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Thermodynamics of de Sitter and Nariai spaces in the 50 years of Bekenstein's black hole entropy

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Outline

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1. 50 years of black hole entropy: Bekenstein 1972 and after

·In 1972, after giving deep thoughts to his supervisor John Wheeler's question, what happens to the entropy when matter goes down a black hole, is the second law of thermodynamics violated?, Bekenstein came up with the solution: a black hole has entropy. The letter with title Black Holes and the Second Law, was received by the journal Il Nuovo Cimento, in May 22, 1972 and published just after. The paper has four pages, in it Bekenstein proposes that the black hole entropy S is given by the expression

$$S = \eta \frac{A}{A_{\rm pl}}.$$

where η is a number of order of unity and $A_{\rm pl}$ is the Planck area, $A_{\rm pl} = \frac{\hbar G}{c^3}$. It was an amazing proposal and as written it is the Bekenstein entropy.

·Since $A_{\rm pl}$ appears in the formula, it was the first time that it was recognized that black holes and quantum gravity were entangled.

1. 50 years of black hole entropy: Bekenstein 1972 and after

LETTERE AL NUOVO CIMENTO VOL. 4, N. 15 12 Agosto 1972

Black Holes and the Second Law (*).

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(ricevuto il 22 Maggio 1972)

Black-hole physics seems to provide at least two ways in which the second law of thermodynamics may be transcended or violated:

a) Let an observer drop or lower a package of entropy into a black hole; the entropy of the exterior world decreases. Furthermore, from an exterior observer's point of view a black hole in equilibrium has only three degrees of freedom: mass, charge and angular momentum (1). Thus, once the black hole has settled down to equilibrium, there is no way for the observer to determine its interior entropy. Therefore, he cannot exclude the possibility that the total entropy of the universe may have decreased in the process. It is in this sense that the second law appears to be transcended (2).

b) A method for violating the second law has been proposed by GEROCH (3): By means of a string one slowly lowers a body of rest mass m and nonzero temperature toward a Schwarzschild black hole of mass M. By the time the body nears the horizon, its energy as measured from infinity, $E = m(1-2M/r)^{\frac{1}{2}}$, is nearly zero; the body has already done work m on the agent which lowers the string. At this point the body is allowed to radiate into the black hole until its rest mass is $m - \Delta m$. Finally, by expending work $m - \Delta m$, one hauls the body back up.

The net result: a quantity of heat Δm has been completely converted into work. Furthermore, since the addition of the radiation to the black hole takes place at a point where $(1-2M/r)^{\frac{1}{2}} \approx 0$, the mass of the black hole is unchanged. Thus the black hole appears to be unchanged after the process. This implies a violation of the second law: « One may not transform heat entirely into work without compensating changes taking place in the surroundings. »

In this note we formulate the second law in a form suitable for black-hole physics which resolves the transcendence problem. We also indicate why Geroch's gedanken experiment does not, in fact, violate the second law.

(*) Work supported by National Science Foundation Grant No. GP 30799 X.
(**) National Science Foundation Predoctoral Fellow.
(1) C. W. MISNER, K. S. THORNE and J. A. WHEELER: *Gravitation* (San Francisco, 1972).
(2) For transcendence of the law of baryon conservation see J. D. BEKENSTEIN: *Phys. Rev. Lett.*, **28**, 452 (1972); *Phys. Rev. D*, **5**, 1239 (1972) and to be published.
(3) R. GEROCH: *Colloquium at Princeton University* (Dec. 1971).

1. 50 years of black hole entropy: Bekenstein 1972 and after

We state the second law as follows: «Common entropy plus black-hole entropy never decreases.» By common entropy we mean entropy in the black-hole's exterior. By black-hole entropy $S_{\text{b.h.}}$ we mean

$$S_{\text{b.h.}} = \eta k L_P^{-2} A \,, \tag{1}$$

where A is the area of the black hole (4) in question, L_P is the Planck length $(\hbar G/c^3)^{\frac{1}{2}} = 1.6 \cdot 10^{-33}$ cm, k is Boltzmann's constant $1.38 \cdot 10^{-16}$ erg/°K and η is a constant number of order unity.

The introduction of a black-hole entropy is necessitated by our first example. Without it the second law is definitely transcended. With black-hole entropy the second law becomes a well-defined statement susceptible to verification by an exterior observer. The choice of the area of a black hole as a measure of its entropy is motivated by the results of CHRISTODOULOU (5) and of HAWKING (6) that the area of a black hole never decreases, and that it increases for all but a very special class of black-hole transformations (5). The area of a black hole appears to be the only one of its properties having this entropylike behavior which is so essential if the second law as we have stated it is to hold when entropy goes down a black hole.

By choosing $S_{\text{b.h.}}$ proportional to A rather than to some monotonically increasing function of A, we ensure that the total black-hole entropy of a system of black holes (the sum of individual $S_{\text{b.h.}}$) depends only on the total horizon area—also a nondecreasing quantity which is of fundamental importance in Hawking's work (6).

The introduction of k and of a length squared in (1) is necessitated by dimensional considerations. We choose Planck's length because it is the only truly universal constant with units of length. (We shall show later on that no larger scale of length will do.) Although a black hole is a thoroughly classical entity, black-hole entropy contains $\hbar$ because it relates to the interaction of a black hole with material systems. It is well known that the expression for entropy of any material system always contains $\hbar$; thus, the appearance of $\hbar$ in $S_{\text{b.h.}}$ is understandable. (In what follows we use units with $G = c = 1$; thus $L_P^2 = \hbar$.)

Let us illustrate the operation of the second law with a simple example. Consider the case of a narrow beam of black-body radiation of temperature T which is directed into a black hole of mass M. We are clearly assuming that geometrical optics is applicable. Thus, the characteristic wavelengths of the radiation must be much smaller than M. This implies that

$$T \gg \hbar/kM \,. \tag{2}$$

We note also that, to a given energy E in the beam, there corresponds entropy (7) S, where

$$S = E/T \,. \tag{3}$$

The increase in area of the black hole (4) caused by the infall of radiation with energy E may be calculated from the conservation of energy and angular momentum.

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(4) The area of a (Kerr-Newman) black hole of mass M, charge Q and angular momentum L is $A = 4\pi\{[M + (M^2 - Q^2 - L^2/M^2)^{\frac{1}{2}}]^2 + L^2/M^2\}$. An extreme Kerr black hole has $L = M^2$ and $Q = 0$.

(5) D. CHRISTODOULOU: *Phys. Rev. Lett.*, **25**, 1596 (1970); D. CHRISTODOULOU and R. RUFFINI: *Phys. Rev. D*, **4**, 3552 (1971).

(6) S. W. HAWKING: *Phys. Rev. Lett.*, **26**, 1344 (1971); *Comm. Math. Phys.*, **25**, 152 (1972).

(7) A black-body cavity emits entropy E/T as it emits energy E; the propagation of the beam is a reversible process.

1. 50 years of black hole entropy: Bekenstein 1972 and after

·In November 2, 1972, Bekenstein submitted another paper, now to PRD, published in 1973, with the title Black Holes and Entropy. In this paper he proposed that $\eta = \frac{1}{2}\ln 2$, i.e., $S = \left(\frac{1}{2}\ln 2\right)\frac{A}{A_{\rm pl}}$. Note that $\frac{1}{2}\ln 2 = 0.35$ in first approximation.

·Then Hawking using quantum fields in a star collapsing to a black hole background found that a black hole has temperature $T^{\rm H} = \frac{1}{8\pi}\frac{c^2}{k_{\rm B}}\frac{m_{\rm pl}^2}{M}$, where $m_{\rm pl}^2 = \frac{\hbar c}{G}$. He then found immediately, that $\eta = \frac{1}{4}$ so that $S = \frac{1}{4}\frac{A}{A_{\rm pl}}$, i.e.,

$$S = \frac{1}{4}A.$$

in Planck units. This is the Bekenstein-Hawking entropy, the black hole entropy.

·Black holes extended their realm, they were of interest not only in astrophysics and gravitation, they now also embraced the quantum realm, from elementary particles to quantum gravity.

1. 50 years of black hole entropy: Bekenstein 1972 and after

·We are interested in systems with high Hawking temperature: 10^{-13} cm and 10^{11} K. Quantum effects are important, but not full quantum gravity.

·If the system is a pure black hole, left by itself it evanesces. Due to the Hawking temperature, the black hole would radiate on and on and disappears.

·To understand better black holes, one encloses it inside a heat reservoir at constant temperature T and constant radius R, characterizing the statistical mechanics canonical ensemble. A thermodynamic treatment is then possible.

·Use the Euclidean path integral approach to quantum gravity of Hawking 1979 and York 1986. The idea is that the classical action I contributes to the statistical mechanics partition function Z, and at a semiclassical level one has $Z = \mathrm{e}^{-I}$. From thermodynamics the free energy is $F = -T \ln Z = TI$.

·Now, $I = -\frac{1}{16\pi} \int_M \sqrt{g}\,(R - \Lambda)\, d^4x + \frac{1}{8\pi} \int_{r=R} \left(K - K^0\right) \sqrt{\gamma} d^3x$.

·In the path integral approach to quantum gravity we seek to understand phase transitions involving changes of topology. Here we go one step further and study black holes and cosmology, i.e., the Schwarzschild-de Sitter space.

2. Schwarzschild-de Sitter in the canonical ensemble

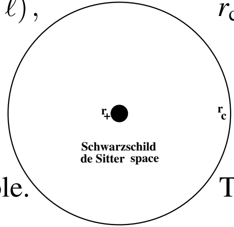
·The Schwarzschild-de Sitter metric is

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2, \quad \text{with} \quad V(r) = 1 - \frac{2m}{r} - \frac{r^2}{3\ell^2},$$

where m is the spacetime mass, ℓ is the typical length, $\ell = \frac{1}{\sqrt{\Lambda}}$, with Λ a positive cosmological constant, $\Lambda > 0$, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. There are horizons for $V(r = r_+) = 0$, i.e., $m = \frac{r}{2}\left(1 - \frac{r^2}{3\ell^2}\right)$. The two positive roots correspond to a black hole horizon r_+ and to a cosmological horizon r_c,

$$r_+ = r_+(m, \ell)\,, \qquad r_\text{c} = r_\text{c}(m, \ell)\,,$$

with $r_+ \leq r_\text{c}$.

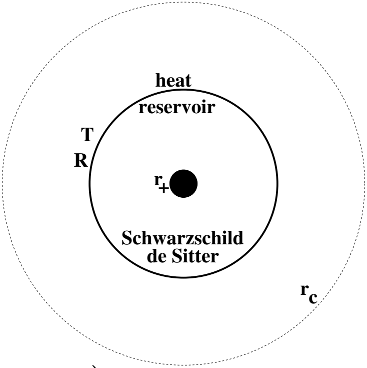


It is a cosmological black hole. The roots coincide for $r_+ = r_\text{c} = 3m = \ell$.

·To do statistical physics Euclideanize time $t \to -it$ and, at some radius R, put a period β given by $\beta = \frac{1}{T}$, where T is the temperature. This defines the canonical ensemble.

2. Schwarzschild-de Sitter in the canonical ensemble

·We have two types of reservoir and so two types of physical situations, the reservoir out for the inside and the reservoir in for the outside. Let us do the reservoir out for the inside. The Euclidean topology is $R^2 \times S^2$, with boundary $S^1 \times S^2$, contrast with trivial topology $S^1 \times R^3$ for hot de Sitter space and the same boundary.



·The action is

$$I = \beta R \left(1 - \sqrt{V(R)}\right) - \pi r_+^2, \qquad V(R) = 1 - \frac{2m}{R} - \frac{R^2}{3\ell^2},$$

with $\beta = \frac{1}{T}$. From the connection with thermodynamics, deduce $E = R\left(1 - \sqrt{V(R)}\right)$ and $S = \pi r_+^2 = \frac{1}{4}A_+$, the Bekenstein-Hawking entropy.

2. Schwarzschild-de Sitter in the canonical ensemble

·The Hawking temperature is $T^{\mathrm{H}} = \frac{\kappa}{2\pi}$. Find $T^{\mathrm{H}} = \frac{1}{2\pi r_+}\left(\frac{3m}{r_+} - 1\right)$. The path integral formalism gives $T = \frac{T^{\mathrm{H}}}{\sqrt{V(R)}}$, so $T = \frac{\frac{1}{2\pi r_+}\left(\frac{3m}{r_+}-1\right)}{\sqrt{1-\frac{2m}{R}-\frac{R^2}{3\ell^2}}}$. T is fixed by the reservoir. There are two solutions, in general,

$$r_{+1} = r_{+1}(R,T,\ell), \qquad r_{+2} = r_{+2}(R,T,\ell)\,,$$

with $r_{+1} \leq r_{+2}$. For small Λ, i.e., for large ℓ, such that $\frac{R}{\ell} \ll 1$, there are no black hole solutions for

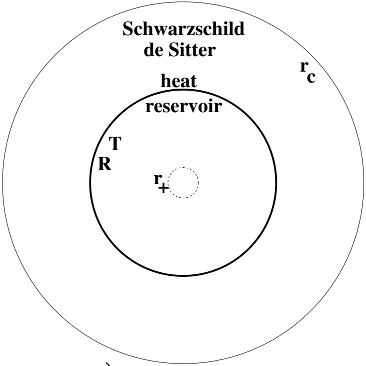
$$RT\left(1 - \frac{415}{81}\frac{R^2}{\ell^2}\right) < \frac{\sqrt{27}}{8\pi}\,,$$

with $\frac{r_{+2}}{R} = \frac{r_{+1}}{R} = \frac{2}{3} - \frac{34}{243}\frac{R^2}{\ell^2}$ at equality. The minus sign is what one expects.

·For high T the system is solvable. Either $T^{\mathrm{H}} \to \infty$, so $r_{+1} \to 0$, it is the small black hole, and is thermodynamically unstable, the heat capacity obeys $C = \left(\frac{dE}{dT}\right)_R < 0$. Or $V(R) \to 0$, then find $r_{+2} = R\left(1 - \frac{1-\frac{R^2}{\ell^2}}{16\pi(RT)^2}\right)$, it is the large black hole, and is thermodynamically stable, the heat capacity obeys $C = \left(\frac{dE}{dT}\right)_R > 0$.

2. Schwarzschild-de Sitter in the canonical ensemble

Let us do now the reservoir in for the outside:



·The action is now

$$I = -\beta R\left(1-\sqrt{V(R)}\right) - \pi r_{\rm c}^2, \qquad V(R) = 1 - \frac{2m}{R} - \frac{R^2}{3\ell^2},$$

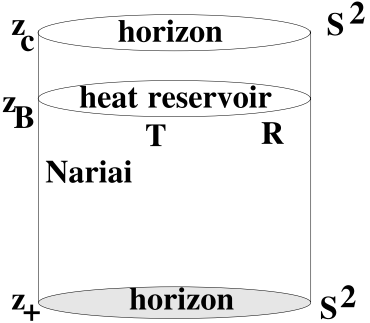
with $\beta = \frac{1}{T}$. Find $E = -R\left(1-\sqrt{V(R)}\right)$ from the connection with thermodynamics, and $S = \pi r_{\rm c}^2 = \frac{1}{4}A_{\rm c}$, the Bekenstein-Hawking entropy for a cosmological horizon. Note the minus sign for E. This system is thermodynamically unstable, $C = \left(\frac{dE}{dT}\right)_R < 0$: m increases, so r_+ increases and $r_{\rm c}$ decreases, and vice versa.

3. Extreme Schwarzschild-de Sitter in the canonical ensemble: Nariai solution

- Want to study black hole horizon region inside the heat reservoir in the extreme Schwarzschild-de Sitter in the canonical ensemble. Get Nariai space.
- Do $r_+ \to R$, $r_c \to R$, so that $T_H \to 0$, with T of the reservoir remaining finite. From $T = \frac{T_H}{\sqrt{V}}$, the numerator is compensated by the denominator. Write $r_+ = R(1-\varepsilon)$ and $\frac{R^2}{\ell^2} = 1-\delta$, obtain $T^H = \frac{\delta+2\varepsilon}{4\pi R}$. Writing $V = 4\pi T^H(R-r_+) - \frac{1}{R^2}(R-r_+)^2$, and making $r - r_+ = \frac{4\pi T^H}{R^2}\sin^2\left(\frac{1}{2}\frac{z}{R}\right)$, $\tilde{t} = 2\pi R T^H t$, obtain
$$ds^2 = +\sin^2\left(\frac{z}{R}\right)d\tilde{t}^2 + dz^2 + R^2 d\Omega^2\,.$$
This is the Nariai space. Note that $z=0$ and $z=\pi R$ are horizons. In general for Nariai, one has $0 < \tilde{t} < 2\pi R$, $0 \le z \le z_B$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$.
- Now from $T = \frac{T^H}{\sqrt{V}}$ get $T = \frac{1}{2\pi R \sin\left(\frac{z_B}{R}\right)}$, so that the new Hawking temperature is $T^H = \frac{1}{2\pi R}$ and
$$z_B = R \arcsin\left(\frac{1}{2\pi T R}\right).$$
Given R and T of the ensemble, one has automatically z_B, the boundary. In turn $0 \le z_B \le \pi R$. For $TR < \frac{1}{2\pi}$ there are no horizons only hot space. Find $E = R =$ constant, and interestingly neutral thermodynamic equilibrium.

3. Extreme Schwarzschild-de Sitter in the canonical ensemble: Nariai solution

·The Nariai solution. The topology is $S^1 \times [a,b] \times S^2$ with boundary $S^1 \times S^2$.



·Note that the three spaces with the reservoir outside, Schwarzschild-de Sitter, Nariai, and hot de Sitter space, have the same type of boundary $S^1 \times S^2$, although different topologies, $R^2 \times S^2$, $S^1 \times [a,b] \times S^2$, and $S^1 \times R^3$. This shows that there are topological phase transitions between the different spaces, in the quantum gravity semiclassical approximation considered here.

4. Conclusions

·Bekenstein was born in Mexico in 1947, went to Princeton for the PhD with Wheeler, and then to Jerusalem. He had several important works on modified Newtonian dynamics as alternative to explain rotation curves of galaxies, and black hole physics, his idea of black hole entropy is most extraordinary.

·In the wake of the idea, 50 years after, we are still exploring it further, and surely it will lead to new phenomena. The holographic principle, that in a way was initiated by York in 1986 with his result that the gravitational thermodynamic energy E, a quasilocal energy, is on the boundary, not in the volume, states that all the fundamental information, coming from the quantum gravitational degrees of freedom, is indeed in the area, spawns from Bekenstein's black hole entropy idea, and it seems correct and to stay.

·We have shown that, black hole and cosmological horizons can be treated in the Euclidean path integral formalism, and that Schwarzschild-de Sitter, Nariai, and hot de Sitter space phases, all are specified by the same boundary data, belong to the same canonical ensemble and can perform topological transitions between them as it is allowed in quantum gravity. Which phase dominates? Puzzling question but not beyond all conjecture.