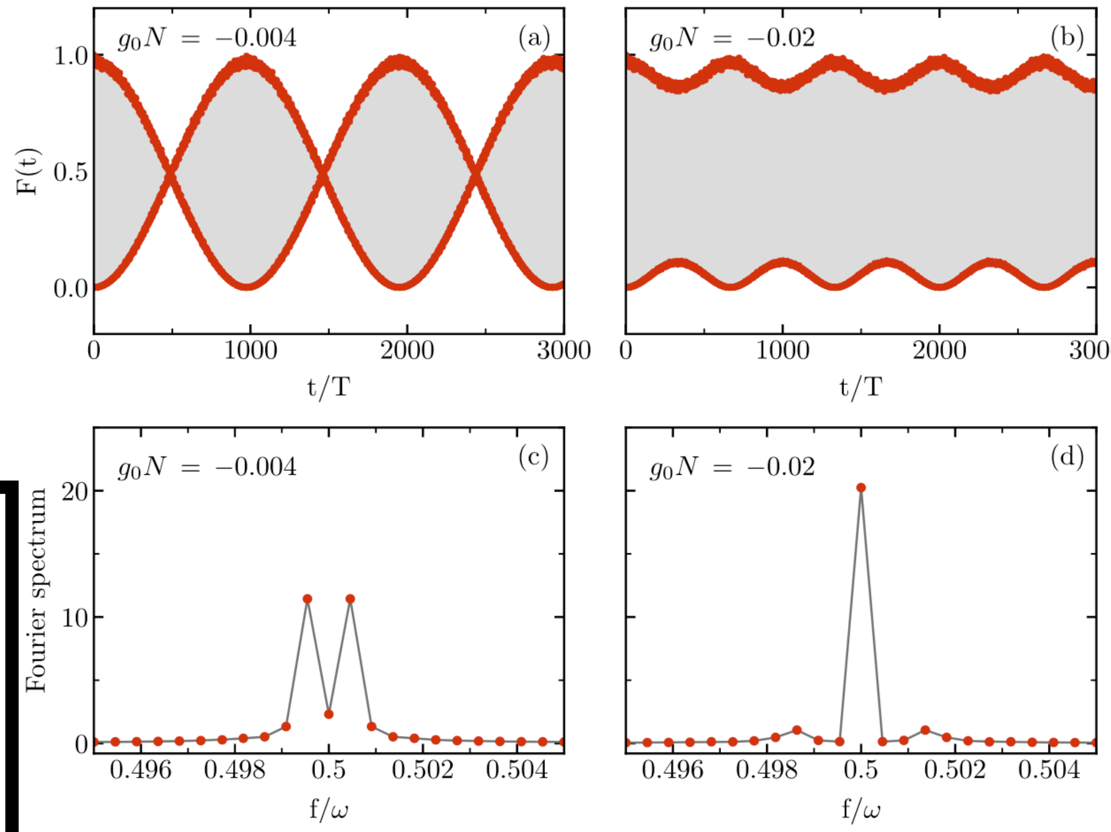
PHYSICAL REVIEW A **91**, 033617 (2015)

Modeling spontaneous breaking of time-translation symmetry

Krzysztof Sacha

$$\boxed{\begin{aligned}\hat{H}(t+T) &= \hat{H}(t)\\ |\psi(t)\rangle &= |\psi(t+2T)\rangle\\ |\psi(t)\rangle &\neq |\psi(t+T)\rangle\end{aligned}}$$

Discrete time-translational symmetry (DTTS)



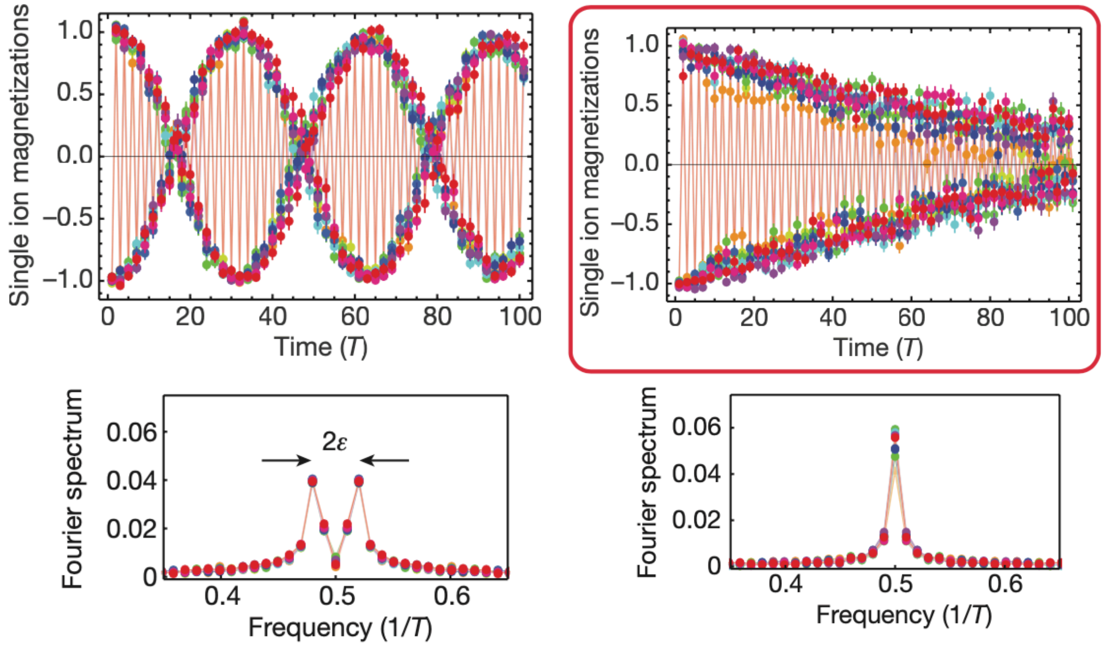
Experimental evidences!

Observation of a discrete time crystal

J. Zhang[1], P. W. Hess[1], A. Kyprianidis[1], P. Becker[1], A. Lee[1], J. Smith[1], G. Pagano[1], I.-D. Potirniche[2], A. C. Potter[3], A. Visl
N. Y. Yao[2] & C. Monroe[1,5]

Observation of discrete time-crystalline order in a disordered dipolar many-body system

Soonwon Choi[1]*, Joonhee Choi[1,2]*, Renate Landig[1]*, Georg Kucsko[1], Hengyun Zhou[1], Junichi Isoya[3], Fedor Jelezko[4], Shinobu Onoda[5], Hitoshi Sumiya[6], Vedika Khemani[1], Curt von Keyserlingk[7], Norman Y. Yao[8], Eugene Demler[1] & Mikhail D. Lukin[1]



Nature 2017 back-to-back papers

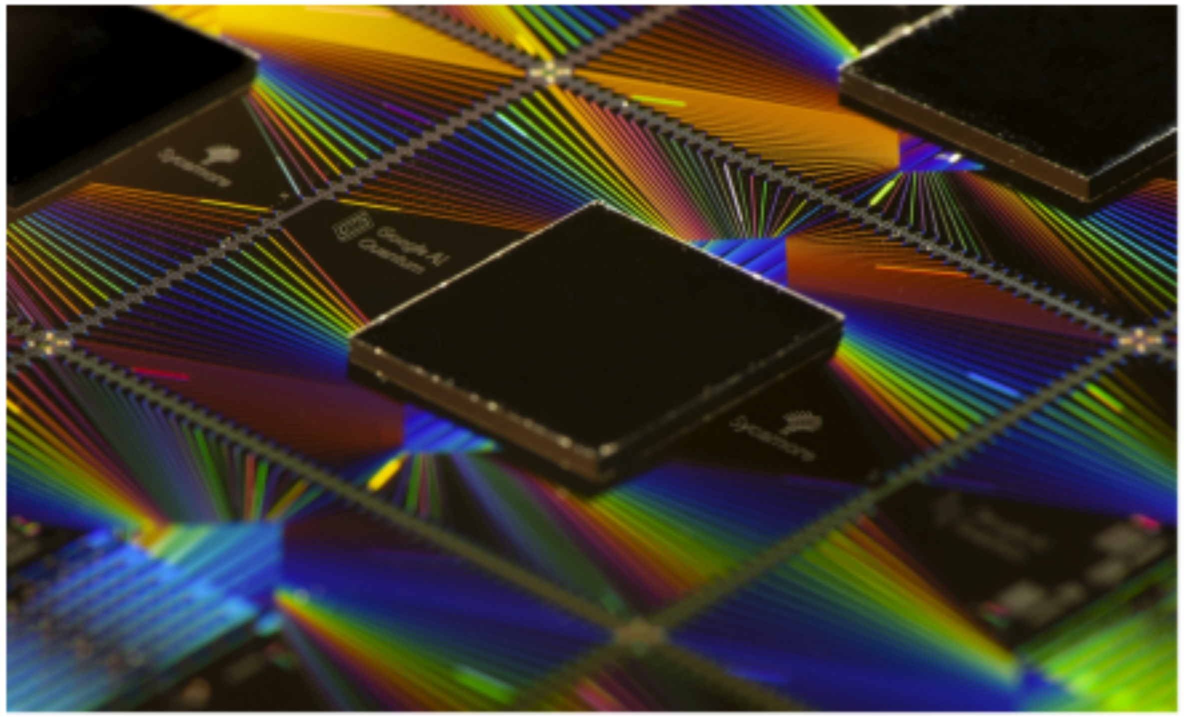
Turning a Quantum Computer into a Time Crystal

Time-crystalline eigenstate order on a quantum processor

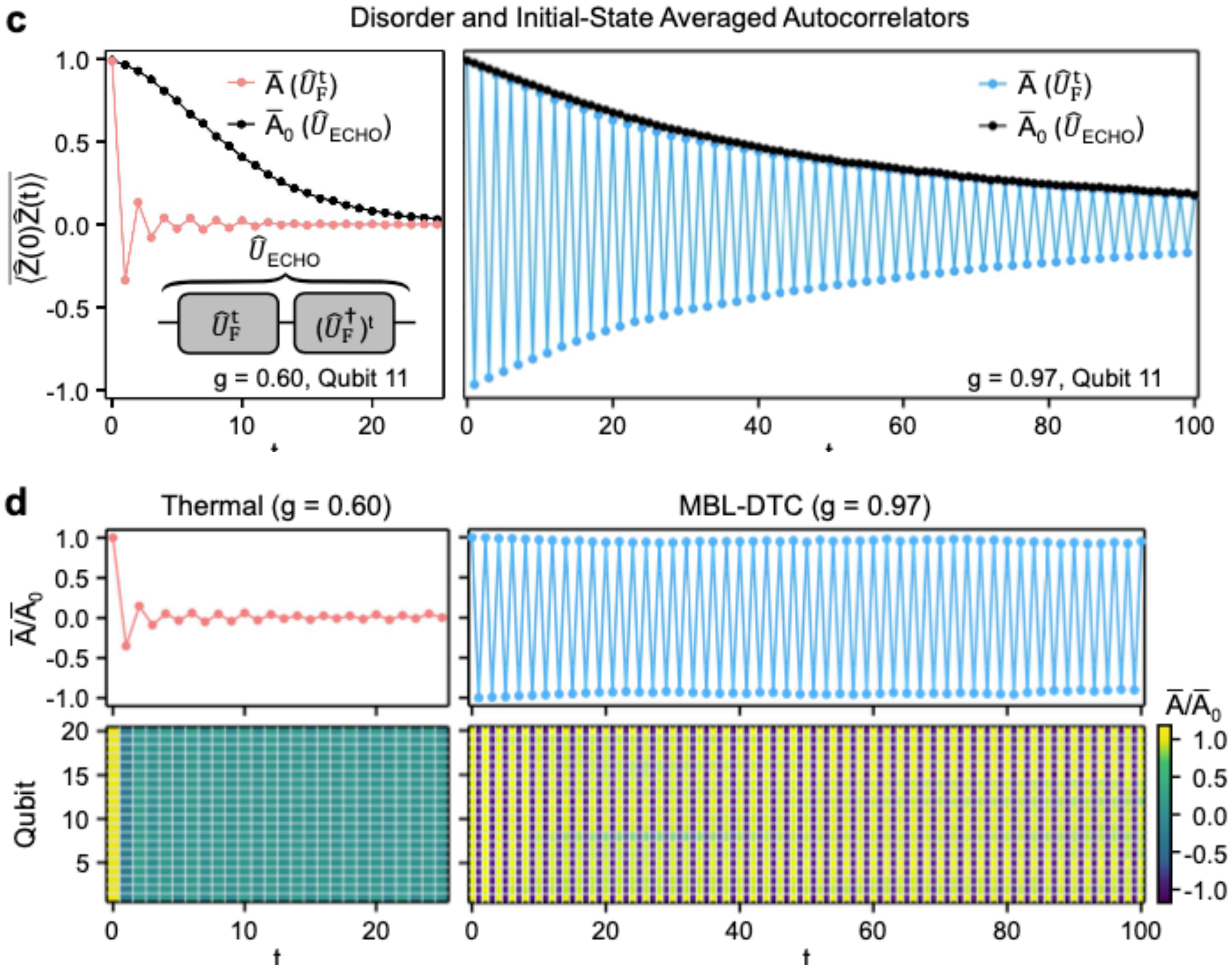
Xiao Mi, Matteo Ippoliti, ... Pedram Roushan ✉ + Show auth

Nature **601**, 531–536 (2022) | Cite this article

58k Accesses | **11** Citations | **1214** Altmetric | Metrics



A time crystal in a chip. Google's Sycamore quantum-computing circuit.
Credit: Google



on IBM's quantum computer:
P. Frey, S Rachel, (U. of Melbourne)
Science Advance 2022 (Thursday Talk)

Spontaneous Symmetry Breaking

Time-independent quantum system $H|\psi_0\rangle = E_0|\psi_0\rangle$

If the Hamiltonian obeys a symmetry: $P^\dagger HP = H \qquad HP = PH$

$$H(P|\psi_0\rangle) = PH|\psi_0\rangle = E_0(P|\psi_0\rangle)$$

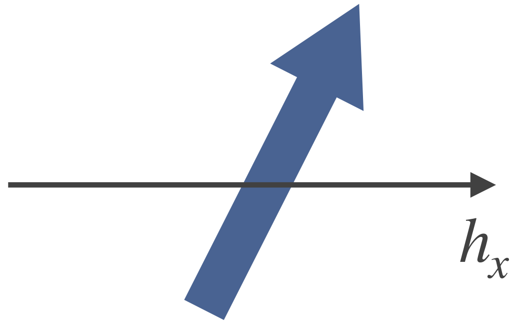
The ground state eigenstate should also obeys the same symmetry:

$P|\psi_0\rangle = |\psi_0\rangle$ up to a global phase.

if the ground state is non-degenerate.

Symmetry breaking can only occur if there are degeneracy.

Nevertheless, real degeneracy are usually forbidden by group theory: for example, Z_2 symmetry has two irreducible representations.



$$H = \begin{array}{cc} \begin{array}{cc} \uparrow & \downarrow \end{array} & \\ \begin{bmatrix} 0 & h_x \\ h_x & 0 \end{bmatrix} & \begin{array}{c} \uparrow \\ \downarrow \end{array} \end{array} \quad E_n = \pm h_x$$

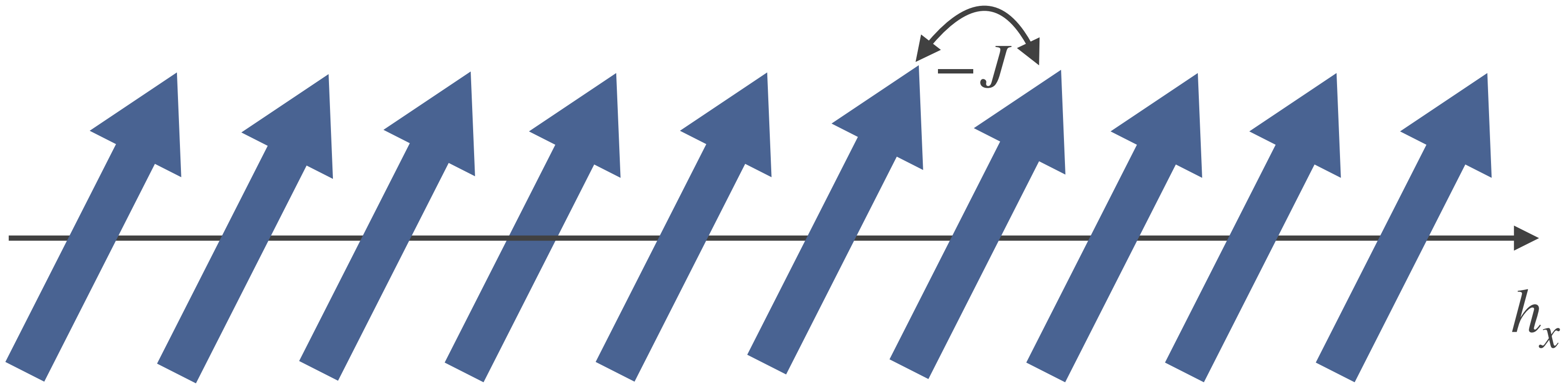
$$|\psi_S\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$$

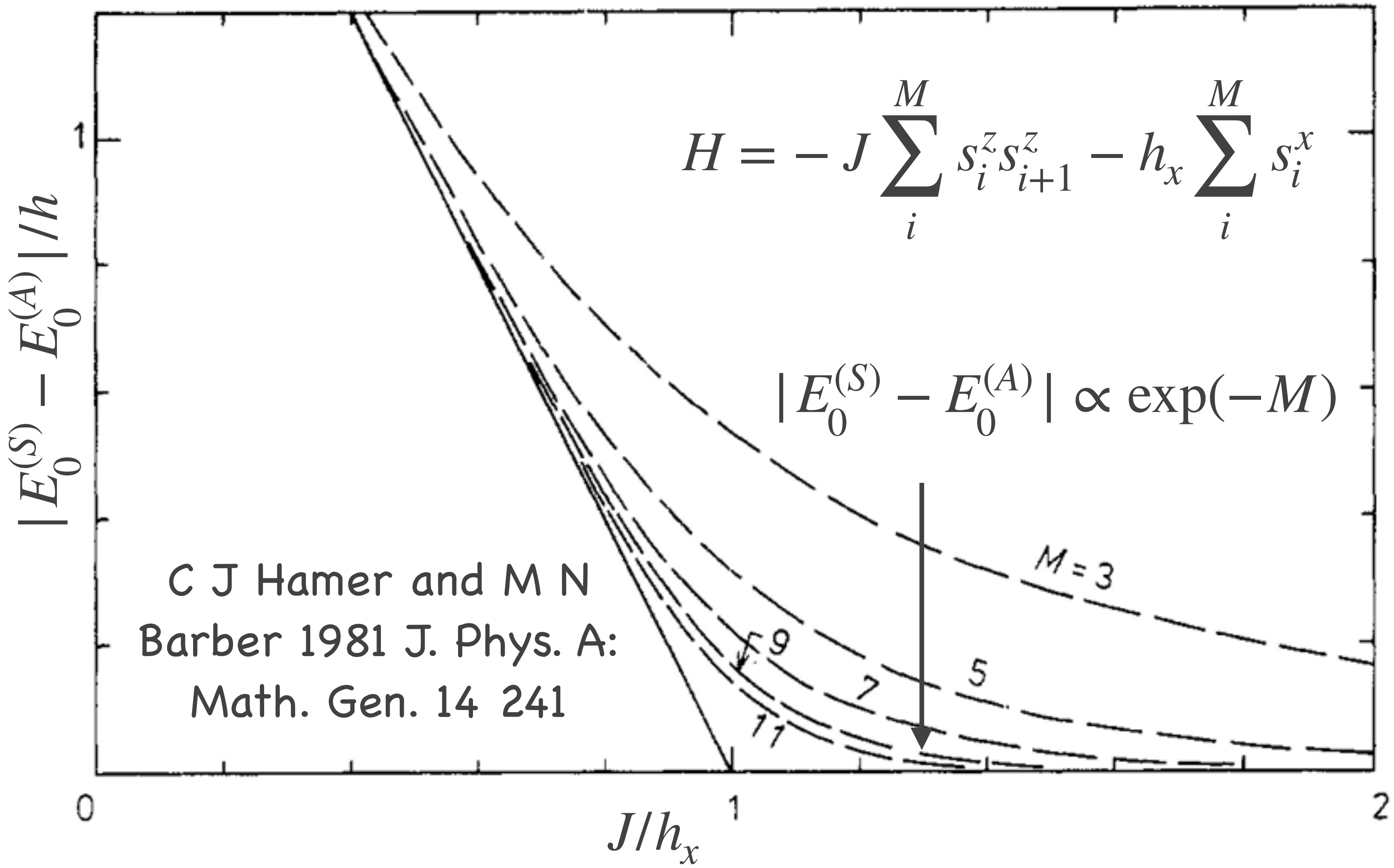
$$|\psi_A\rangle = (|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$$

Spontaneous Symmetry Breaking

Spontaneous symmetry breaking occurs in the thermodynamic limit

Transverse Ising Models with nearest-neighbour interactions:





$$H = -J \sum_{i}^{M} s_i^z s_{i+1}^z - h_x \sum_{i}^{M} s_i^x$$

$$|E_0^{(S)} - E_0^{(A)}| \propto \exp(-M)$$

C J Hamer and M N Barber 1981 J. Phys. A: Math. Gen. 14 241

$$|\psi_0^{(S)}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0^{(\uparrow)}\rangle + |\psi_0^{(\downarrow)}\rangle\right)$$

$$|\psi_0^{(A)}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0^{(\uparrow)}\rangle - |\psi_0^{(\downarrow)}\rangle\right)$$

Long-range correlated cat states

$$|\psi_0^{(\uparrow)}\rangle \approx |\uparrow\uparrow\uparrow\ldots\uparrow\rangle$$

$$|\psi_0^{(\downarrow)}\rangle \approx |\downarrow\downarrow\downarrow\ldots\downarrow\rangle$$

Short-range correlated states

Spontaneous Symmetry Breaking

SSB of time-independent Hamiltonian requires:

Many-body interacting system.

Degenerated ground (or thermal equilibrium) states.

Short-range correlated symmetry broken states.

Spontaneous Symmetry Breaking

SSB of time-independent Hamiltonian requires:	SSB of time-dependent Hamiltonian:
Many-body interacting system.	Driving will induce heating.
Degenerated ground (or thermal equilibrium) states.	No well-defined ground (eigen) states.
Short-range correlated symmetry broken states.	Robustness?

Spontaneous Symmetry Breaking

SSB of time-independent Hamiltonian requires:	SSB of time-dependent Hamiltonian:
Many-body interacting system.	Driving will induce heating.
Degenerated ground (or thermal equilibrium) states.	No well-defined ground (eigen) states.
Short-range correlated symmetry broken states.	Robustness?

For any short-range correlated initial state, the expectation values of local observables relax to those of a steady state that does not obey the symmetry of the Hamiltonian.

Floquet System $\hat{H}(t+T)=\hat{H}(t)$

Floquet formalism: Floquet states and Floquet quasi-eigenenergies:

$$\left[\hat{H}(t)-i\hbar\partial_t\right]\left|\phi_\nu(t)\right\rangle=\epsilon_\nu\left|\phi_\nu(t)\right\rangle \qquad \left|\phi_\nu(t)\right\rangle=\left|\phi_\nu(t+T)\right\rangle$$

The stroboscopic (t = T , 2T , 3T , . . .) dynamics of an isolated periodically driven quantum system are determined by a time-independent Floquet Hamiltonian.

$$\hat{H}_F=\sum_\nu \epsilon_\nu|\phi_\nu(0)\rangle\langle\phi_\nu(0)|$$

Quantum time evolution:

$$|\Psi(sT)\rangle=\sum_\nu c_\nu e^{-i\epsilon_\nu sT}\left|\phi_\nu(0)\right\rangle \qquad c_\nu=\left\langle\phi_\nu(0)\mid\Psi(0)\right\rangle$$

Well defined within a Brillouin zone

$$\epsilon_\nu\rightarrow\epsilon_\nu+m\hbar\omega$$

$$|\phi_\nu(t)\rangle\rightarrow e^{im\omega t}|\phi_\nu(t)\rangle$$

There are No ground state in isolated Floquet systems. SSB can be defined as the situation where the steady state is less symmetrical than its parent Hamiltonian.

Steady states and thermalisation

Equilibrium states of generic quantum systems subject to periodic driving

Achilleas Lazarides, Arnab Das, and Roderich Moessner
Phys. Rev. E **90**, 012110 – Published 11 July 2014

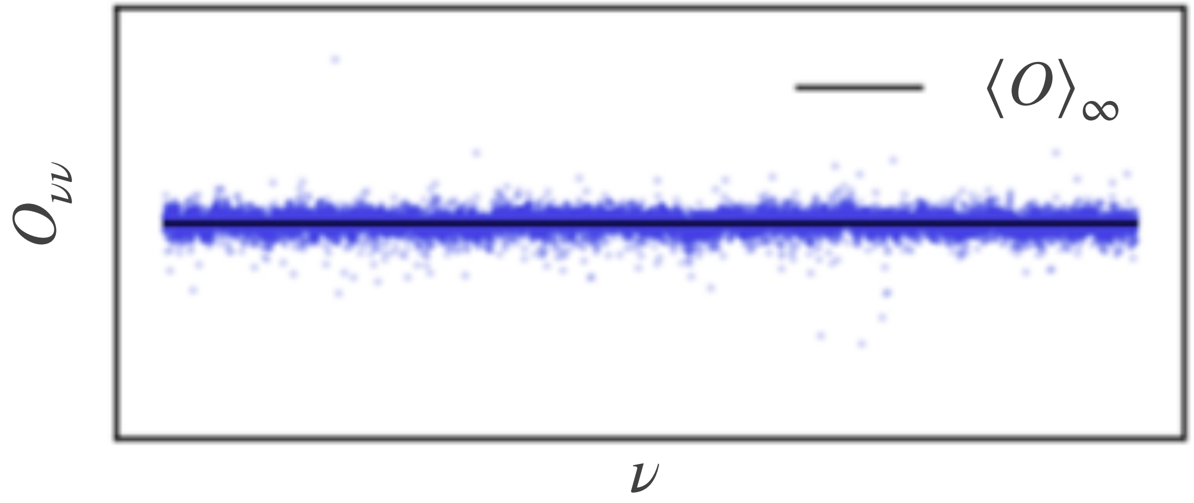
An article within the collection: *Physical Review E* 25th Anniversary Milestones

When a closed quantum system is driven periodically with period T, it approaches a periodic state synchronized with the drive in which any local observable measured stroboscopically approaches a steady value … here we show that for generic nonintegrable interacting systems, local observables become independent of the initial state entirely.

Expectation value of local observable: $\langle \hat{O}(t)\rangle \equiv \langle \Psi(t)|\hat{O}|\Psi(t)\rangle = \sum_{\nu,\mu} c_\nu^* c_\mu e^{i(\epsilon_\nu-\epsilon_\mu)t} O_{\nu\mu}(t)$ $\quad O_{\nu\mu}(t) = \langle \phi_\nu(t)|\hat{O}|\phi_\mu(t)\rangle$

Longtime average: $\overline{\langle \hat{O}(nT)\rangle} = \sum_\nu |c_\nu|^2 O_{\nu\nu}(nT)$

One Floquet eigenstates consist of a mixture of the exponentially many eigenstates of the undriven Hamiltonian.



Floquet-ETH: $\overline{\langle \hat{O}(nT)\rangle} = O_{\nu\nu}(nT) = \langle O\rangle_\infty$

Spontaneous Symmetry Breaking

SSB of time-independent Hamiltonian requires:

Many-body interacting system.

Degenerated ground (or thermal equilibrium) states.

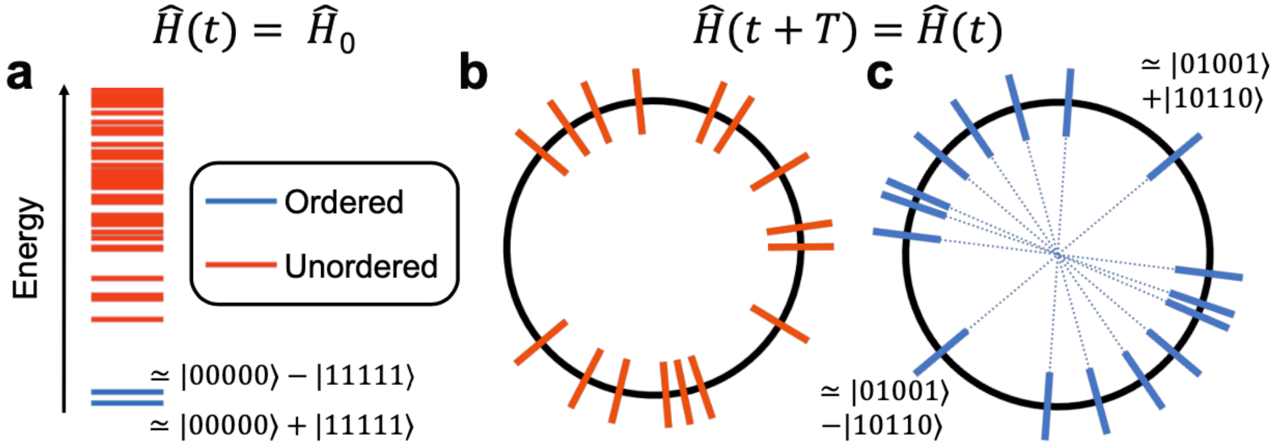
Short-range correlated symmetry broken states.

SSB of time-dependent Floquet Hamiltonian:

Driving will induce heating.

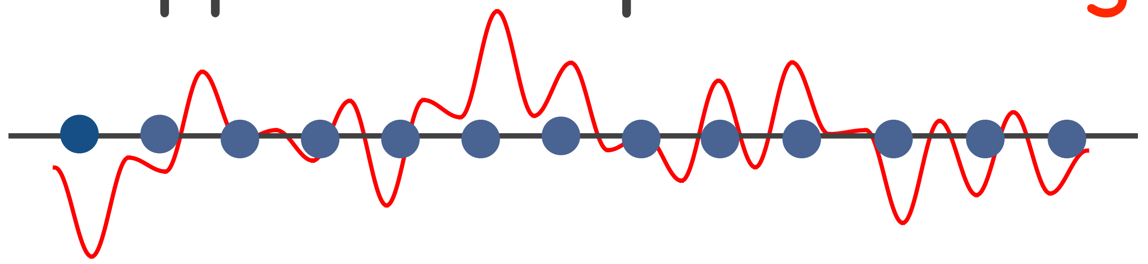
ALL (many) Floquet states are pairing.

Short-range correlated symmetry broken Floquet states.



$$\langle \hat{O}(t)\rangle \equiv \langle\Psi(t)|\hat{O}|\Psi(t)\rangle = \sum_{\nu,\mu} c_\nu^* c_\mu \boxed{e^{i(\epsilon_\nu-\epsilon_\mu)t}} O_{\nu\mu}(t)$$

Suppress Floquet heating.



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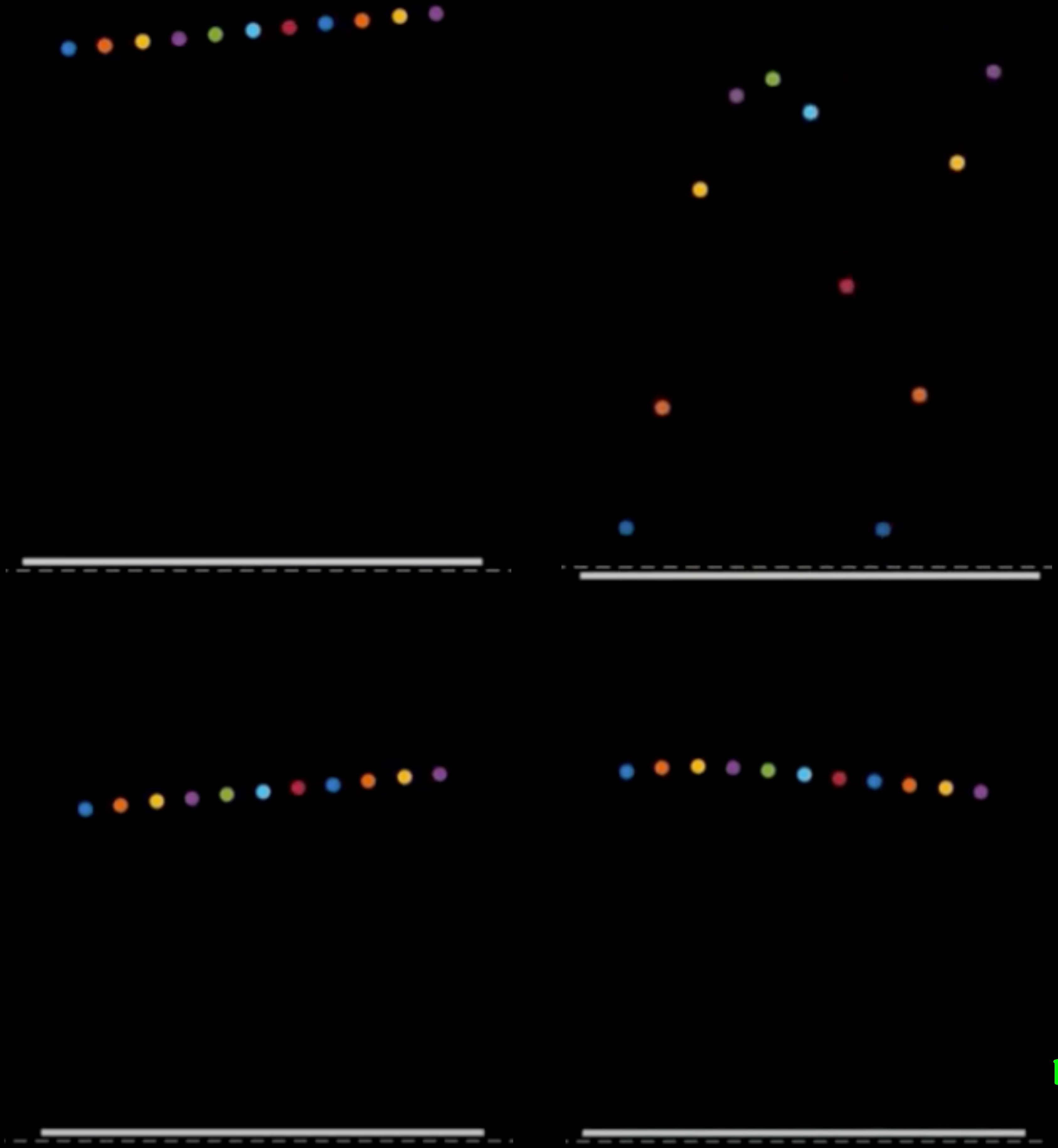
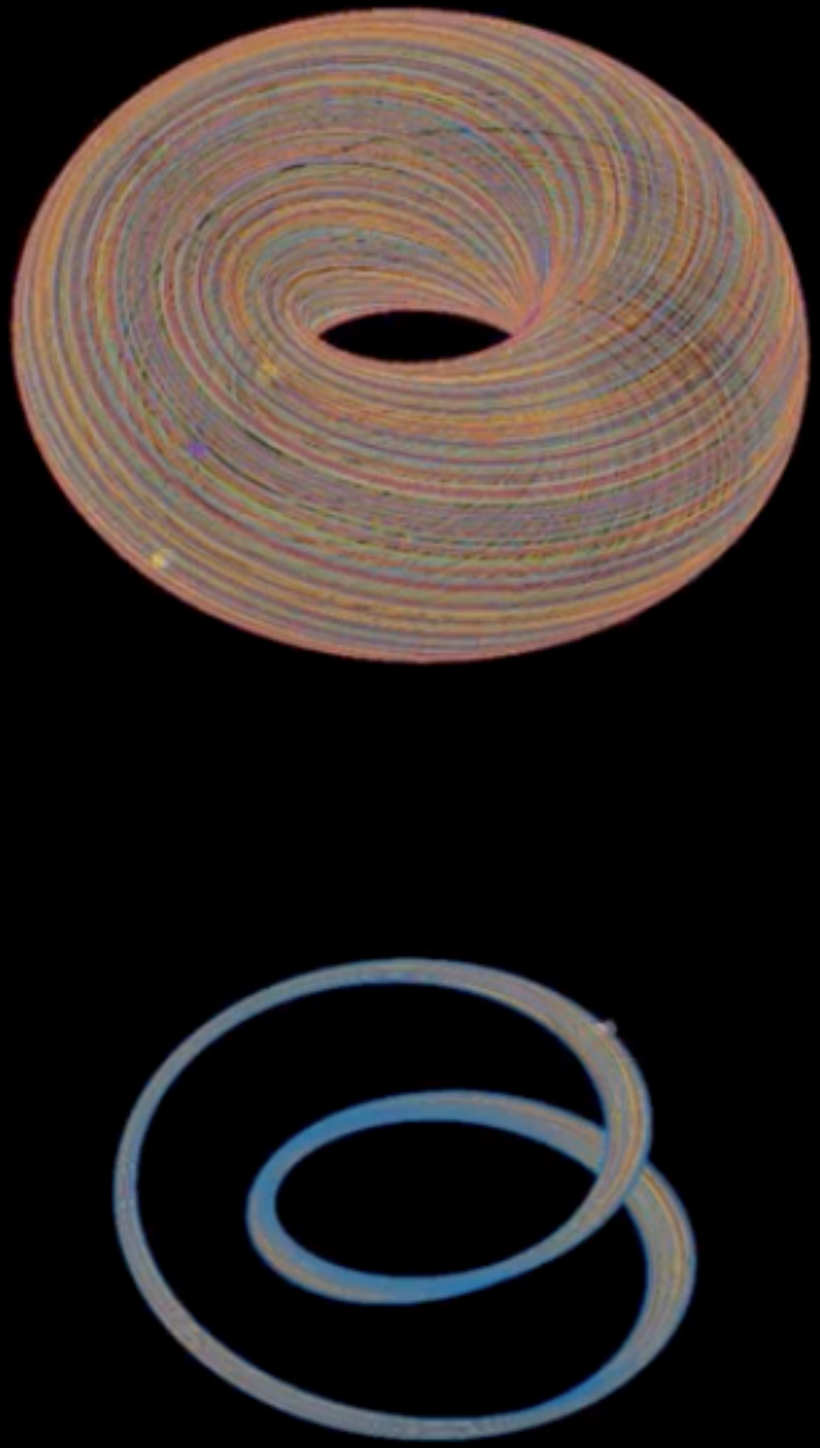




Exhibition “UNDUPLICATED”
at Hong Kong,
https://recfro.github.io/unduplicated

Floquet and Wannier Mode

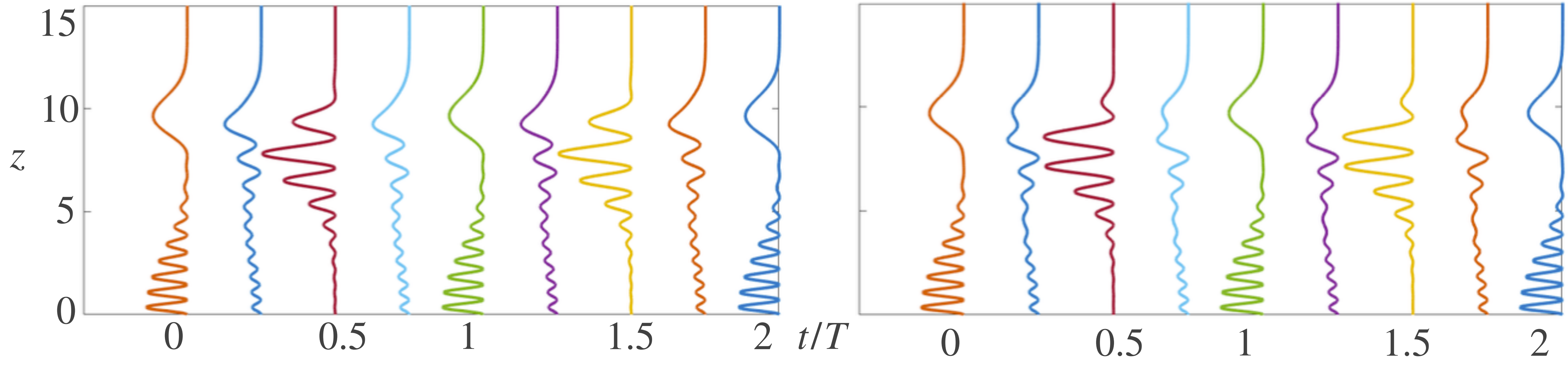
Single-particle Floquet Modes (T-periodic):

$\phi_1(z, t)$

$J = \epsilon_1 - \epsilon_2 - \omega/2$

small but finite.

$\phi_2(z, t)$



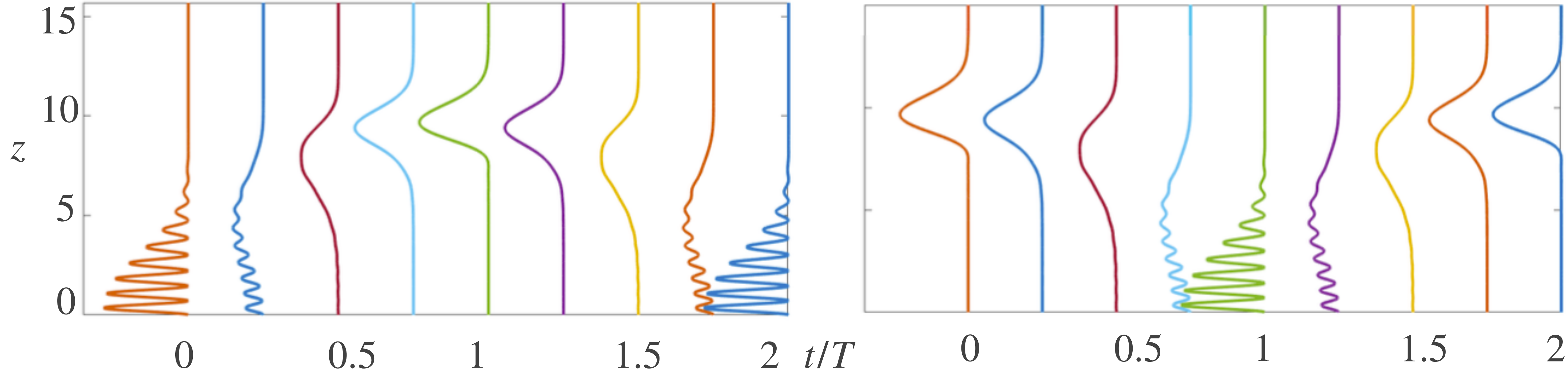
Single-particle Wannier Modes (2T-periodic):

Mode 1

$\Phi_1(z, t)$

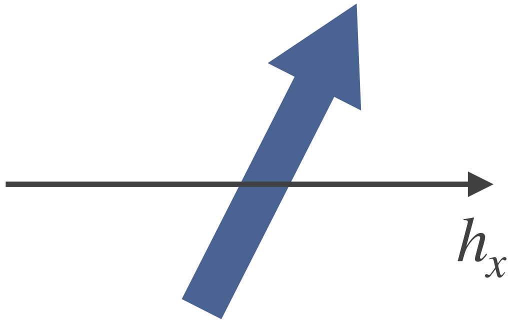
Mode 2

$\Phi_2(z, t)$



Single-particle picture

Single particle system cannot have symmetry breaking:



$$H = \begin{array}{cc} \uparrow & \downarrow & \\ \left[\begin{array}{cc} 0 & h_x \\ h_x & 0 \end{array}\right. & & \!\!\!\left.\begin{array}{c} \\ \\ \end{array}\right] \begin{array}{c} \uparrow \\ \downarrow \end{array} \end{array}$$

$$|\psi_S\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$$

$$|\psi_A\rangle = (|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$$

Symmetry broken states $|\uparrow\rangle$ and $|\downarrow\rangle$ are NOT degenerated and NOT eigenstates.

$$H = \begin{array}{ccc} \Phi_1 & \Phi_2 & \\ \left[\begin{array}{c} 0 \\ J \end{array}\right. & \left.\begin{array}{c} J \\ 0 \end{array}\right] & \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \end{array}$$

$$\Phi_1(z,t) = [\phi_1(z,t) + e^{-i\pi t/T}\phi_2(z,t)]/\sqrt{2}$$

$$\Phi_2(z,t) = [\phi_1(z,t) - e^{-i\pi t/T}\phi_2(z,t)]/\sqrt{2}$$

Symmetry broken states Φ_1 and Φ_2 are NOT pi-pairing and NOT Floquet states.

Symmetry broken in the many-body states with interaction:

$$|\uparrow\uparrow\ldots\uparrow\rangle \qquad \Phi_2\Phi_2\ldots\Phi_2 = |N_1 = 0, N_2 = N\rangle$$

Previous studies

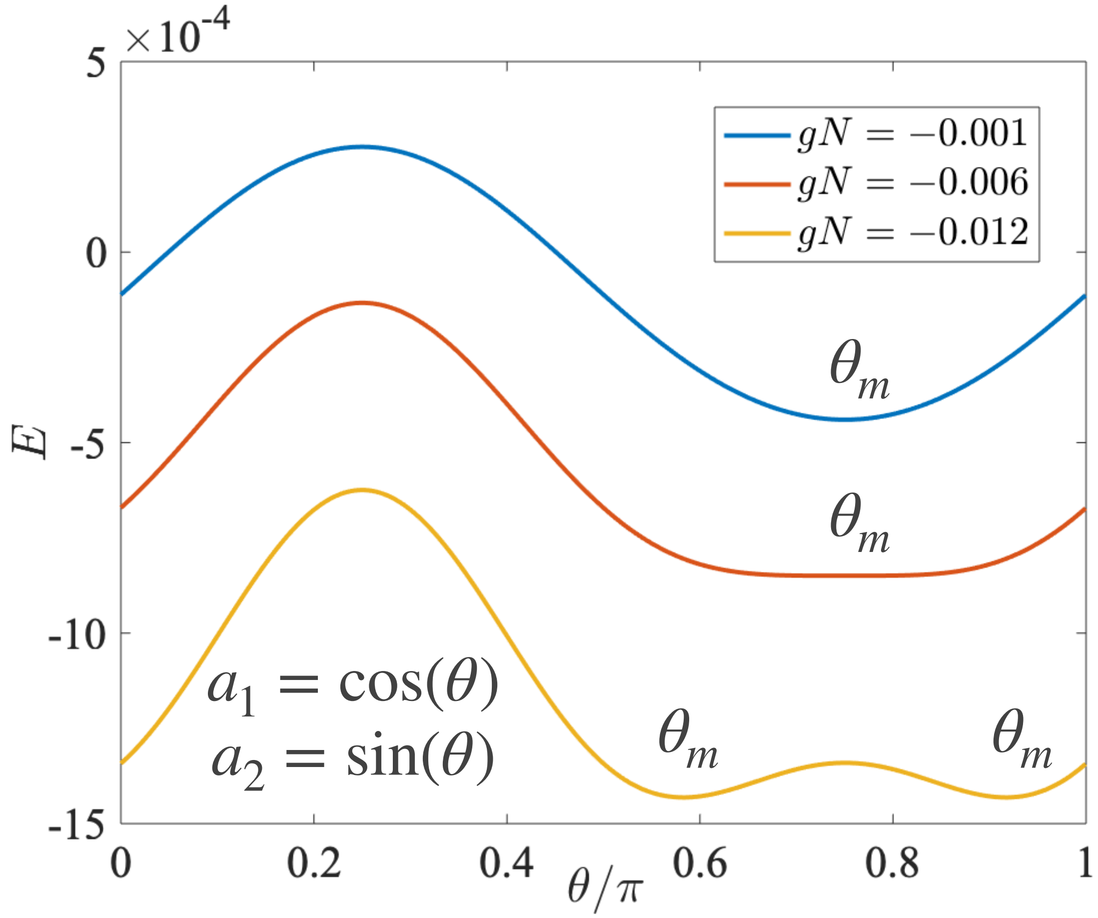
Mean-field GPE: K. Sacha PRA 2015 and Kuroś NJP 2020.

Time-dependent Bogoliubov: Kuroś NJP 2020.

Two-mode model: K. Sacha PRA 2015.

$$\psi \approx \phi_1 a_1 + \phi_2 a_2$$

$$E = \int_0^\infty dz \int_0^{4\pi/\omega} dt\psi^* \left(H_0 - i\partial_t + \frac{g_0 N}{2}|\psi|^2\right)\psi$$
$$\approx -\frac{J}{2}(a_1^* a_2 + a_2^* a_1) + \frac{UN}{2}\left(|a_1|^4 + |a_2|^4\right)$$
$$+ 2U_{12}N|a_1|^2|a_2|^2 + \text{const},$$



A concern in applying a mean-field (single-mode) or a few-mode approach to study time crystals and discrete time-translation symmetry breaking is whether the lack of thermalisation and decay of the condensate in such studies is an artefact imposed by the adopted approximations

Truncated Wigner Approximation

Many-body Hamiltonian: $\hat{H} = \int dz \Big[\hat{\Psi}(z)^\dagger \hat{H}_{\rm sp} \hat{\Psi}(z) + \frac{g}{2} \hat{\Psi}(z)^\dagger \hat{\Psi}(z)^\dagger \hat{\Psi}(z) \hat{\Psi}(z) \Big]$

(1) The Liouville von-Neumann Equation for the density operator is mapped onto the Functional Fokker-Planck Equation (FFPE) for the Wigner distribution functional.

(2) Neglecting the third-order functional derivatives, the FFPE is solved by the equivalent Ito Stochastic Field Equation.

$$\boxed{\frac{\partial}{\partial t}\widetilde{\psi}(z,t) = -\frac{i}{\hbar}\Big[H_{\rm sp}\widetilde{\psi}(z,t) + g\Big\{\widetilde{\psi}^{+}(z,t)\widetilde{\psi}(z,t) - \delta_C(z,z)\Big\}\widetilde{\psi}(z,t)\Big]}$$

(3) Initial stochastic field functions

$$\widetilde{\psi}(z) = \tilde{\gamma}_0 \psi_c(z) + \sum_{k\neq 0}\Big[u_k(z)\tilde{\beta}_k - v_k(z)^*\tilde{\beta}_k^{+}\Big]$$

$\psi_c(z)$ is the condensate mode

$u_k(z), v_k(z)$ are Bogoliubov modes

(4) Observables are obtained via stochastic average.

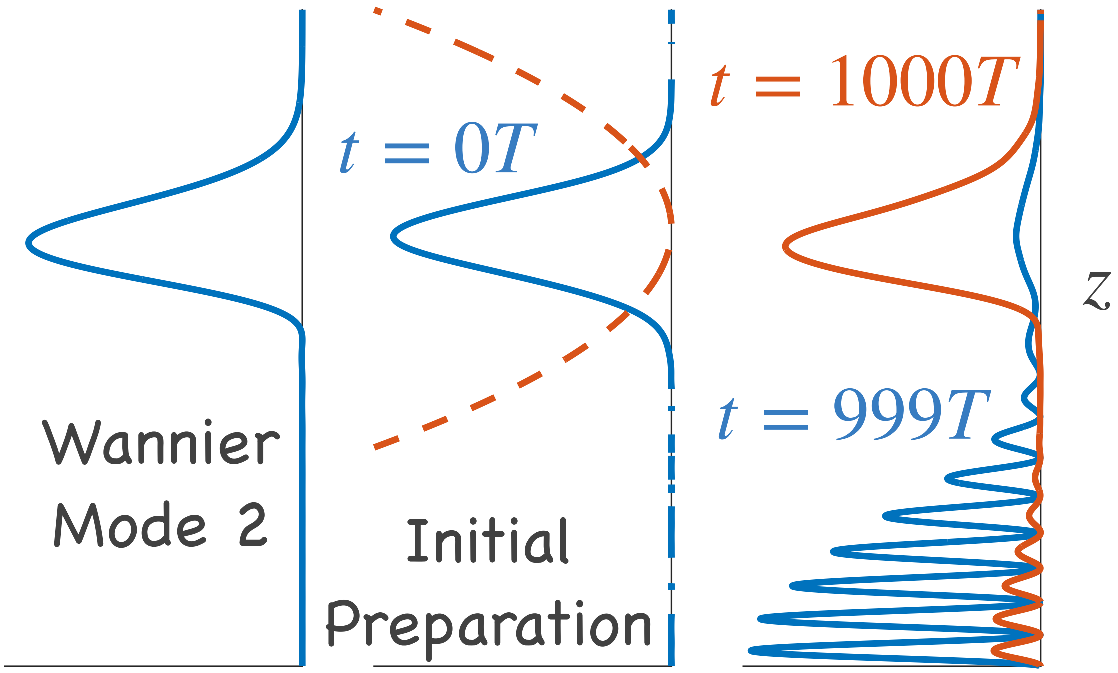
$$F(z,t) = \mathrm{Tr}\Big(\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)\rho(t)\Big) = \overline{\widetilde{\psi}(z,t)\widetilde{\psi}^{+}(z,t)} - \frac{1}{2}\delta_C(z,z) \quad \delta_C(z,z^{\#}) = \sum_k \phi_k(z)\phi_k(z^{\#})^*$$

Key feature:

(a) Asymptotic exact in the large boson particle number limit

(b) Multi-mode theory that can treat thermalisation.

TWA Results



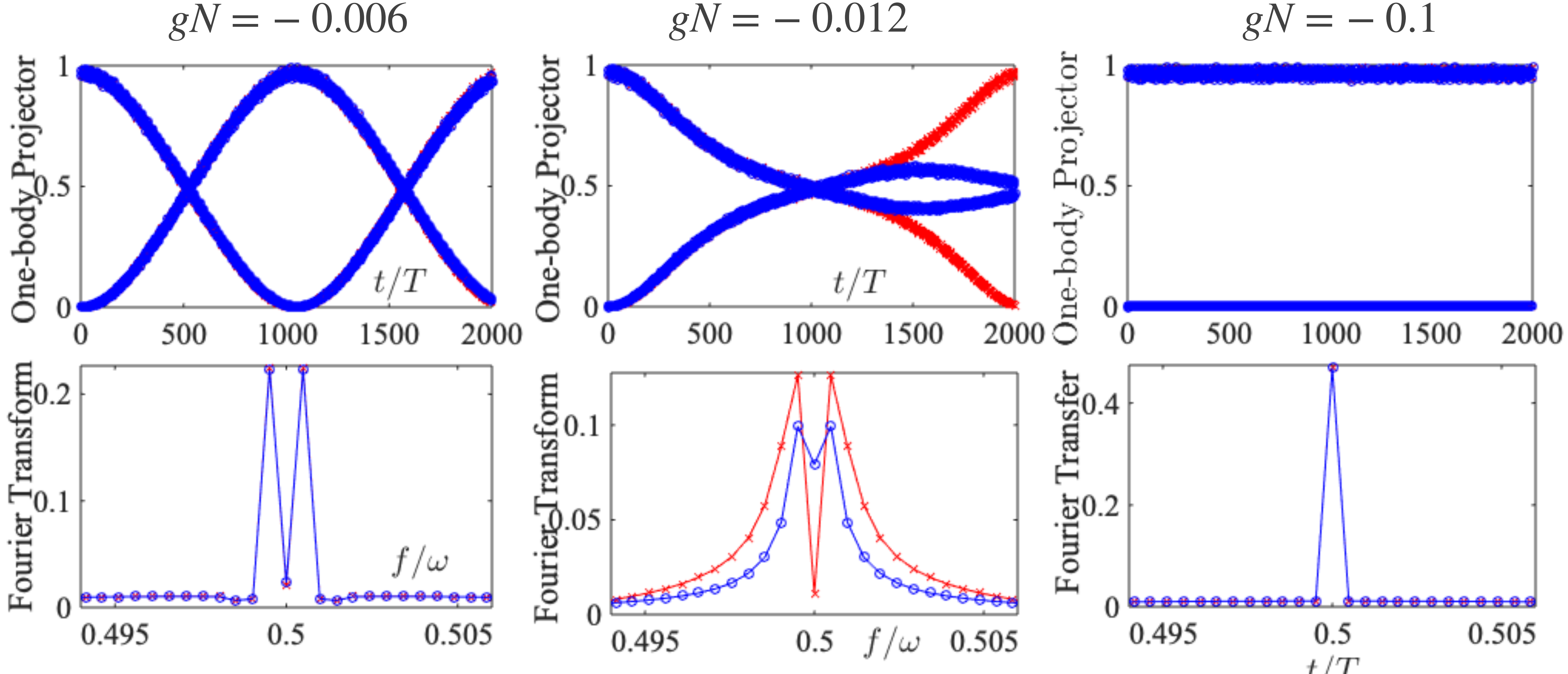
JW, Peter Hannaford, and Bryan Dalton

New J. Phys. **23** 063012 (2021)

One-body projector:

$$P(z,z^{\#},t) = \widetilde{\psi}(z,t)\widetilde{\psi}^{+}(z\#,t) - \frac{1}{2}\delta_C(z,z^{\#})$$

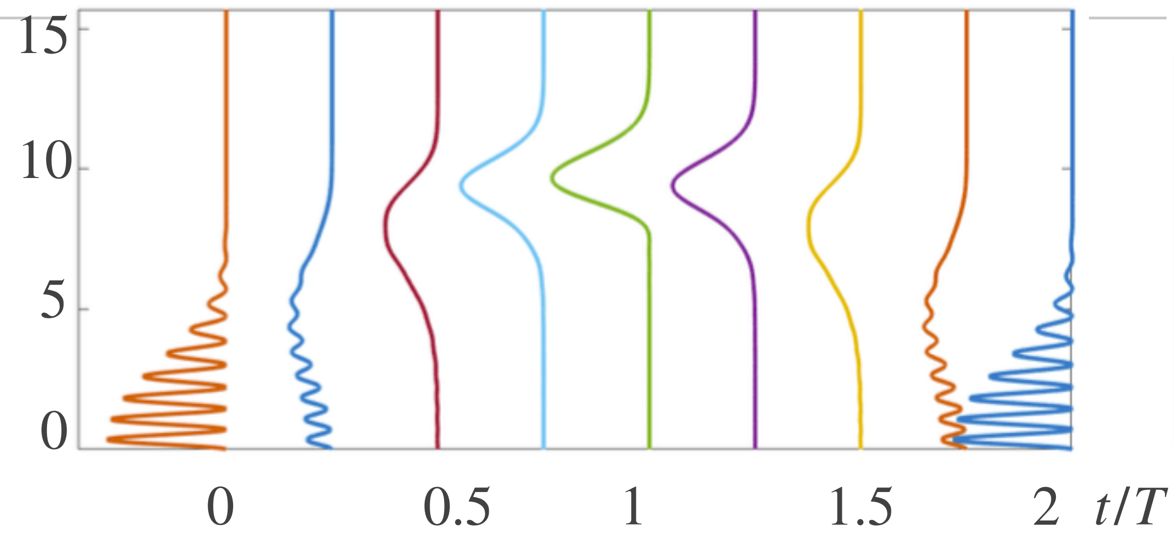
$$M_c(t) = \iint dzdz^{\#}\psi_c(z^{\#},0)\psi_c^{*}(z,0)P(z,z^{\#},t)$$

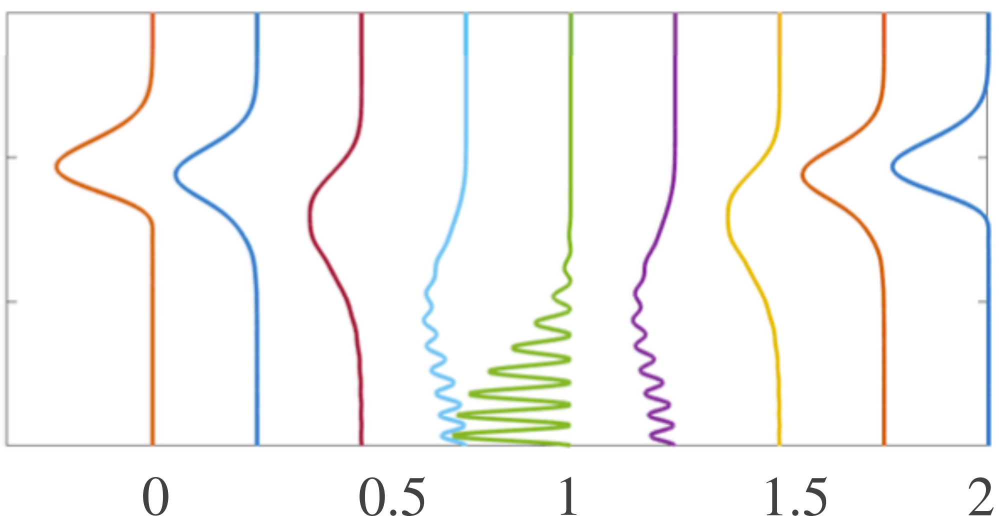


Only Two Modes

Mode 1

$\Phi_1(z, t)$



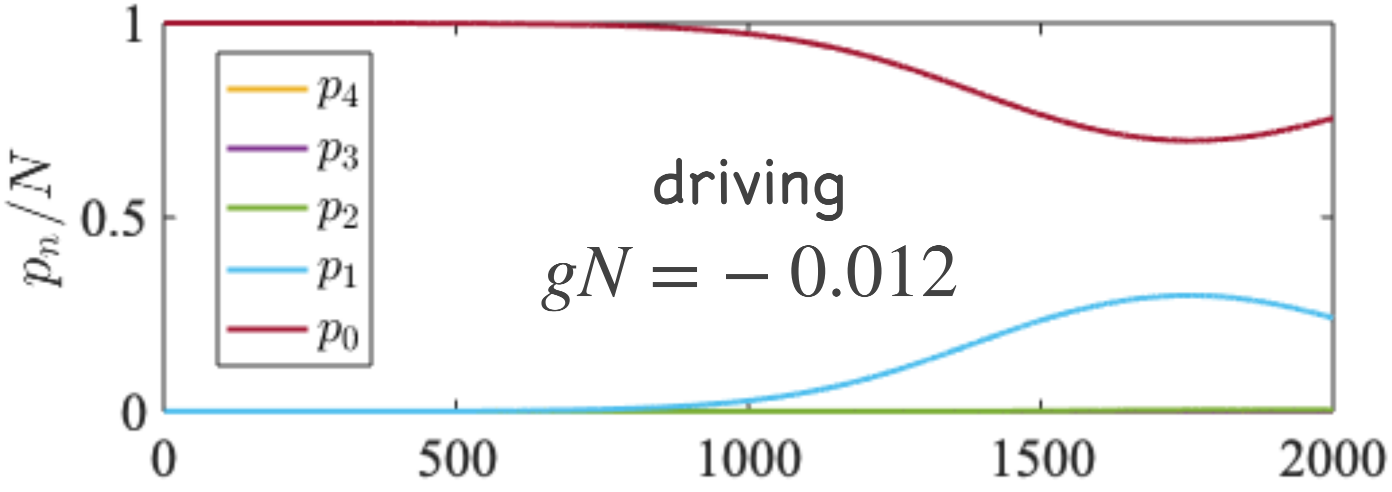


Mode 2

$\Phi_2(z, t)$

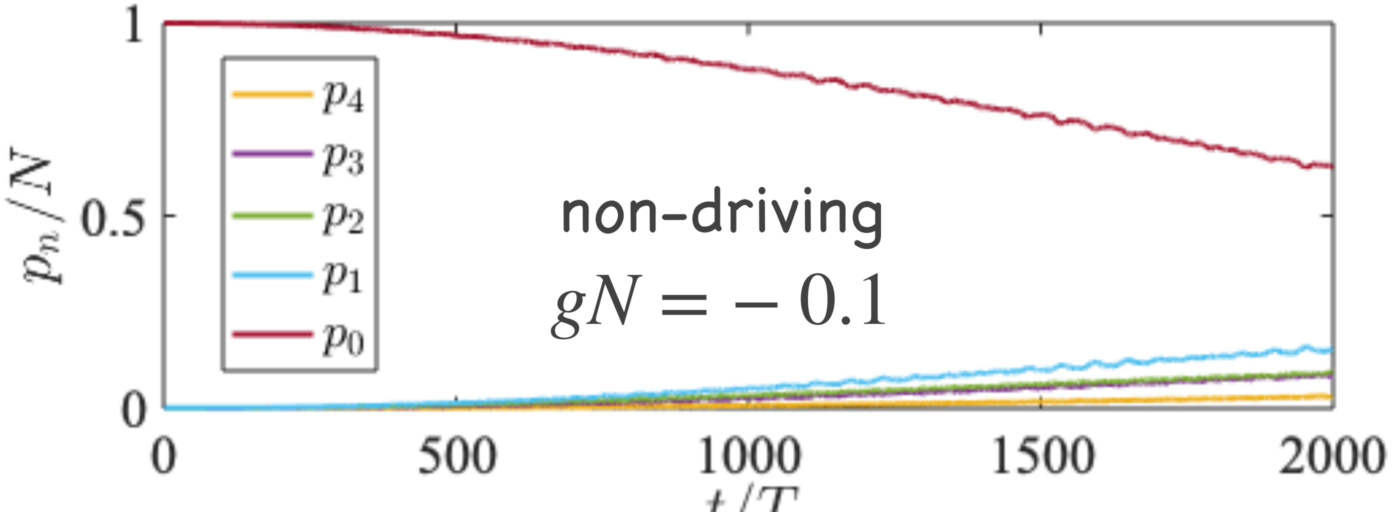
N_1, N_2 : Number of particles in Mode 1 and 2

$N_s = N_1 + N_2$



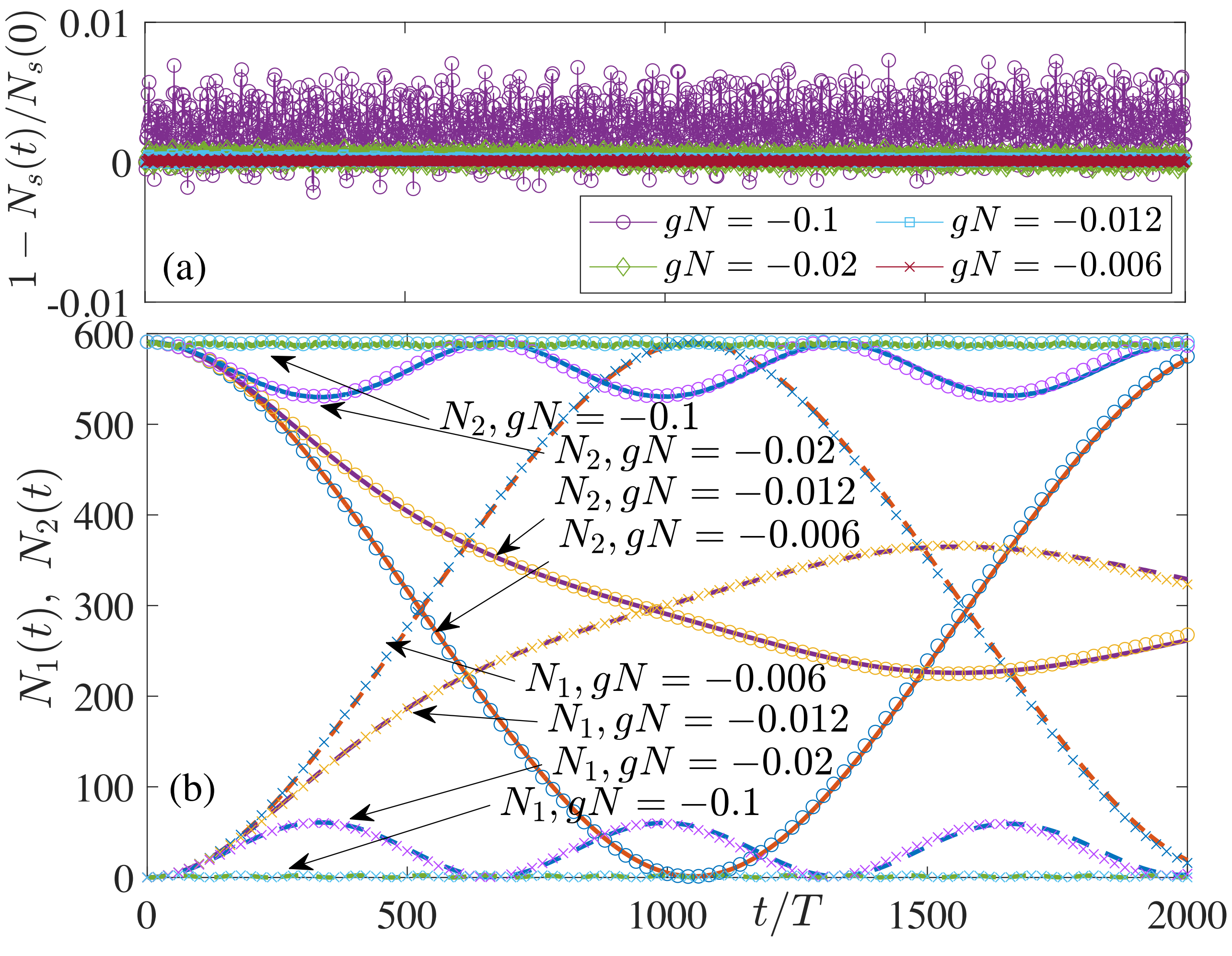
driving

$gN = -0.012$

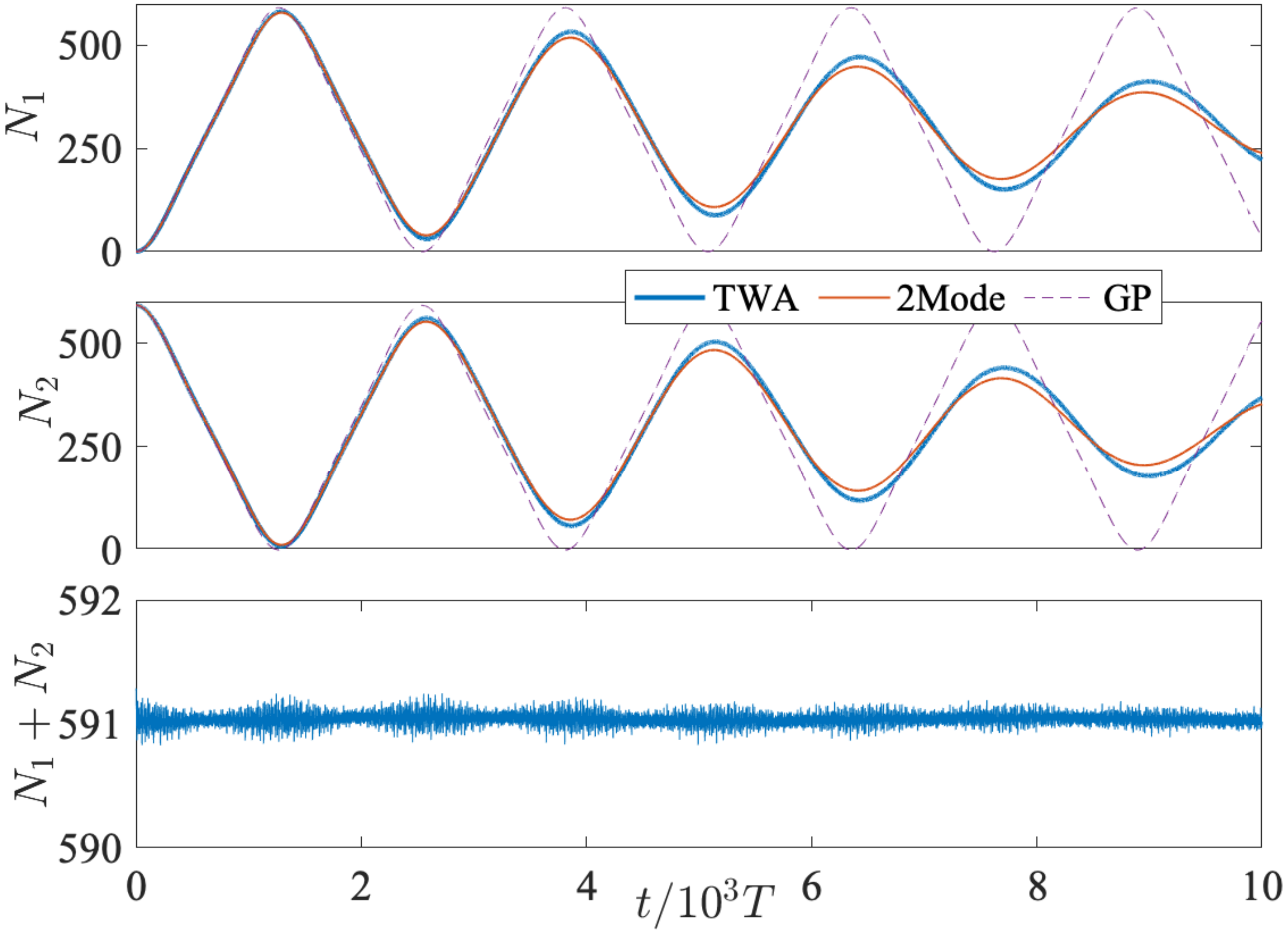


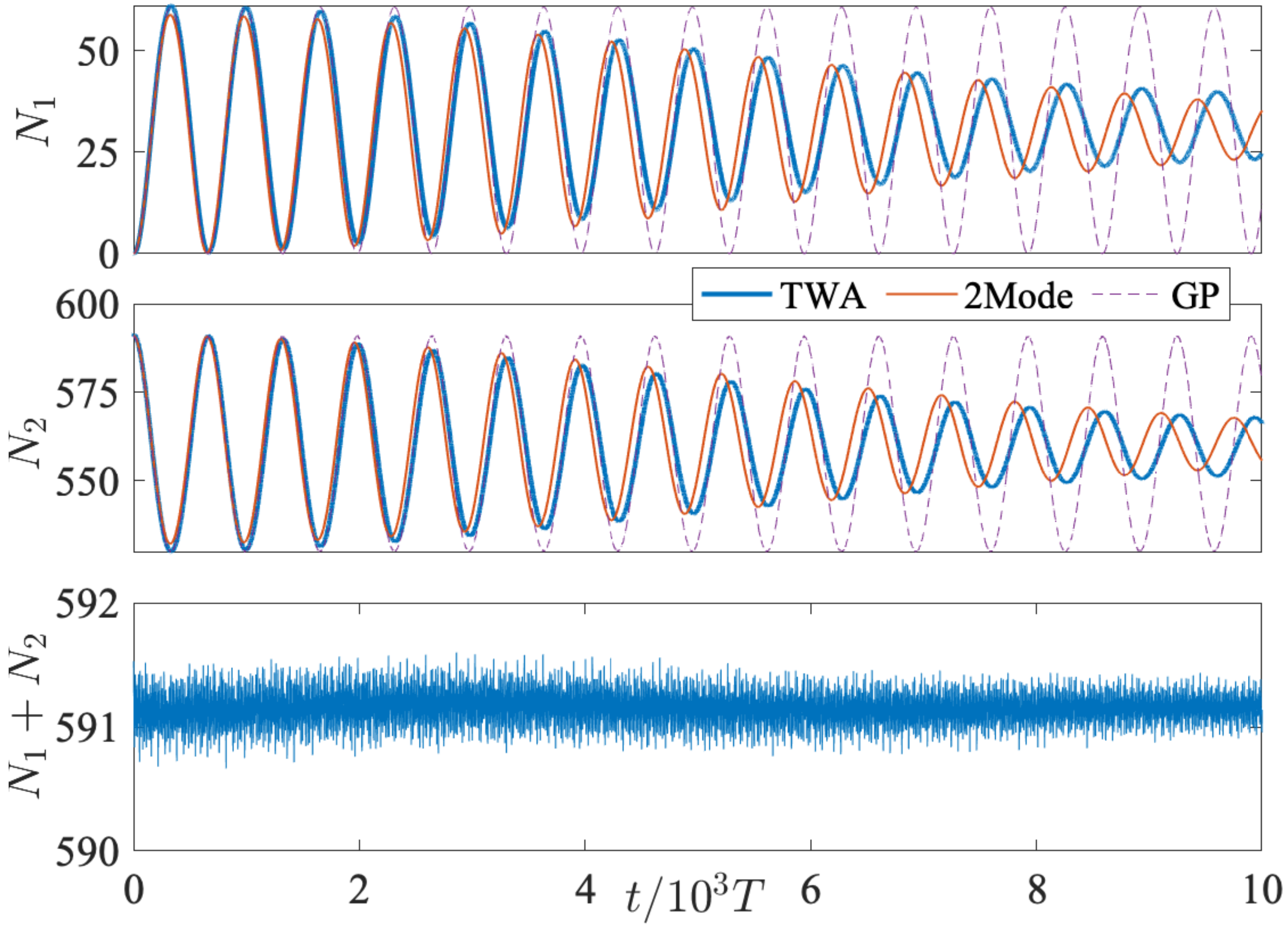
non-driving

$gN = -0.1$



TWA vs 2Mode





TWA

Include many modes

Asymptotic exact in the large boson particle number limit. In practise, limited by finite particle number and initial state sampling

Two-mode

Only includes two modes

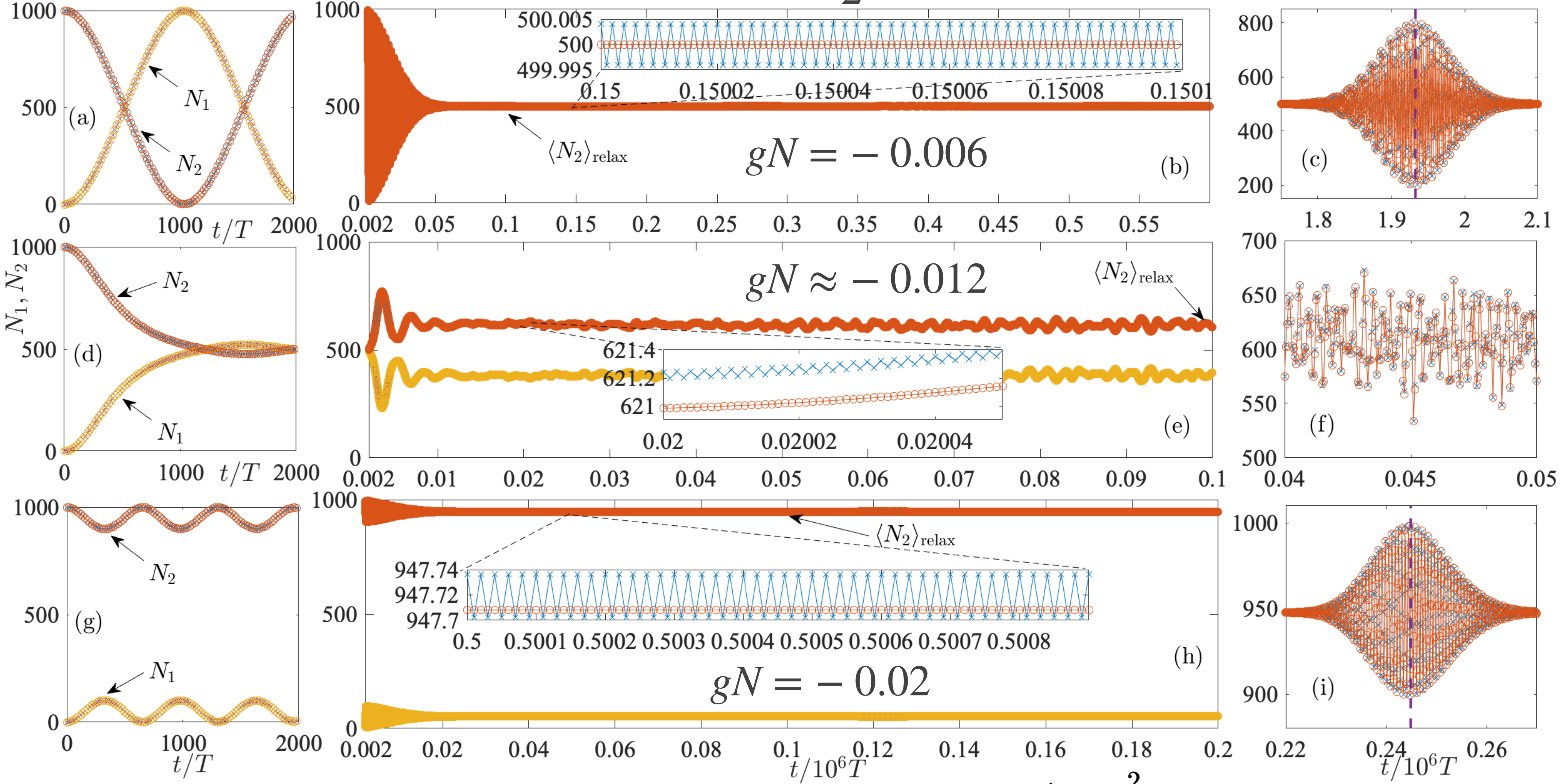
Exact quantum dynamical evolution

JW, Krzysztof Sacha, Peter Hannaford, and Bryan J. Dalton PRA **104**, 053327 (2021)

Two-mode Results

Initial state: $N_1 = 0, N_2 = 1000$ $\qquad \left\langle N_2(nT) \right\rangle = \frac{N}{2} \Rightarrow F(z, kT) = F(z, kT + T)$ symmetric

$\Phi_1(z, kT) = \Phi_2(z, kT + T)$ $\qquad \left\langle N_2(nT) \right\rangle = \text{const} \neq \frac{N}{2} \Rightarrow F(z, kT) = F(z, kT + 2T) \neq F(z, kT)$ broken

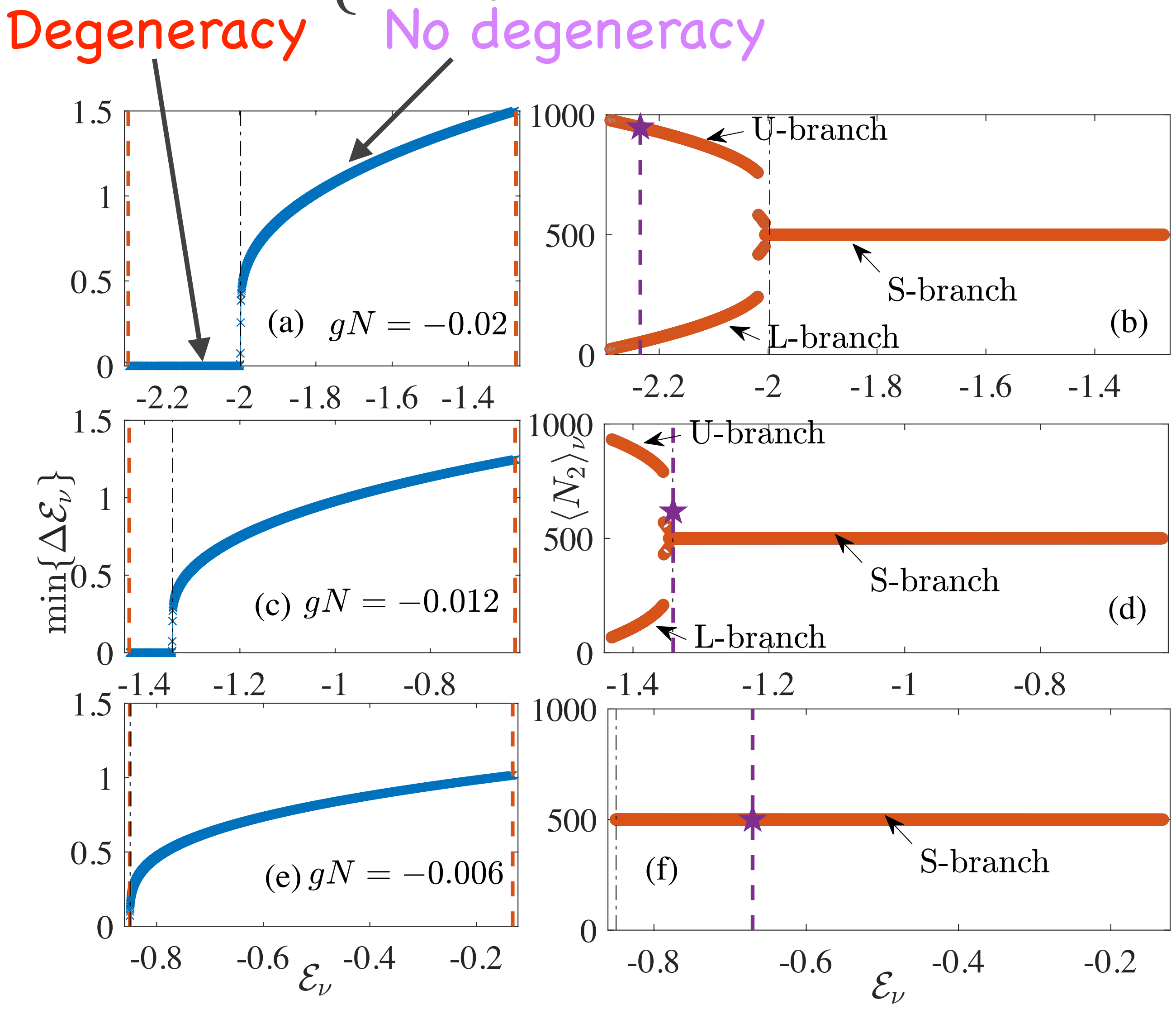


High frequency expansion: $\overline{\hat{\mathcal{H}}} = J\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right) + \frac{1}{2} g \sum_{ijkl=1}^{2} \overline{U}_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$

Eigenenergies and degeneracy

$$\min\left\{\Delta\mathscr{E}_\nu\right\} = \min\left(\mathscr{E}_\nu - \mathscr{E}_{\nu-1}, \mathscr{E}_{\nu+1} - \mathscr{E}_\nu\right)$$

$$\langle N_2\rangle_\nu = \begin{cases} \langle N_2\rangle_\nu^{(U)} \text{ or } \langle N_2\rangle_\nu^{(L)} & \mathscr{E}_\nu < \mathscr{E}_{\text{edge}} \\ \langle N_2\rangle_\nu^{(S)} & \mathscr{E}_\nu \geq \mathscr{E}_{\text{edge}} \end{cases}$$

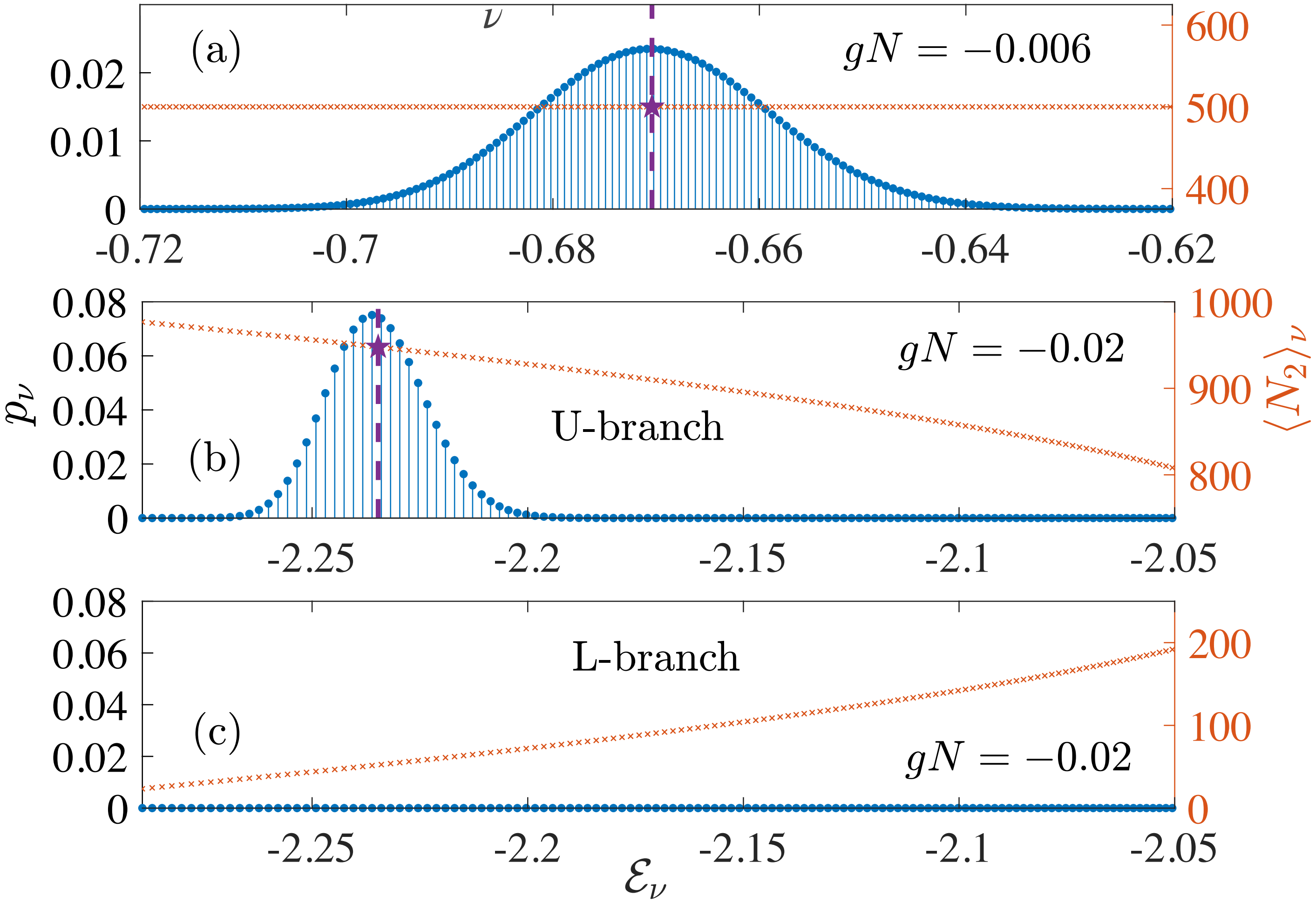


Effective Hamiltonian can be mapped to the Lipkin-Meshkov-Glick model.

Double degeneracy below an analytical edge

p_ν: Projection of initial state

$$\langle N_2\rangle_{\text{relax}} = \sum_\nu p_\nu \langle N_2\rangle_\nu \quad |N_1 = 0, N_2 = 1000\rangle$$



Symmetry broken edge

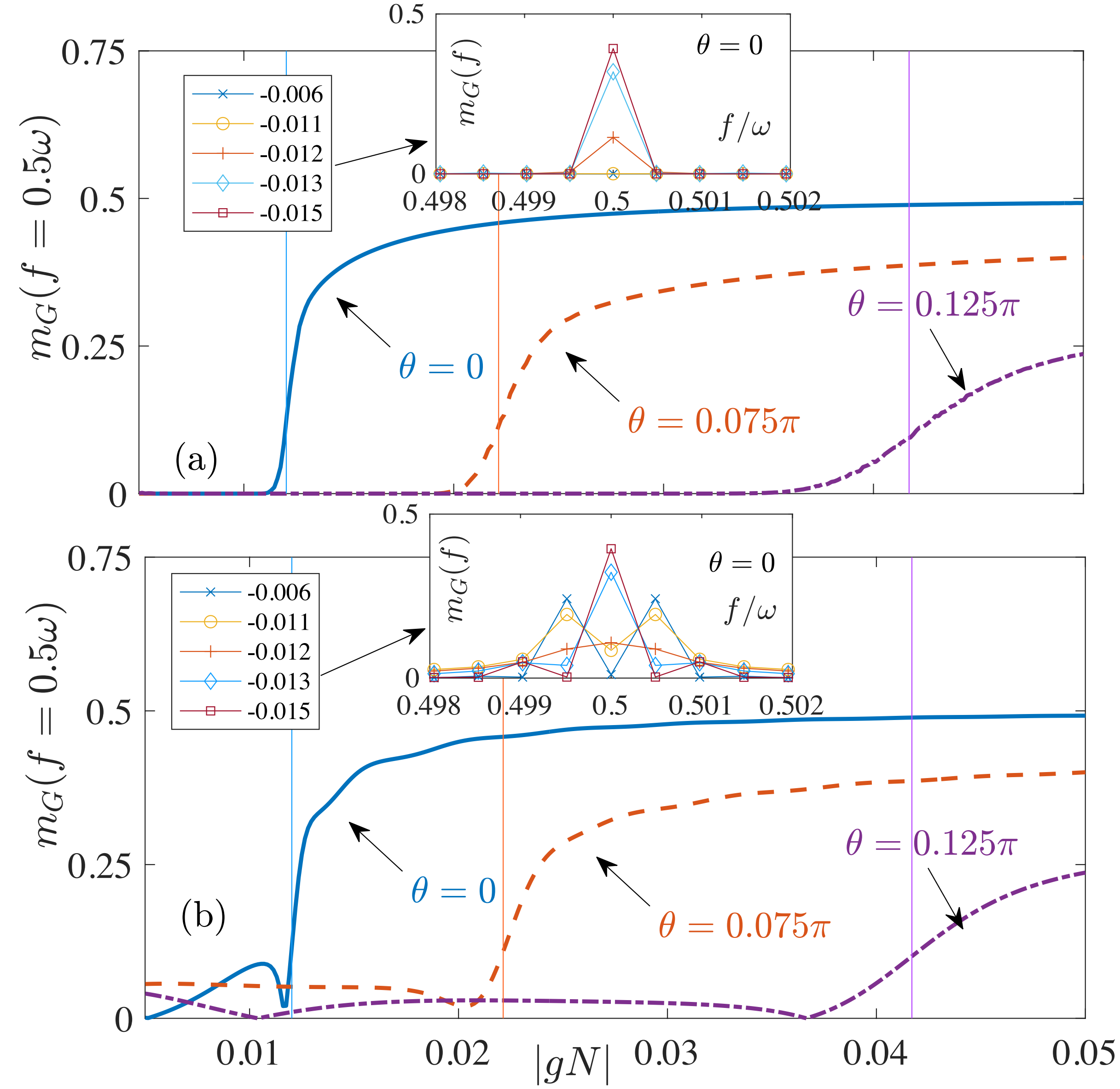
$$\hat{M}_G = \sum_{i=1}^{N} (|\chi\rangle\langle\chi|)_i \qquad \langle z\,|\,\chi\rangle = \Phi_2(z,0)$$

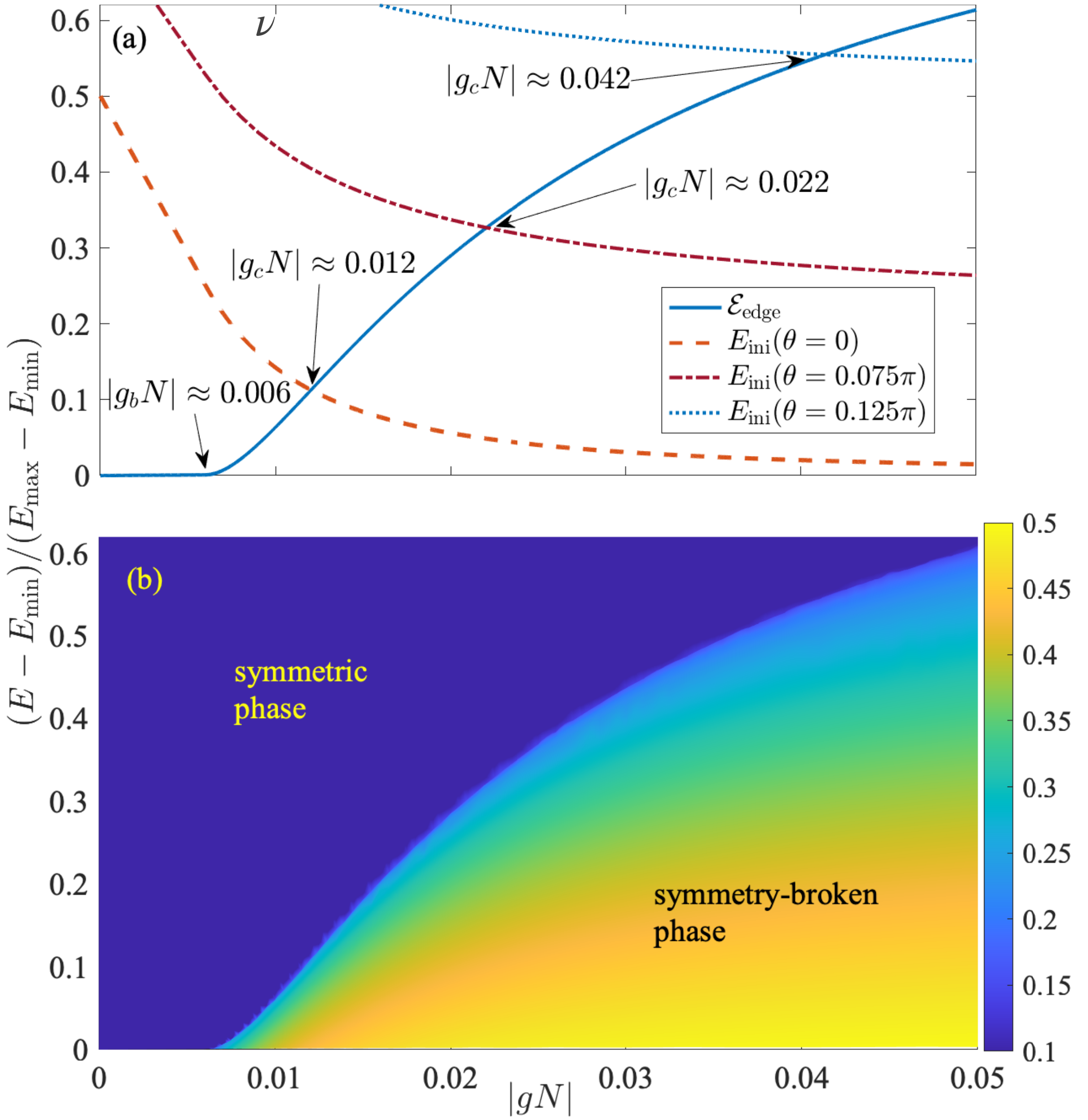
$$M_G(kT) = \begin{cases} N_1(kT) = N - N_2(kT) & k = 1,3,5,\dots \\ N_2(kT) & k = 0,2,4,\dots \end{cases}$$

$$|\{\theta,\varphi\}\rangle = \left|\Lambda_2\right\rangle_1 \left|\Lambda_2\right\rangle_2 \dots \left|\Lambda_2\right\rangle_N$$

$$\Lambda_2 = \sin\theta e^{i\varphi}\Phi_1(t=0) + \cos\theta\Phi_2(t=0)$$

$$E_{\text{ini}} = \sum_{\nu} p_\nu \mathscr{E}_\nu$$

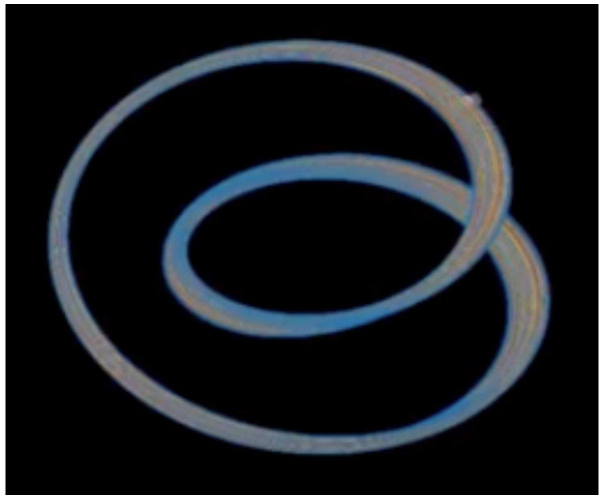




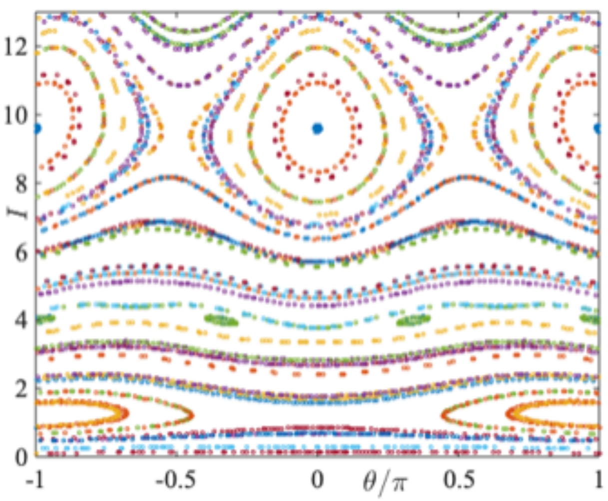
Conclusion

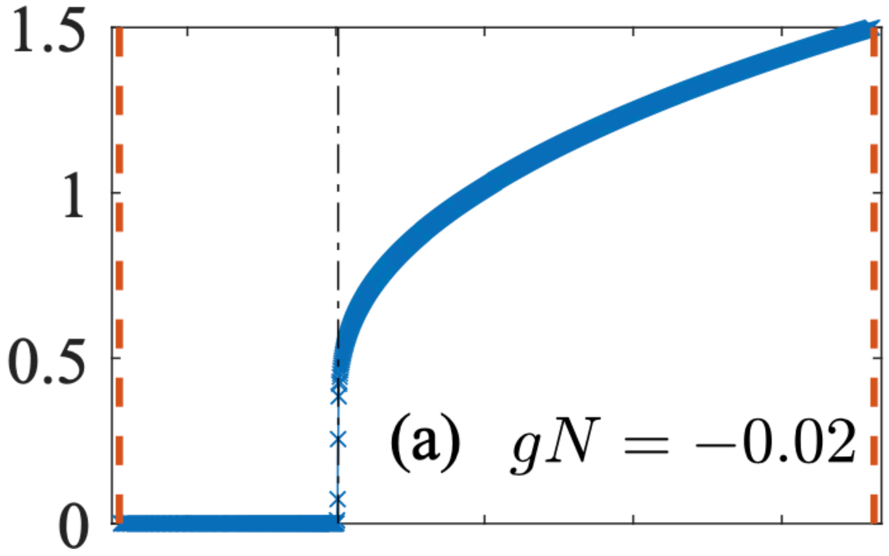
DTC: long-time steady state of an interacting many-body system. TWA, two-mode, …

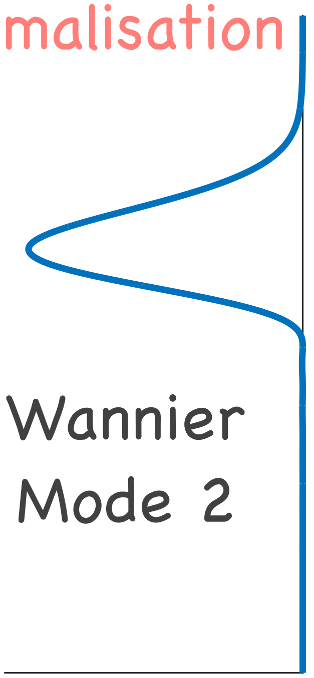
SSB: many degenerated/pi-pairing (quasi-)eigen energies and short-range correlated (quasi-)eigen states → suppress/absence of quantum thermalisation



Poincaré section







Future extension with TWA:

- Finite temperature, "bigger" time crystal, higher dimension, dissipation
- DTC in kicked Lieb-Liniger model.

Thank You!

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Ray RC (CityU HK)

Krzysztof Giergiel

[1] New J. Phys. **23** 063012 (2021)

[2] PRA **104**, 053327 (2021)

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