Universality of curvature invariants in critical collapse of axisymmetric gravitational waves

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Outline

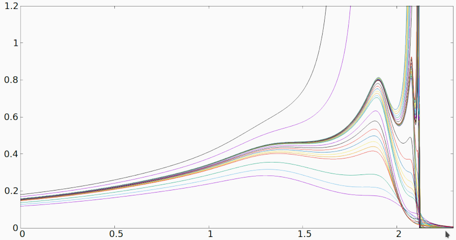
1. Introduction

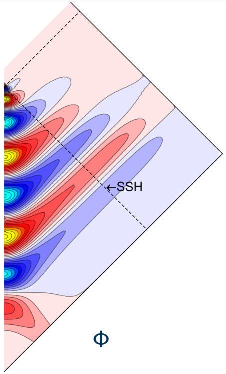
Introduction

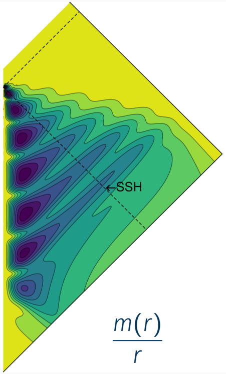
- Massless scalar field Φ evolution [Choptuik1993]
- Einstein gravity provides arena $G_{ab}[g_{cd}] = 8\pi T_{ab}[\Phi]$
- Spherical symmetry + Numerical solution
- Various initial data (ID) families, varying 'strength' p
- Possible outcomes: Black hole or empty spacetime
- Fine-tuning $ID[p]$ in-between these outcomes — p_*
- Surprise: For massless Φ initial data $\Phi_0[p]$ provide

$$M_{BH} \sim |p - p_*|^\gamma$$

- Universality — γ for any initial data
- Discrete self-similarity — near p_* we see 'echoes'
- Various matter models lead to similar behavior with specific γ (see [Gundlach1999])







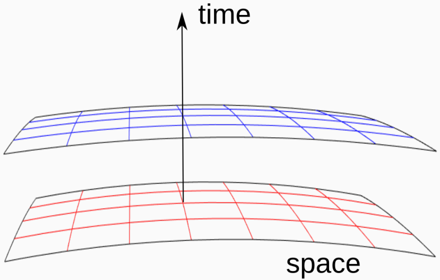
Introduction (continued)

- Gravitational waves (GW) instead of matter
  (see [Abrahams1993] [Abrahams1994] [Alcubierre2000] [Garfinkle2001] [Rinne2008] [Sorkin2011] [Hilditch2013] [Hilditch2017] [Khirnov2018] [Ledvinka2021])
- Spherical symmetry no longer possible (in 3+1D)
  $\implies$ even more demanding computations
- Hard to navigate in dynamic vacuum spacetime
- Larger space of outcomes (e.g. several BHs)
- Early results hard to reproduce by 'BBH merger codes'
- But recent results show that
  — 'moving puncture' gauge not working
  (and has to be improved)
  — off-center curvature maxima observed on the symmetry axis
  — no universal γ observed
- 2021 — universality and partial self-similarity identified

2. Field equations, Coordinates

Evolution equations

- 3+1 splitting of spacetime into $t = \text{const.}$ hypersurfaces Σ_t
- Each Σ_t is 3D manifold with metric γ_{ij}
- Its embedding into 4D spacetime yields
  — lapse α and shift β^i, unit normal n^i
  — extrinsic curvature K_{ij}
- Standard BSSN equations
  — $\gamma_{ij} \to \left\{\gamma = \det \gamma_{ij}, \tilde{\gamma}_{ij} = \gamma^{-1/3}\gamma_{ij}\right\}$
  — $K_{ij} \to \left\{K = \mathrm{tr} K_{ij}, \tilde{A}_{ij} = \gamma^{-1/3}\left(K_{ij} - \frac{1}{3}K\gamma_{ij}\right)\right\}$
  — $\tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk}$ as independent field
- Initially we used 1+log (Bona-Massó, moving puncture) slicing
- No shift ($\beta^i = 0$)
- Evolution equations (1T-2D approach) $\partial_t U = F(U, \partial_i U, \partial_{ij} U)$
  — with $U = [\log \gamma, \alpha, \tilde{\gamma}_i j, K, \tilde{A}_{ij}, \tilde{\Gamma}^i]$



Brill initial data

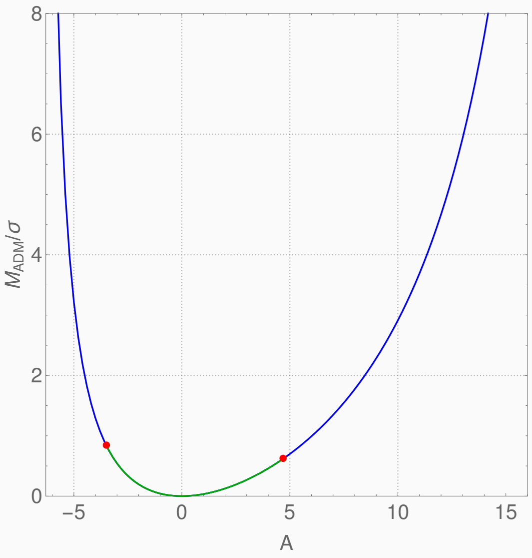
- Time symmetric $K_{ij} = 0$,
  $\gamma_{ij} dx^i dx^j = \psi^4 \left[e^{2q}(dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta d\varphi^2 \right]$
- Deformation of the initial slice determined by

$$q(x^i) = A \, \frac{r^2}{\sigma^2} e^{-\frac{r^2}{\sigma^2}} \sin^2 \theta$$

- $A = 0 \implies$ Minkowski spacetime
- Hamilton constraint $\implies \psi$

$$\Delta\psi + \frac{1}{4} \left(\partial_{\rho\rho} q + \partial_{zz} q \right) \psi = 0$$

- $M_{\mathrm{ADM}} = \int \frac{|\nabla\psi|^2}{\psi^2} d^3x$
- $A > 0$ and $A < 0$ are different
- Why $M_{\mathrm{ADM}}(A) \to \infty$ see [Eppley77]

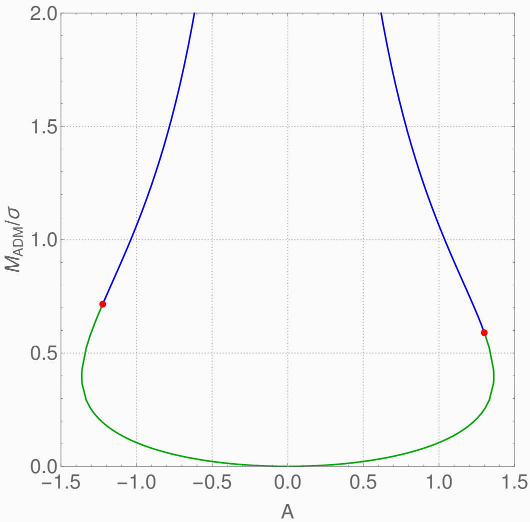


Time-asymmetric initial data

- Time asymmetric $\tilde{\gamma}_{ij} = 0$
  — inspired by [Abrahams1994]
- Complementary to Brill ID
- Deformation of the initial slice determined by

$$K^r_\theta(x^i, t=0) = A\frac{r^2}{\sigma^3}(\sigma - r)e^{-\frac{r^2}{\sigma^2}}\sin 2\theta$$

- Hamilton and momentum constraints
  $\Longrightarrow$ ψ, $K^r{}_r$ and $K^\varphi{}_\varphi$
- $A = 0 \Longrightarrow$ Minkowski spacetime
- Two branches of $M(A)$ dependence
  —we use e.g. $A = -\overline{1.1}$ for upper branch

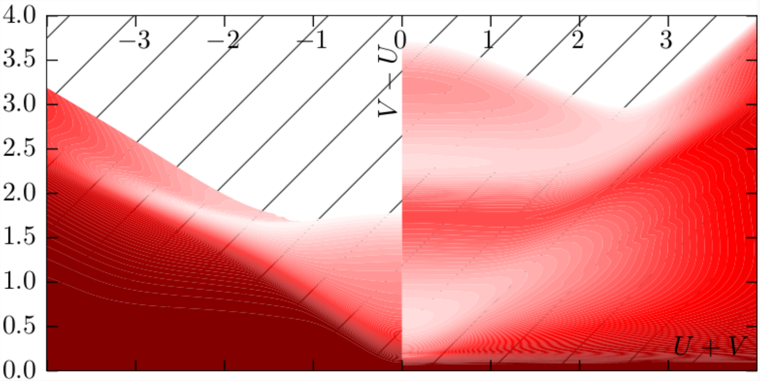


Slicing conditions

- $1 + \log$ (Bona-Massó, generalized harmonic) slicing

$$\partial_t \alpha = -2\alpha K$$

- Important for strong hyperbolicity of evelution PDEs
- Gauge waves ($c_\alpha = \sqrt{2}$, still assuming $\beta^i = 0$)
- Important ingredient of 'Moving puncture' approach
- Efficient (no additional overhead)
- Known to break down for collapsing GW spacetimes



Quasi-maximal slicing

- So-called maximal slicing $K = 0$ known to behave well
- Because

$$K = -D^i D_i \alpha + K_{ij} K^{ij} \alpha \tag{1}$$

  $K = 0$ becomes an elliptic equation for α

- Elliptic solver in each time step $\rightarrow$ slow
- PDE hyperbolicity requires enforcing $K = 0$ in evolution equations
- Any numerical error in solving (1) spoils simulation
- We suggest alternative

$$\partial_t \alpha = -2\alpha K + W$$

  where W is given by time derivative of (1)

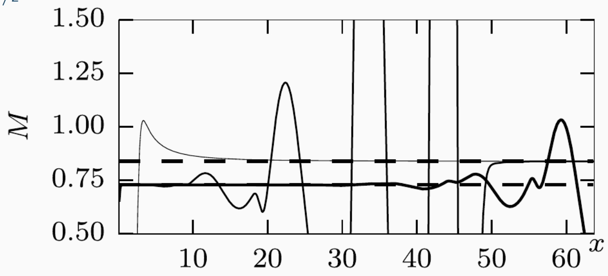
$$D^k D_k W - K_{ij} K^{ij} W = -2\alpha K^{ij} D_i D_j \alpha + \gamma^{ij} \left(\partial_t \Gamma^k_{ij} \right) \partial_k \alpha + \ldots$$

- Still elliptic equation, but (truncation) error does not spoil simulation, only changes gauge
- Suppresses gauge waves, hyperbolicity of $1 + \log$ slicing unchanged

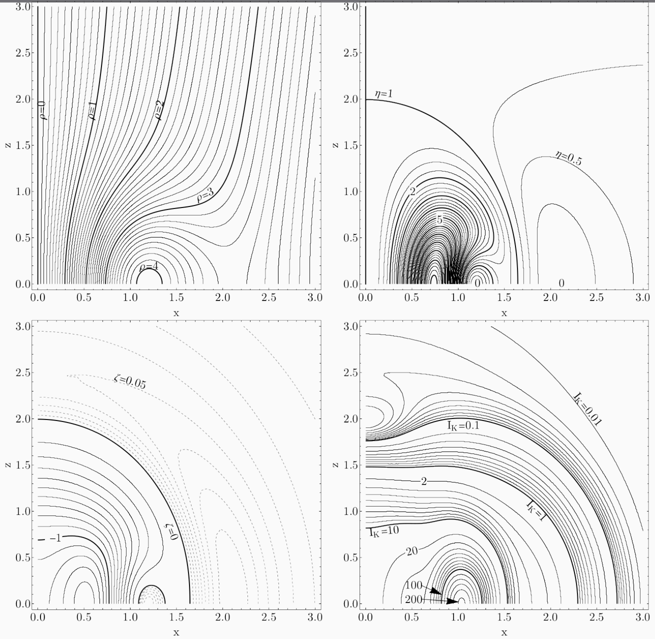
3. Navigation in the vacuum spacetime

Gauge-invariant description of vacuum collapse

- Vacuum — Metric + Riemann=Weyl (+ derivatives)
- Simplest scalar: Kretschmann $I_K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$
- Axisymmetry — $\xi_\mu \equiv (\partial/\partial\phi)^\mu$, $\rho \equiv (\xi^\mu \xi_\mu)^{1/2}$
- Additional scalars:
  - — $\eta \equiv \rho^{,\mu}\rho_{,\mu}$ (vanishes at the axis)
  - — $\zeta \equiv (1-\eta)/\rho^2$
- In the equatorial plane:
  - — $\eta \to 0$ at settled AH
  - — Schwarzschild has $\eta = 1 - 2m/\rho$
- On the axis:
  - — $I_K = 12\zeta^2$, also $\zeta = 2\Psi_2$
  - — Only one nonzero Riemann component



$M_{\bar\rho}$ along the x-axis at times $t = \{0, 50, 100\}$ (thin to thick).



Brill initial data geometry

(Additional material for discussion)

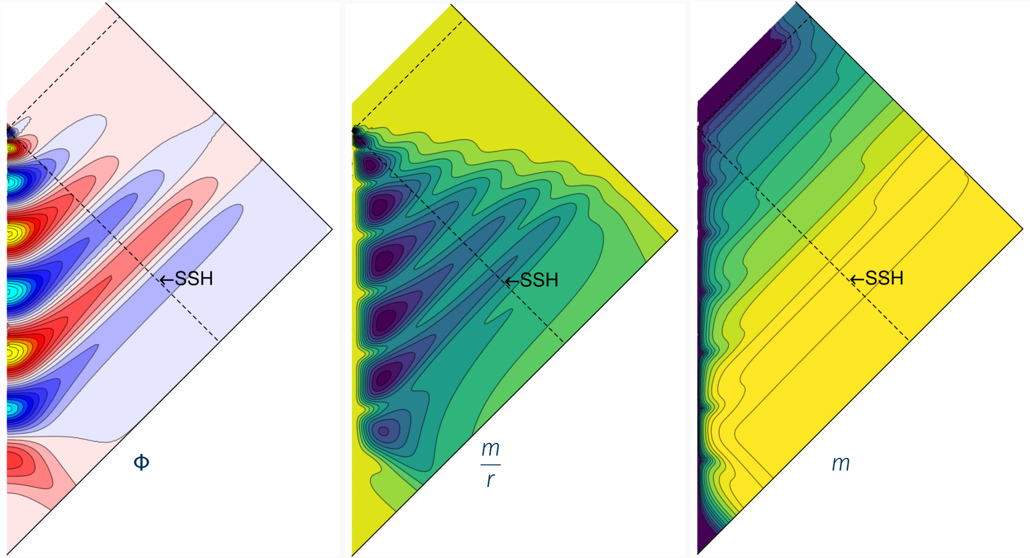
Geometry of the initial slice with Brill initial data illustrated by the values of spacetime invariants.

The Brill initial data with $\sigma = 1, A = -3.6$ have $K_{ij} = 0$ and the GW 'seed' appears in γ_{ij}. This can be seen in the top plots — the initial γ_{ij} makes ρ non-monotonous and $\eta = 0$ where $\partial_i \rho$ vanishes.

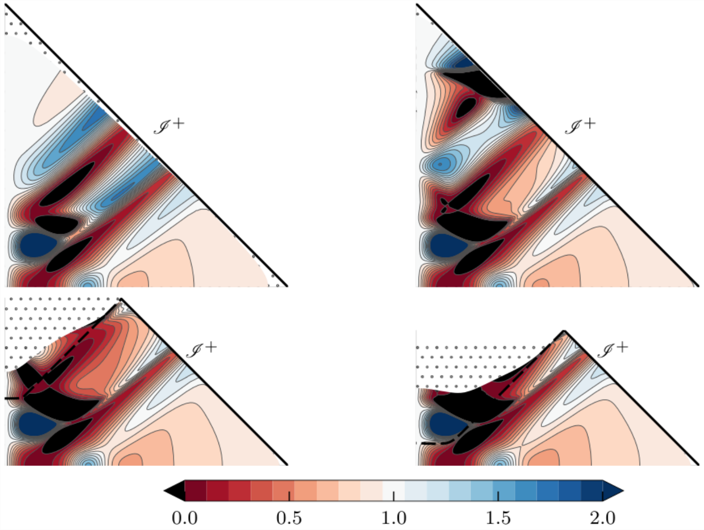
At larger coordinate radii the spacetime resembles the Schwarzschild geometry and ζ and I_K become approximately spherically symmetric.

4. Results

What we expect: Compactified diagrams of massless spherically symmetric near-critical spacetime



Evolution of Brill initial data



Conformal diagrams for the equatorial plane of the spacetimes produced by evolving the Brill waves.

The amplitudes are $A = \{4.0, 4.6, 4.8, 5.25\}$ for, respectively top left, top right, bottom left and bottom right. The top two spacetimes are subcritical, while the bottom two are super-critical, as seen from the presence of an event horizon. The color denotes the value of η.

Note the range the quantity η spans: for the spherically symmetric sub-critical spacetimes of [Choptuik93] it stays between 0.4 and 1.

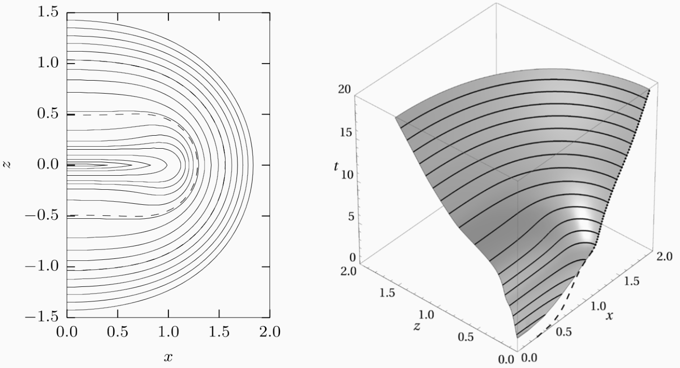
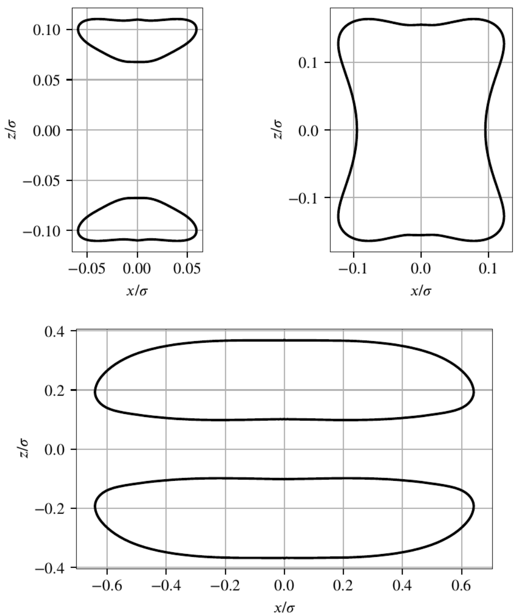


Figure 5. Formation of the event horizon for $A = 5$ Brill waves, zero shift. It is constructed by evolving the apparent horizon section from the $t = 25$ slice as a null surface back in time. Left: solid lines show cross-sections of the event horizon in the $x - z$ plane at simulation times $t = \{2, 3, \cdots, 14\}$ (inner to outer). Dashed lines show the apparent horizon at time $t = 10$ (when it first appears) and $t = 14$. At later times both horizons cannot be distinguished in this plot. Right: the event horizon as a surface in $x - z - t$ coordinates. When it first appears it is not smooth at the equator, which is represented by the x coordinate in this plot. It remains non-smooth until the radial null geodesic (dashed line in the $z = 0$ plane) enters the event horizon as its generator (dotted line).

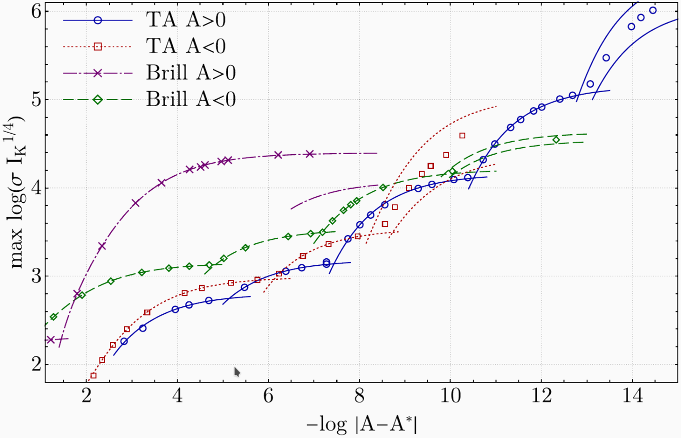
Bifurcation of echoes and AHs

- Massless scalar field is known to have DSS near-critical spacetimes.
- GW $\implies$ prohibited spherical symmetry
- Observation: Generic axisymmetric and **plane-symmetric** vacuum critical collapse leads to 2 'centers' [Hilditch2017]
- Single mass $\to$ (mass, separation, velocity)
- Limitation: MOTS searched during post-processing in saved 2D data
- Near-critical evolution leads to two black holes. Right: MOTS for different ID families
  top left: TA_+ $A = \overline{1.3008075}$, $t = 20.75$;
  top right: TA_- $A = -\overline{1.22436524}$, $t = 26$;
  bottom: Brill_+ $A = 4.698$, $t = 20.5$



Scaling (curvature)

- No apparent horizons for subcritical evolution
- Curvature invariants provide scale, e.g. $|I_K|^{-1/4}$
- Extremes appear on symmetry axis
- Extremes form repeating 'echoes'
- More and stronger echoes appear closer to A_*
- Scaling used as DSS indicator
- No universal slope observed



Scaling (AH mass)

- Apparent horizon is more involved
- Needs 2D data
  its mass changes fast fist
  limited data for post-processing
  → approximate M_{AH}
- Slicing-dependent
- Bifurcation

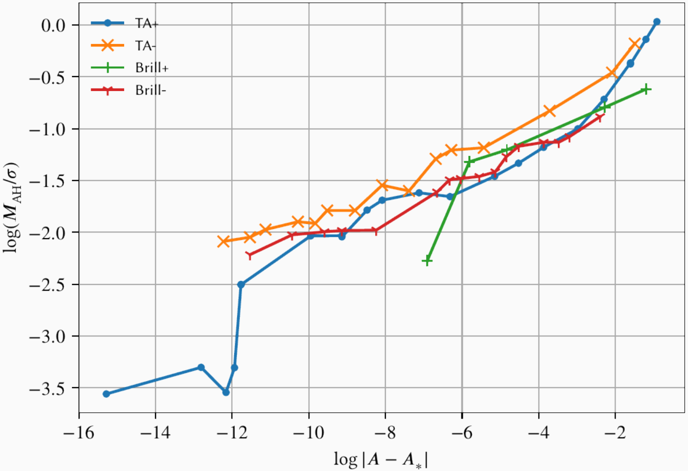


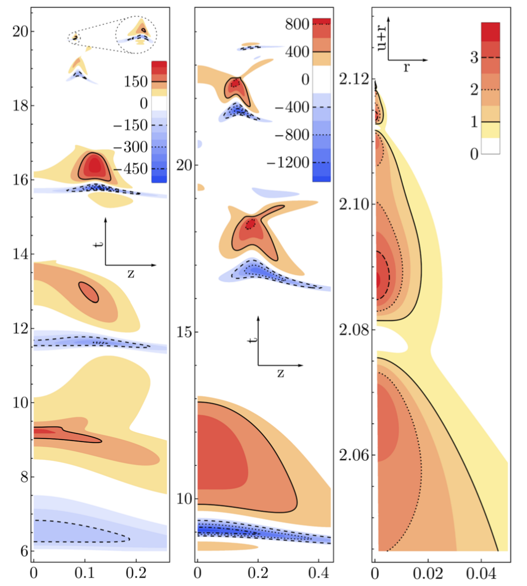
FIGURE 12.14: Apparent horizon mass scaling for different ID families. Plotted is the mass M_{AH} of the earliest AH located in each simulation. The sharp drop in the TA+ and Brill+ curves corresponds to the appearance of pairs of "off-center" horizons, where we show the mass of one horizon in the pair.

Discrete self-symmetry?

- Massless scalar field (right plot) is known to have DSS near-critical spacetimes.
- With prohibited spherical symmetry axisymmetric and plane-symmetric vacuum critical collapse leads to 2 'centers'
- Subcritical evolution forms 'bifurcated' curvature extremes
- Echoes in the curvature invariant ζ between different collapse scenarios.
- Since its extrema span several orders of magnitude, we plot contours of a dimensionless (but coordinate-dependent) quantity $(\tau - \tau_*)^2\zeta$ in the t-z plane.
- A DSS near-critical massless scalar-field collapse in the right panel.

  Left: TA_+ $A = \overline{1.30080828}$, $\tau_* = 3.88\sigma$

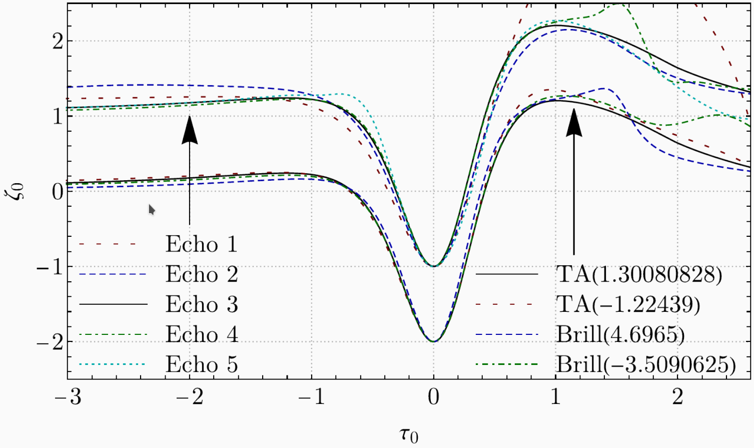
  Center: Brill_$A = 3.5090625$, $\tau_* = 5.9\sigma$

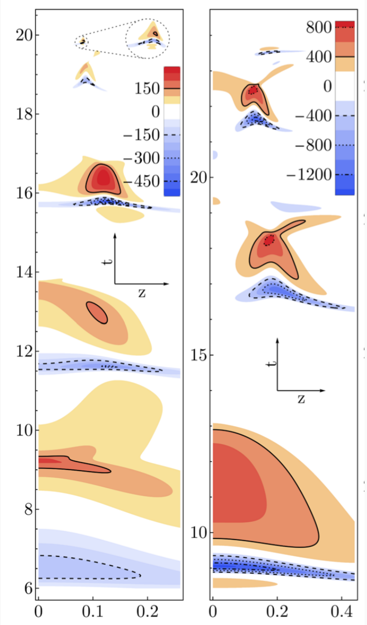


Universality of curvature extremes

- Observers in echoes see self-similar echoes
- Appropriately scaled experience

  $\zeta_0 = \langle \text{scale} \rangle^2 \zeta, \ \tau_0 = \tau / \langle \text{scale} \rangle$





Conclusions

- Even with axisymmetry near critical vacuum collapse simulations are still hard
- Spacetime curvature is more severe then for scalar field case
- We observe ‘local’ self-similarity of curvature extremes (echoes) in subcritical spacetimes
- Scaling does not seem universal, slopes are different between ID families
- More work needed

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Thank you