Epicyclic Frequencies for a Generic Ultracompact Object

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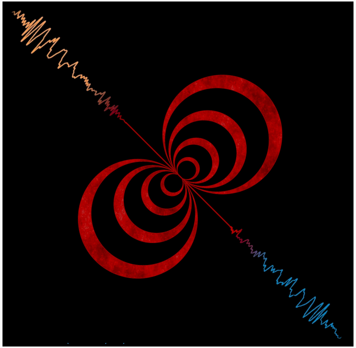
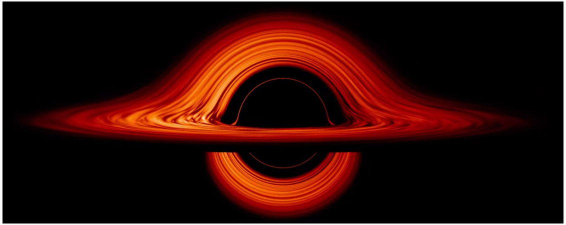
Jorge F. M. Delgado[1]

jorgedelgado@ua.pt

[1]University of Aveiro, Portugal



Motivation



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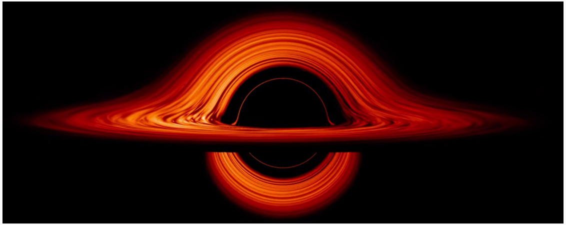

Timelike Circular Orbits

Null Circular Orbits

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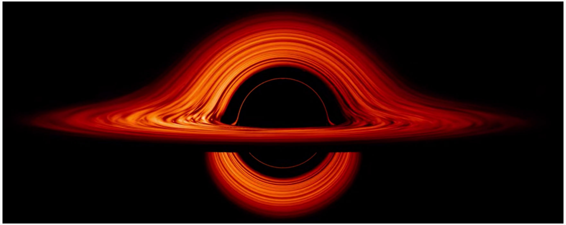

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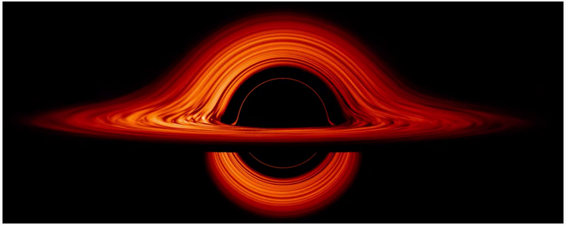

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→ Inner edge of an accretion disk.

→ Cut-off frequency of the emitted synchrotron radiation.

→ Cut-off frequency of the GW produced by EMRIs.

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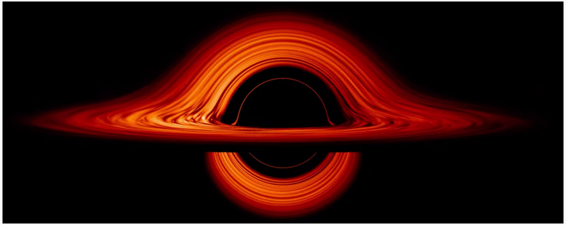

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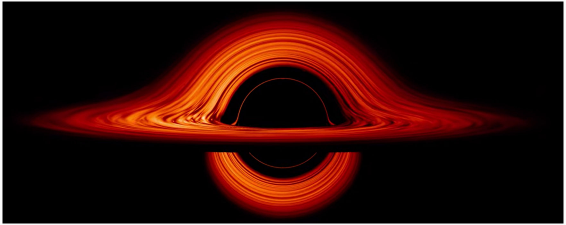

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Null Circular Orbits

Light-Ring

→ Inner edge of the shadow.

→ Real part of the frequency of quasi-normal modes.

Previous Result

For a generic stationary, axisymmetric, asymptotically flat compact object with a $\mathbb{Z}_2$ symmetry,

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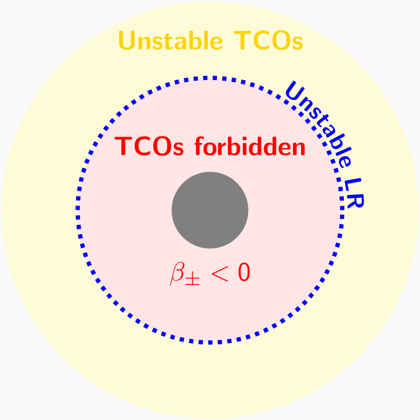

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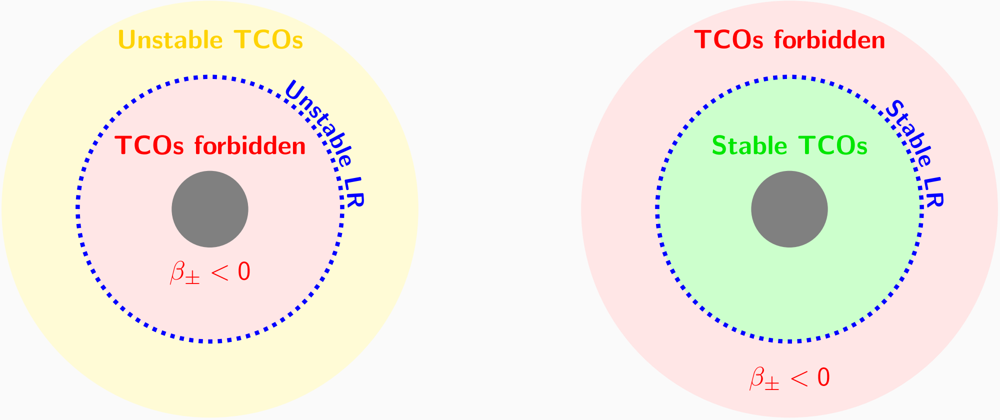

Vertical Stability?

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Generic spacetime,

$$ds^2 = g_{tt}(r,\theta)dt^2 + 2g_{t\varphi}(r,\theta)dtd\varphi + g_{\varphi\varphi}(r,\theta)d\varphi^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2$$

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Through the Lagrangian of a test particle and the constants of motion associated to the Killing vector fields, we introduce,

$$\boxed{V_\xi(r,\theta) \equiv g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 = \xi + \frac{A(r,\theta,E,L)}{B(r,\theta)}}$$

$$A(r,\theta,E,L) = g_{\varphi\varphi}E^2 + 2g_{t\varphi}EL + g_{tt}L^2 \ , \quad B(r,\theta) = g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}$$

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Circular motion can be found by solving,

$$V_\xi(r,\theta) = 0 \ , \quad \partial_r V_\xi(r,\theta) = 0 \ , \quad \partial_\theta V_\xi(r,\theta) = 0 \ .$$

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$$r = r^{\text{cir}} + \delta r \;\Rightarrow\; \dot{r} = \dot{\delta}_r$$

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Epicyclic Frequencies

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Null particles, $\xi = 0$:

Radial: $$(\nu_0^r)^2 = -\frac{1}{2}\left[\frac{\partial_r^2 g_{\varphi\varphi}\sigma_\pm^2 + 2\partial_r^2 g_{t\varphi}\sigma_\pm + \partial_r^2 g_{tt}}{g_{rr}}\right]$$

Vertical: $$(\nu_0^\theta)^2 = -\frac{1}{2}\left[\frac{\partial_\theta^2 g_{\varphi\varphi}\sigma_\pm^2 + 2\partial_\theta^2 g_{t\varphi}\sigma_\pm + \partial_\theta^2 g_{tt}}{g_{\theta\theta}}\right]$$

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Timelike particles, $\xi = -1$:

Radial: $$(\nu_{-1}^r)^2 = -\frac{1}{2}\left[\frac{\left(\partial_r^2 g_{\varphi\varphi}\Omega_\pm^2 + 2\partial_r^2 g_{t\varphi}\Omega_\pm + \partial_r^2 g_{tt}\right) B - 2C\beta_\pm}{Bg_{rr}}\right]$$

Vertical: $$(\nu_{-1}^\theta)^2 = -\frac{1}{2}\left[\frac{\partial_\theta^2 g_{\varphi\varphi}\Omega_\pm^2 + 2\partial_\theta^2 g_{t\varphi}\Omega_\pm + \partial_\theta^2 g_{tt}}{g_{\theta\theta}}\right]$$

Connection between Null and Timelike Circular Orbits

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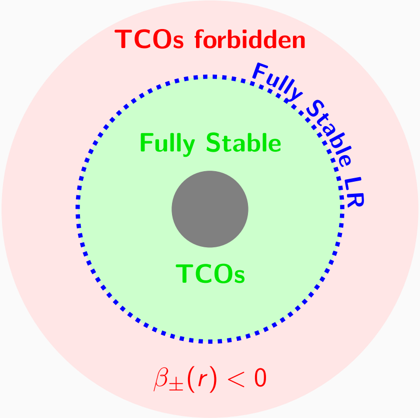
New Main Result

The *radial* stability of a light-ring determines the possibility and *radial* stability of timelike circular orbits around it.

The *vertical* stability of a light-ring determines only the *vertical* stability of timelike circular orbits around it.

Fully stable LR:



Illustrations

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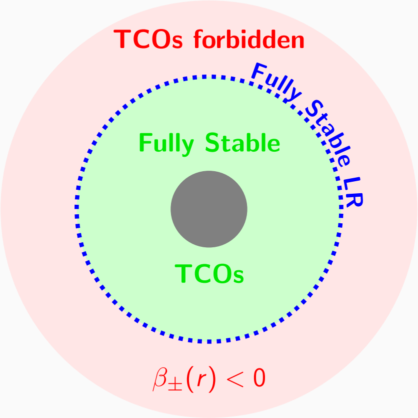

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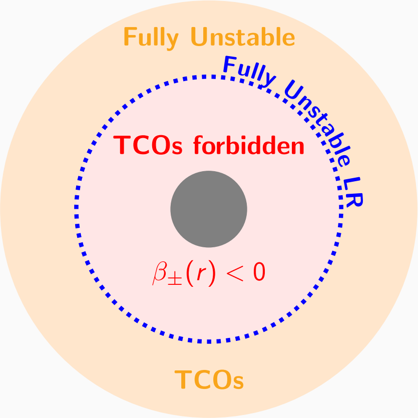

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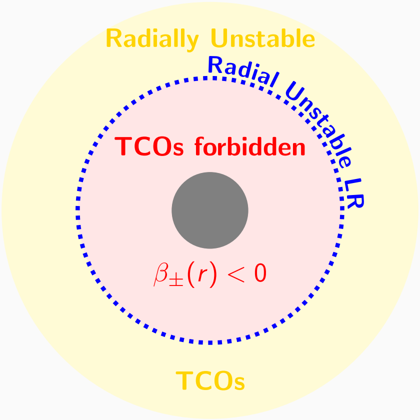
Radially unstable and vertically stable LR:



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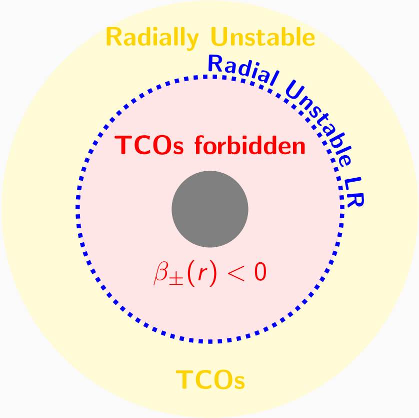

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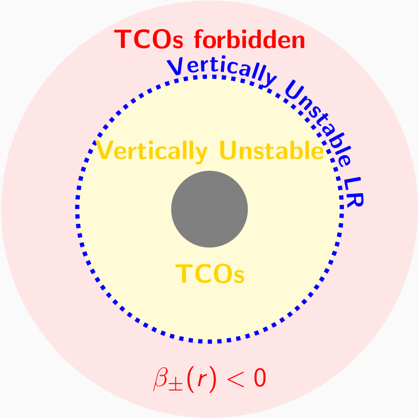

Epicyclic Frequencies and Energy Conditions

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$$\boxed{(\nu_0^r)^2 + (\nu_0^{\theta})^2 = \frac{1}{b^2} T_{\mu\nu}k^{\mu}k^{\nu}}$$

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Timelike Circular Orbits and the Strong Energy Condition, $R_{\mu\nu}t^\mu t^\nu \geq 0$.

Consider a timelike vector field tangent to a TCO, $t^\mu = a(\partial_t^\mu + \Omega_\pm \partial_\varphi^\mu)$. Then,

$$\boxed{(\nu_{-1}^r)^2 + (\nu_{-1}^\theta)^2 = \frac{1}{a^2} R_{\mu\nu}t^\mu t^\nu + \frac{1}{2g_{rr}} \frac{C\beta_\pm}{B}}$$

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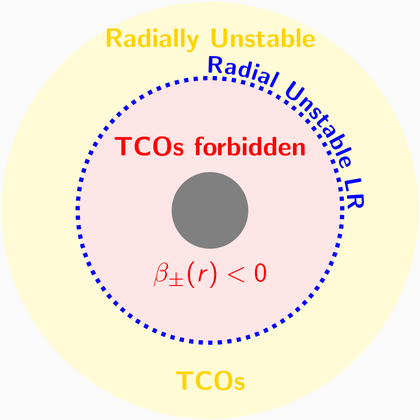
Radially unstable and vertically stable LR:



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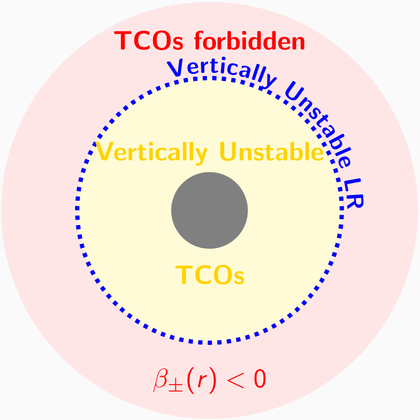

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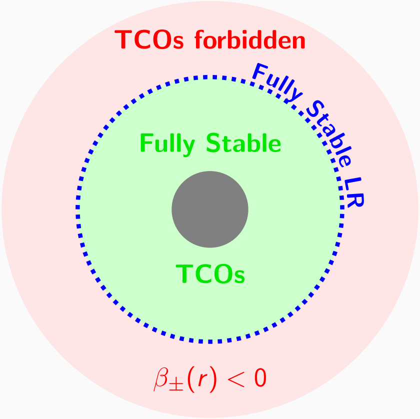

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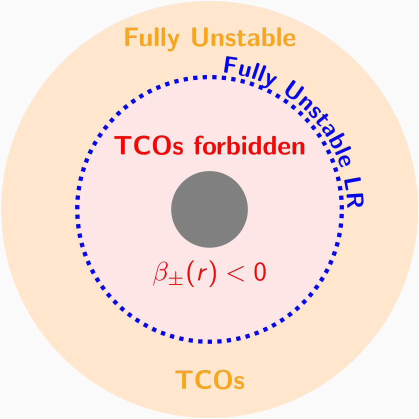

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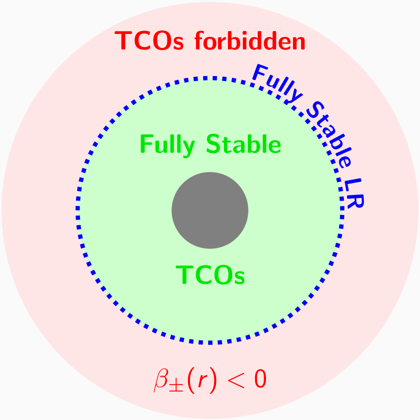

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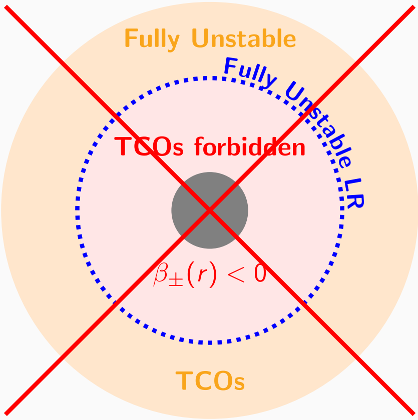

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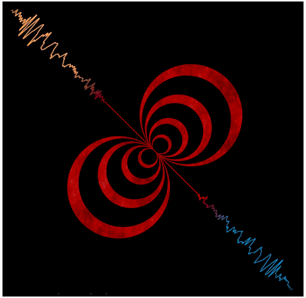

jorgedelgado@ua.pt

universidade de aveiro

MARIE CURIE ACTIONS

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

Generic Spacetime

$(\mathcal{M}, g)$ is a stationary, axi-symmetric, asymptotically flat and 1+3 dimensional spacetime.

- Two Killing vectors: $\{\eta_1, \eta_2\} \xrightarrow[\text{flatness}]{\text{asymptotically}} [\eta_1, \eta_2] = 0.$
- Appropriated coordinate system (t, r, θ, φ) such that $\eta_1 = \partial_t$ and $\eta_2 = \partial_\varphi$.

We assume,

1. A north-south $\mathbb{Z}_2$ symmetry.
2. Circularity. $\longrightarrow \quad g_{\rho t} = g_{\rho\varphi} = 0 \ , \ \rho = \{r, \theta\}$

Gauge choice:

- ⋆ r and θ are orthogonal.
- ⋆ Horizon located at constant radial coordinate: $r = r_H$.

$\longrightarrow \quad g_{r\theta} = 0 \ , \ g_{rr} > 0 \ , \ g_{\theta\theta} > 0$

Causality implies $g_{\varphi\varphi} \geq 0$

$$ds^2 = g_{tt}(r,\theta)dt^2 + 2g_{t\varphi}(r,\theta)dtd\varphi + g_{\varphi\varphi}(r,\theta)d\varphi^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2$$

Circular Causal Orbits on the Equatorial Plane $\theta = \pi/2$

Effective Lagrangian of a test particle,

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \xi \ , \quad \xi \equiv \begin{cases} -1 \ , & \text{timelike} \\ 0 \ , & \text{null} \\ 1 \ , & \text{spacelike} \end{cases}$$

Constants of motion associated to the two Killing vectors: $E = g_{t\mu}\dot{x}^\mu$ and $L = g_{\varphi\mu}\dot{x}^\mu$.

$$2\mathcal{L} = -\frac{A(r,E,L)}{B(r)} + g_{rr}\dot{r}^2 = \xi$$

$$A(r,E,L) = g_{\varphi\varphi}E^2 + 2g_{t\varphi}EL + g_{tt}L^2 \ , \quad B(r) = g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}$$

Effective potential $V_\xi(r)$,

$$\boxed{V_\xi(r) \equiv g_{rr}\dot{r}^2 = \xi + \frac{A(r,E,L)}{B(r)}}$$

Circular Causal Orbits on the Equatorial Plane $\theta = \pi/2$

A particle will follow a circular orbit at $r = r^{\text{cir}}$ iff,

$$V_\xi(r^{\text{cir}}) = 0 \quad \longrightarrow \quad A(r^{\text{cir}}, E, L) = -\xi B(r^{\text{cir}})$$
$$V'_\xi(r^{\text{cir}}) = 0 \quad \longrightarrow \quad A'(r^{\text{cir}}, E, L) = -\xi B'(r^{\text{cir}})$$

The *radial* stability of such orbit can be verified by the sign of $V''_\xi(r^{\text{cir}})$,

$$V''_\xi(r^{\text{cir}}) = \frac{A''(r^{\text{cir}}, E, L) + \xi B''(r^{\text{cir}})}{B(r^{\text{cir}})}$$

$V''_\xi(r^{\text{cir}}) > 0$
Unstable Circular Orbits

$V''_\xi(r^{\text{cir}}) < 0$
Stable Circular Orbits

For a generic stationary spacetime we can find pairs of solutions corresponding to *prograde* orbits $(r^{\text{cir}}_+, E_+, L_+)$ and *retrograde* orbits $(r^{\text{cir}}_-, E_-, L_-)$.

Light-Rings $\xi = 0$

For null particles, circular orbits are light-rings.

$$V_0(r^{\rm LR}) = 0 \quad \longrightarrow \quad \left[g_{\varphi\varphi}\sigma_\pm^2 + 2g_{t\varphi}\sigma_\pm + g_{tt}\right]_{\rm LR} = 0$$

$$V_0'(r^{\rm LR}) = 0 \quad \longrightarrow \quad \left[g'_{\varphi\varphi}\sigma_\pm^2 + 2g'_{t\varphi}\sigma_\pm + g'_{tt}\right]_{\rm LR} = 0$$

Solving both equations gives the *inverse impact parameter* $\sigma_\pm = E_\pm / L_\pm$ and the radial coordinate of the light-ring, $r = r^{\rm LR}$.

The *radial* stability of the light-ring is evaluated by checking the sign of $V_0''(r^{\rm LR})$,

$$V_0''(r^{\rm LR}) = L_\pm^2 \left[\frac{g''_{\varphi\varphi}\sigma_\pm^2 + 2g''_{t\varphi}\sigma_\pm + g''_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}\right]_{\rm LR}$$

Positive Numerator	Negative Numerator
Unstable Light Ring	Stable Light Ring

Timelike Circular Orbits $\xi = -1$

Angular velocity of timelike particles, $\Omega = \dfrac{d\varphi}{dt} = -\dfrac{E g_{t\varphi} + L g_{tt}}{E g_{\varphi\varphi} + L g_{t\varphi}}$

First equation: $V_{-1}(r^{\text{cir}}) = 0$,

$$E_\pm = -\frac{g_{tt} + g_{t\varphi}\Omega_\pm}{\sqrt{\beta_\pm}}\Bigg|_{r^{\text{cir}}}, \quad L_\pm = \frac{g_{t\varphi} + g_{\varphi\varphi}\Omega_\pm}{\sqrt{\beta_\pm}}\Bigg|_{r^{\text{cir}}}$$

$$\beta_\pm = -g_{tt} - 2g_{t\varphi}\Omega_\pm - g_{\varphi\varphi}\Omega_\pm^2$$

Second equation: $V'_{-1}(r^{\text{cir}}) = 0$,

$$\left[g'_{\varphi\varphi}\Omega_\pm^2 + 2g'_{t\varphi}\Omega_\pm + g'_{tt}\right]_{r^{\text{cir}}} = 0 \quad \longrightarrow \quad \Omega_\pm = \left[\frac{-g'_{t\varphi} \pm \sqrt{(g'_{t\varphi})^2 - g'_{tt}g'_{\varphi\varphi}}}{g'_{\varphi\varphi}}\right]_{r^{\text{cir}}}$$

Radial Stability, $V''_{-1}(r^{\text{cir}}) = \left[\dfrac{g''_{\varphi\varphi}E_\pm^2 + 2g''_{t\varphi}E_\pm L_\pm + g''_{tt}L_\pm^2 - (g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi})''}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}\right]_{r^{\text{cir}}}$