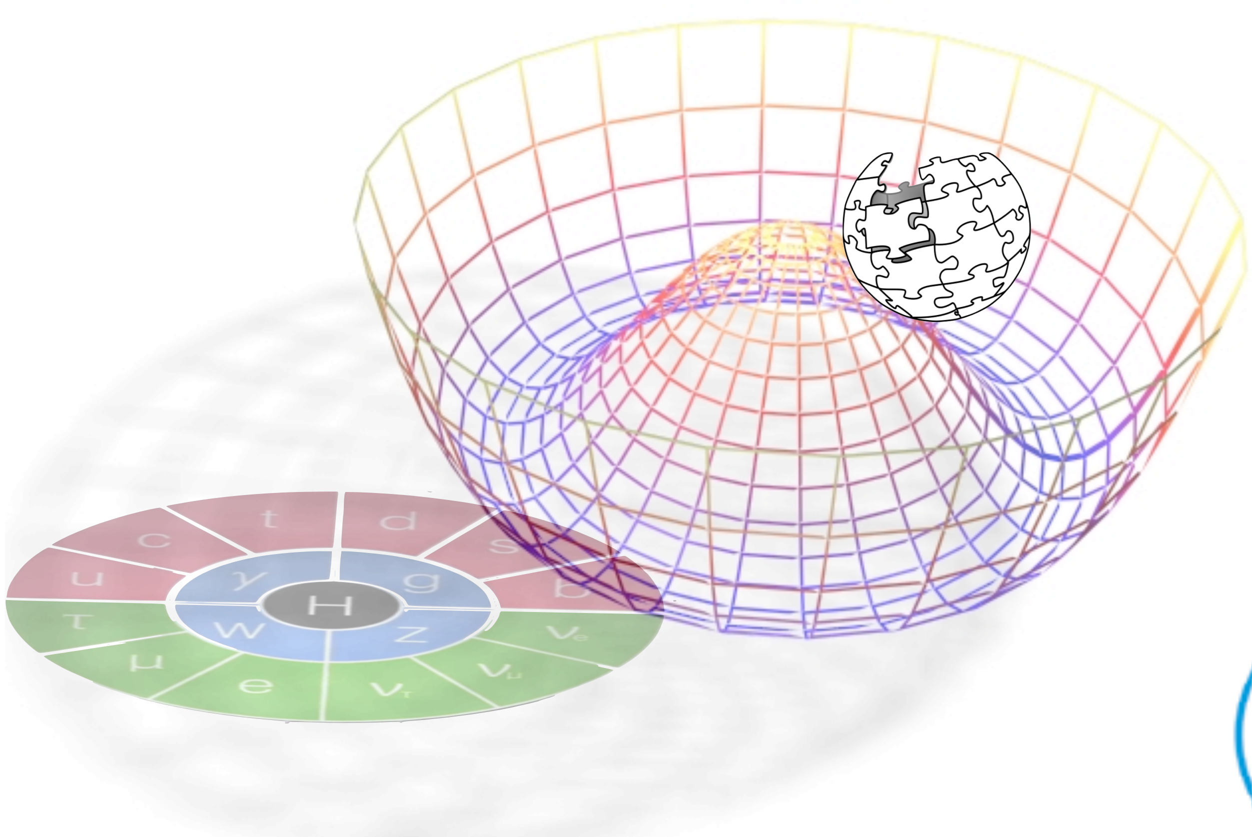
The Standard Model
of particle physics

CERN summer student lectures 2022

Lecture 1/5



Christophe Grojean
DESY (Hamburg)
Humboldt University (Berlin)
(christophe.grojean@desy.de)

Citius, Altius, Fortius

Often, the athletes, as the physicists, push the frontiers/break the records.
How high can a human jump with a pole?
Physics (energy conservation) tells us that longer poles don't help!

$$\Delta h = \frac{v^2}{2g}$$

footspeed: 44.72km/h
(Usain Bolt, Berlin, August 2009, between 60m and 80m)

$$\Delta h = 7.62\,\mathrm{m}$$

Over the years, we have learnt a few other **conservation laws** that tell us what an athlete/a particle can do or cannot do.

— Remarkable breakthrough in the understanding of Nature: —
forces among particles are associated to symmetries

- conservation of E → invariance by (time)-translation
- electro-magnetic forces → (local) invariance by phase rotation of particle wavefunctions

The Standard Model of Particle Physics
Lorentz symmetry + internal SU(3)xSU(2)xU(1) symmetry

Outline

- **Monday**
  - Lagrangians
  - Lorentz symmetry - scalars, fermions, gauge bosons
  - Dimensional analysis: cross-sections and life-time.
- **Tuesday**
  - Gauge interactions
  - Electromagnetism: U(1)
  - Nuclear decay, Fermi theory and weak interactions: SU(2)
  - Strong interactions: SU(3)
- **Wednesday**
  - Chirality of weak interactions
  - Pion decay
- **Thursday**
  - Spontaneous symmetry breaking and Higgs mechanism
  - Quark and lepton masses
  - Neutrino masses
- **Friday**
  - Running couplings
  - Asymptotic freedom of QCD
  - Anomalies cancelation

Intro: SM= S(R+Q)M

The fundamental constituents of matter obey the laws of **Quantum Mechanics** and **Special Relativity**
They are described in the framework of **Quantum Field Theory (QFT)**

QFT offers a way
1) To organise our knowledge
2) To parametrise our ignorance

→ Describe collider data

→ Play a crucial role in the evolution of the Universe

→ Likely to fail to describe gravity @ quantum level

"Before breaking the rules, you first need to master them"

Goals of the lectures

1. Explain QFT to describe the SM particles and their interactions
2. Explain how to compute cross-section and decay rate
3. Introduce the principles to build a model of Nature
4. Unveil clues where the SM might fail

Lagrangians

The Newton law of classical mechanics

$$\vec{F} = m\vec{a} \quad \text{or} \quad V'(x) = -m\ddot{x}$$

can be obtained by requiring the least action principle

$$\delta S = 0$$

where

the action: $S = \int_{t_1}^{t_2} dt \mathcal{L}(x, \dot{x})$ with the (classical) Lagrangian: $\mathcal{L}(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)$

(Hamiltonian/energy: $\mathcal{H} = \dot{x} \frac{\delta \mathcal{L}}{\delta \dot{x}} - \mathcal{L} = \frac{1}{2} m \dot{x}^2 + V(x)$)

Euler-Lagrange equations

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\delta \mathcal{L}}{\delta x} - \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{x}} \right) \delta x + \text{boundary terms} = 0 \quad \Rightarrow \quad \frac{\delta \mathcal{L}}{\delta x} = \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{x}}$$

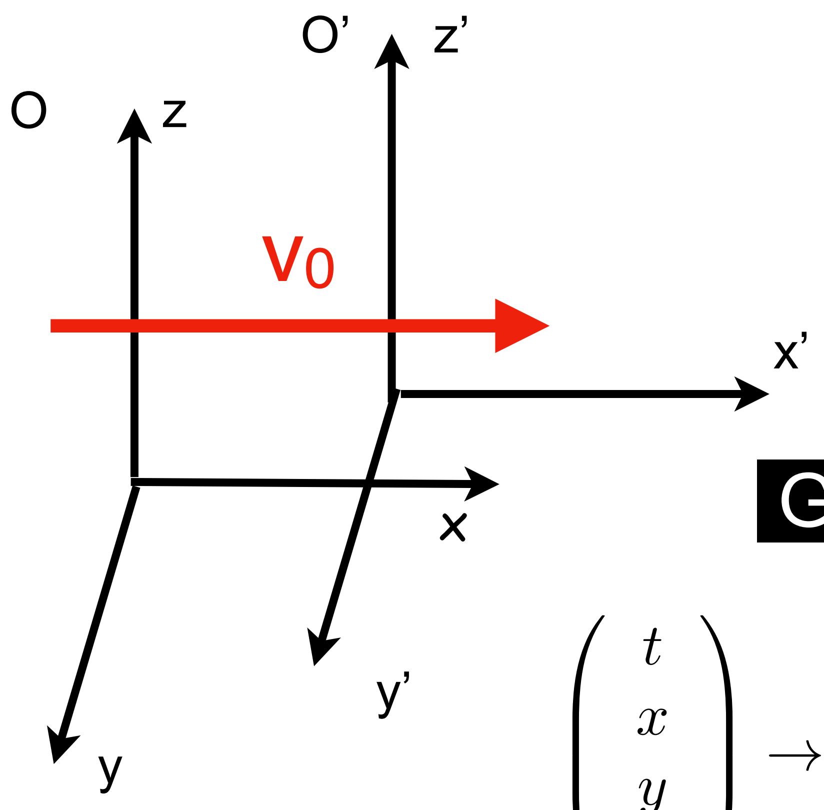
For the classical Lagrangian: $-V'(x) = m\ddot{x}$

Questions we will address in the lectures

What is the Lagrangian that describes the dynamics of the SM particles?

What are the rules to construct such a Lagrangian?

Lorentz Transformations



Consider two observers

in relative motion with a constant speed v_0 along the x-axis

they use their own systems of coordinates (t,x,y,z) and (t',x',y',z')

Galilean transformations

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} t' = t \\ x' = -\beta_0 ct + x \\ y' = y \\ z' = z \end{pmatrix} \quad \text{with} \quad \beta_0 = \frac{v_0}{c}$$

in particular

$$v' = v - v_0$$

The speed can

be arbitrarily large

Lorentz transformations

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} ct' = \gamma_0 \left(ct - \beta_0 x\right) \\ x' = \gamma_0 \left(-\beta_0 ct + x\right) \\ y' = y \\ z' = z \end{pmatrix} \quad \text{with} \quad \beta_0 = \frac{v_0}{c}, \quad \gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}$$

in particular

$$v' = \frac{v - v_0}{1 - v \cdot v_0/c^2}$$

The speed of light is

the same for all observers:

if v=c than v'=c too

Equations of Motion of Elementary Particles

Schrödinger Equation (1926): $$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\Delta-V\right)\Phi=0$$

$$E=\frac{p^2}{2m}+V$$

classical ↔ quantum correspondance

$E\to i\hbar\frac{\partial}{\partial t}$ & $p\to i\hbar\frac{\partial}{\partial x}$

Klein-Gordon Equation (1927): $$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta+\frac{m^2c^2}{\hbar^2}\right)\Phi=0$$

$$\frac{E^2}{c^2}=p^2+m^2c^2$$

More on Thursday lecture

Dirac Equation (1928): $$\left(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar}\right)\Psi=0$$

$$E=\begin{cases}+\sqrt{p^2c^2+m^2c^4} & \text{matter}\\ -\sqrt{p^2c^2+m^2c^4} & \text{antimatter}\end{cases}$$

$$E=\vec{\alpha}\vec{p}c+\beta mc^2$$

$$\gamma^0=\beta,\ \gamma^i=\beta\alpha^i,\ \{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$$

positron (e^+) discovered by C. Anderson in 1932

Scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

ϕ (real) scalar field
describes a spin-0 particle
when quantised

- **Equation of motion:**

$$0 = \delta\mathcal{L} = \left(-\partial_\mu \partial^\mu \phi - \frac{\partial V}{\partial \phi}\right)\delta\phi$$

up to boundary terms (that should vanish at infinity)

Klein-Gordon equation $\Box\phi = -V'(\phi)$

- **Lorentz invariant Lagrangian:**

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\phi(x) \to \phi'(x') = \phi(x)$$

with

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Then $\partial_\mu \phi = \Lambda^\nu{}_\mu \, \partial'_\nu \phi'$

And $\partial_\mu \phi \partial^\mu \phi = \underbrace{\eta^{\mu\nu} \Lambda^{\mu'}{}_\mu \, \Lambda^{\nu'}{}_\nu}_{\eta^{\mu'\nu'}} \, \partial'_{\mu'} \phi' \partial'_{\nu'} \phi' = \eta^{\mu'\nu'} \partial'_{\mu'} \phi' \partial'_{\nu'} \phi'$

$\eta^{\mu'\nu'}$ for a Lorentz transformation

Fermion Lagrangian

$$\mathcal{L} = \psi^\dagger \gamma^0 \left(i\gamma^\mu \partial_\mu - m \right) \psi$$

ψ 4-component Dirac spinor describes a spin-1/2 particle when quantised

$\gamma^\mu \ (\mu = 0, 1, 2, 3)$ are four 4x4 matrices

- **Equation of motion:**

$$0 = \delta\mathcal{L} = \psi^\dagger \gamma^0 \left(i\gamma^\mu \partial_\mu - m \right) \delta\psi$$

Dirac equation $\left(i\gamma^\mu \partial_\mu - m \right) \psi = 0$

- **Lorentz invariance:** (see technical slides at the end of the lecture)

$$x^\mu \to x'^\mu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu \quad \text{with} \quad \omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

$$\psi(x) \to \psi'(x') = \left(1_4 + \frac{1}{8} \omega_{\mu\nu} [\gamma^\mu, \gamma^\nu] \right) \psi(x)$$

- **Dirac algebra:**

For this equation to be consistent with Einstein equation ($m^2=E^2-p^2$) or Klein-Gordon eq., the γ^μ matrices have to obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

- **Dirac matrices:** One particular realisation of the Dirac algebra (not unique)

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}$$

Natural & Planck Units

- $[G_N]$=mass^{-1} L^3 T^{-2}
- $[\hbar]$=mass L^2 T^{-1}
- $[c]$=L T^{-1}

- Planck mass: $M_{\rm Pl} = \sqrt{\frac{\hbar c}{G_{\rm N}}} \sim 10^{19}\,{\rm GeV/c^2} \sim 2\times 10^{-5}\,{\rm g}$
- Planck length: $l_{\rm Pl} = \sqrt{\frac{\hbar G_{\rm N}}{c^3}} \sim 10^{-33}\,{\rm cm}$
- Planck time: $\tau_{\rm Pl} = \sqrt{\frac{\hbar G_{\rm N}}{c^5}} \sim 10^{-44}\,{\rm s}$

In High Energy Physics, it is a current practise to use a system of units for which $\hbar$=1 and c=1

Mass ~ distance^{-1} ~ time^{-1}

Unit conversion: SI ↔ HEP

E	T	L
1eV	10^{-16}s	10^{-7}m
10^{-16}eV	1s	10^{9}m
10^{-7}eV	10^{-9}s	1m

- The string theorists will remember:

$$M_{\rm Pl} \sim 10^{19}\,{\rm GeV} \quad \leftrightarrow \quad \tau_{\rm Pl} \sim 10^{-44}\,{\rm s} \quad \leftrightarrow \quad l_{\rm Pl} \sim 10^{-33}\,{\rm cm}$$

- The nuclear physicists will remember:

$$\hbar c \sim 200\,{\rm MeV}\cdot{\rm fm}$$
$$10^8\,{\rm eV} \quad \leftrightarrow \quad 10^{-15}\,{\rm m} \quad \leftrightarrow \quad 10^{-24}\,{\rm s}$$

- The others will remember:

average mosquito m~10^{-3}g=100$M_{\rm Pl}$
Compton wavelength 0.01$L_{\rm Pl}$=10^{-35}cm, Schwarzschild radius 100$L_{\rm Pl}$=10^{-31}cm
(much smaller than its physical size, so a mosquito is not a Black Hole)

Dimensional Analysis

$$[S]_m = 0 \quad \Rightarrow \quad [\mathcal{L}]_m = 4$$

$$S = \int d^4x \, \mathcal{L}$$

Scalar field	$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + \ldots$	➡	$[\phi]_m = 1$
Spin-1/2 field	$\mathcal{L} = \psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi$	➡	$[\psi]_m = 3/2$
Spin-1 field	$\mathcal{L} = F_{\mu\nu} F^{\mu\nu} + \ldots$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \ldots$	➡	$[A_\mu]_m = 1$

Particle lifetime of a (decaying) particle: $[\tau]_m = -1$ Width: $[\Gamma = 1/\tau]_m = 1$

Cross-section ("area" of the target): $[\sigma]_m = -2$

Lifetime "Computations"

muon and neutron are unstable particles

$$\mu \to e \nu_\mu \bar{\nu}_e$$

$$n \to p\, e\, \bar{\nu}_e$$

We'll see that the interactions responsible for the decay of muon and neutron are of the form

$$\mathcal{L} = G_F\, \psi^4 \quad \Rightarrow \quad \Gamma \propto G_F^2 m^5$$

$\mathcal{L}$: $[\text{mass}]^4$; G_F: $[\text{mass}]^{-2}$; ψ^4: $[\text{mass}]^{3/2\times 4}$; m: $[\text{mass}]$

G_F = Fermi constant: $G_F \sim \dfrac{10^{-5}}{m_{\text{proton}}} \sim 10^{-5}\,\text{GeV}^{-2}$

$1 = \hbar c \sim 200\,\text{MeV}\cdot\text{fm}$

E	T	L
1eV	10^{-16}s	10^{-7}m

For the **muon**, the relevant mass scale is the muon mass m_μ=105MeV:

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} \sim 10^{-19}\,\text{GeV} \quad \text{i.e.} \quad \tau_\mu \sim 10^{-6}\,\text{s}$$

For the **neutron**, the relevant mass scale is $(m_n\text{-}m_p)\approx$1.29MeV:

$$\Gamma_n = \mathcal{O}(1)\frac{G_F^2 \Delta m^5}{\pi^3} \sim 10^{-28}\,\text{GeV} \quad \text{i.e.} \quad \tau_n \sim 10^3\,\text{s}$$

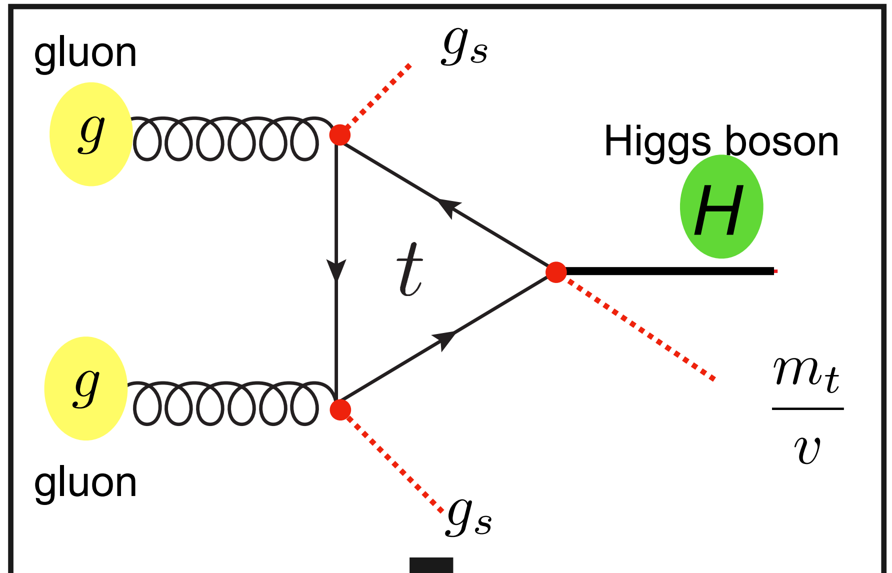
Higgs production "Computation"

At the LHC, the dominant Higgs production mode is gluon fusion

strong coupling constant $g_s \sim 1$

$$\mathcal{L} = g_s\, g^a_\mu\, \bar{t}\gamma^\mu T^a t$$

[mass]⁰ [mass]¹ [mass]^3/2x2



Higgs coupling

$$g_{ttH}\, \bar{t}t H$$

[mass]⁰ [mass]^3/2x2 [mass]¹

$$g_{ttH} = \frac{m_t}{v}$$ v=246 GeV

dimensional analysis allows us to compute the Higgs production cross-section

$$\sigma = \frac{1}{8\pi}\,\frac{1}{16\pi^2}\, g_s^4\, \frac{m_t^2}{v^2}\, \frac{1}{m_t^2}$$

flux — loop — couplings — dimensionally $[\sigma]_m=-2$

i.e. $\sigma \sim 10^{-25}\,\mathrm{eV}^{-2} \sim 10^{-39}\,\mathrm{m}^2 = 10\,\mathrm{pb}$

E	L
1eV	10^{-7}m

$1\,\mathrm{barn} = 10^{-28}\,\mathrm{m}^2$

$1\,\mathrm{pb} = 10^{-12}\,\mathrm{barn}$

One could think that all the quarks should give a similar contribution to the Higgs production since m_t factors cancel.
But it can be shown that this cancelation holds only for quarks heavier than the Higgs
(value of Higgs production xs excludes a heavy fourth generation of quarks).
LHC collides protons which contain gluons. How many gluons are inside the quarks depends on the energy of the protons. The production cross-section depends on the energy of the collider (40 pb at 14TeV, at 100TeV)

How many Higgs bosons produced at LHC? $\sigma \times \int dt\, L = 10\,\mathrm{pb} \times 100\,\mathrm{fb}^{-1} \sim 10^6$

(L = luminosity= nbr of collisions)

Higgs Lifetime "Computation"

exercice:

Using similar dimensional analysis arguments,
compute the Higgs boson lifetime (or its inverse aka as the Higgs decay width)

— Hints —

Higgs couplings proportional are proportional to the mass of the particles it couples to.
It will therefore decay predominantly decay into the heaviest particle that is lighter than $m_H/2$

Weak Interactions and Higgs Physics

The decays of the neutrons and muons are controlled by the value of the Fermi constant

G_F = Fermi constant: $G_F \sim \frac{10^{-5}}{m_{\mathrm{proton}}} \sim 10^{-5}\,\mathrm{GeV}^{-2}$

Higgs physics is controlled by the value of the Vacuum Expectation Value

v = 246 GeV

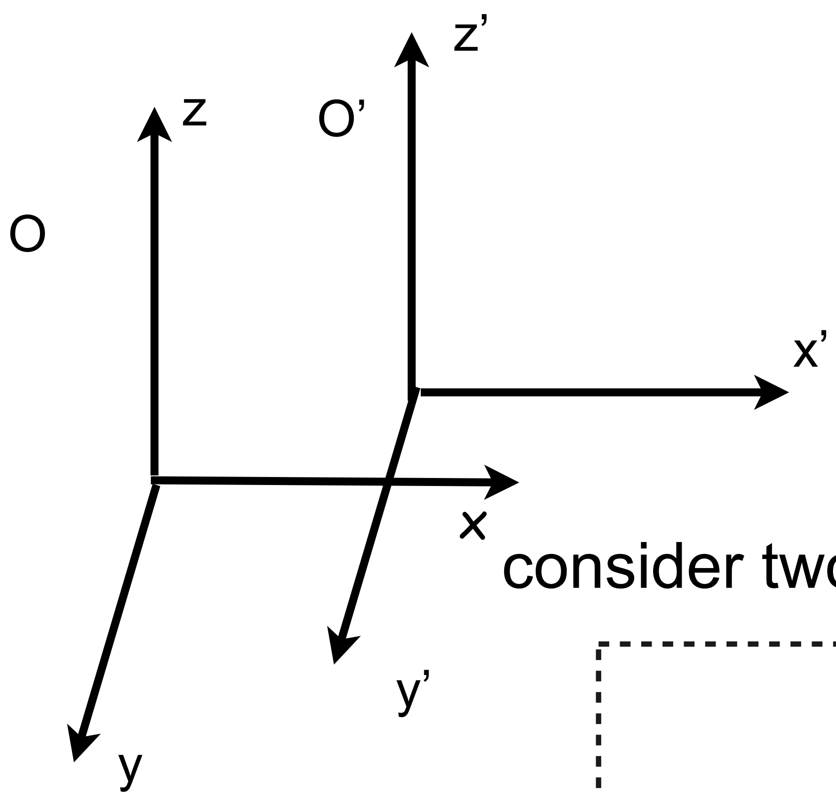
We can notice that

$$G_F = \frac{1}{\sqrt{2}v^2}$$

This relation is the consequence of the description of the weak interactions in terms of a local internal symmetry and its spontaneous breaking in the vacuum.
That's what we'll figure out in the next lectures.

Technical Details
for Advanced Students

Time-ordering ≠ Causality



$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} ct' = \gamma_0\left(ct - \beta_0 x\right) \\ x' = \gamma_0\left(-\beta_0 ct + x\right) \\ y' = y \\ z' = z \end{pmatrix}$$

"time dilation + space contraction"

consider two events E_1 and E_2 characterised by their space-time coordinates

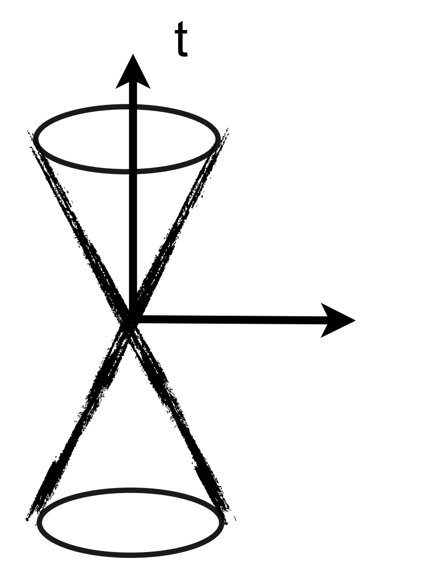
E_1		E_2	
$t_1 = 0$	$t'_1 = 0$	$t_2 > 0$	$ct'_2 = \gamma\left(ct_2 - \beta x_2\right)$
$x_1 = 0$	$x'_1 = 0$	$x_2 > 0$	$x'_2 = \gamma\left(-\beta ct_2 + x_2\right)$

t'_2 can be positive or negative
causality ≠ time ordering

Proper space-time distance Δ is independent of the observer:

$$\Delta'^2 = \left(ct'_2\right)^2 - \left(x'_2\right)^2 = \left(ct_2\right)^2 - x_2^2 = \Delta^2$$

Only events inside the past/future light cones are causally connected
The light cones are invariant under Lorentz transformations



Compton vs Schwarzschild Scales

Compton radius: for an object of mass m, one can define a length scale that will measure its quantum size

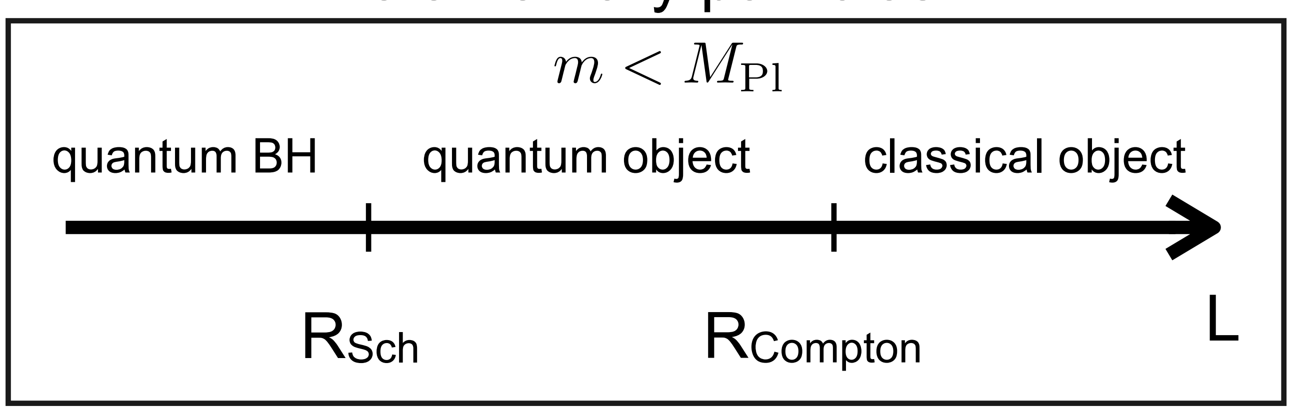
$$R_{\text{Compton}} = \frac{\hbar}{mc}$$

Schwarzschild radius: for an object of mass m, one can define a length scale that will measure its gravitational strength

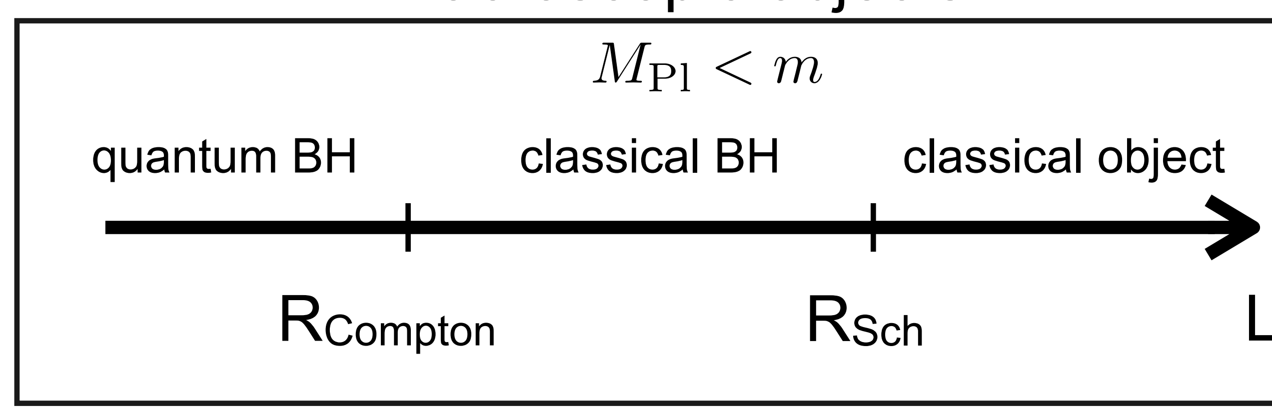
$$R_{\text{Sch}} = \frac{G_{\text{N}} m}{c^2} = \frac{m}{M_{\text{Pl}}} l_{\text{Pl}}$$

$$R_{\text{Compton}} < R_{\text{Sch}} \ \text{ iff } \ M_{\text{Pl}} < m$$

— elementary particles —



— macroscopic objects —

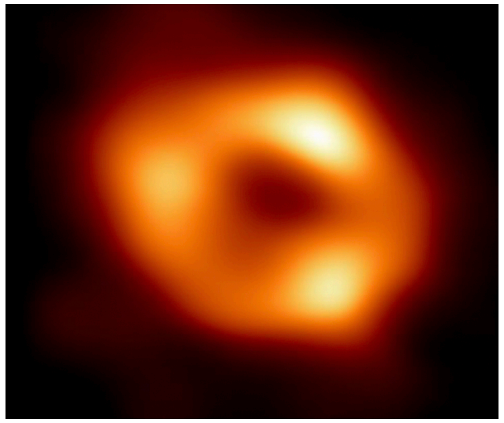


Black Holes

Neutron stars: m~10^{30}kg, R~10^4m (density of human population concentrated in a sugar cube): R_{Sch}~10^3m: ~~BH~~

Stellar BHs: m~10^{31}kg, R~10^4m: R_{Sch}~10^4m: BH

Supermassive BHs: m~10^{37}kg, R~10^{10}m: R_{Sch}~10^{10}m: BH



Event Horizon Telescope
Sagittarius A*
m = 4.3x10^6 M_{sun}
R =23.5x10^6 km

LHC Black Holes: m~1TeV, R~10^{-19}m: $R_{Compton}$~10^{-19}m, R_{Sch}~10^{-51}m (ordinary gravity) but

R_{Sch}~10^{-19}m if M_{Pl} is lowered to 1TeV as in models with large extra dimensions

Einstein Algebra

$$x^\mu = (ct,\ x,\ y,\ z) \quad \mu = 0,1,2,3$$

Lorentz-invariant distance

$$\Delta^2 = c^2t^2 - x^2 - y^2 - z^2 = \eta_{\mu\nu}x^\mu x^\nu$$

with

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Minkowski metric

Useful notations: $x_\mu = \eta_{\mu\nu}x^\nu = (ct, -x, -y, -z)$ such that $\Delta^2 = x_\mu x^\mu$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

Lorentz transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \text{ with } \eta_{\mu\nu}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'} = \eta_{\mu'\nu'}$$

For example:

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

since

$$\gamma^2(1-\beta^2) = 1$$

➡ Minkowski metric is invariant under Lorentz transformations

Lorentz Transformations

Covariant form of a Lorentz transformation: $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$

The invariance of the line element: $\Delta^2 = \eta_{\mu\nu} x^{\mu} x^{\nu} \to \Delta'^2 = \eta_{\mu\nu} x'^{\mu} x'^{\nu}$ imposes the following condition

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$$

We always raise and lower the space time indices with the metric:

$$\Lambda_{\mu\nu} = \eta_{\mu\rho} \Lambda^{\rho}{}_{\nu} \qquad \Lambda_{\mu}{}^{\nu} = \eta_{\mu\rho} \eta^{\nu\sigma} \Lambda^{\rho}{}_{\sigma} \qquad \Lambda^{\mu\nu} = \eta^{\nu\sigma} \Lambda^{\mu}{}_{\sigma}$$

Transformation inverse: $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ $\qquad x^{\mu} = \Lambda_{\nu}{}^{\mu}\, x'^{\nu}$

Transformation of the space-time derivatives:

$$\partial_{\mu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\nu}} = \Lambda^{\nu}{}_{\mu} \partial'_{\nu}$$

$$\partial'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} = \Lambda_{\mu}{}^{\nu} \partial_{\nu}$$

Small Lorentz transformations: $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}$

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma} \quad \Leftrightarrow \quad \omega_{\mu\nu} = \omega_{\nu\mu}$$

Spinor Transformation

Transformation law: $\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)$

We want the Dirac equation to take the same form in the two systems of coordinates x and x'

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad\qquad (i\gamma^\mu \partial'_\mu - m)\psi' = 0$$

This implies the condition: $S\gamma^\mu \Lambda^\nu{}_\mu S^{-1} = \gamma^\nu$

We consider small Lorentz transformations: $\Lambda_\mu{}^\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$ $\qquad S = 1 - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}$

The covariance of the Dirac equation then implies that the matrices $\sigma_{\mu\nu}$ have to satisfy the relation

$$[\gamma^\nu, \sigma^{\rho\sigma}] = 2i(\eta^{\nu\rho}\gamma^\sigma - \eta^{\nu\sigma}\gamma^\rho)$$

It is easy to check that the following matrices fit the bill: $\sigma^{\rho\sigma} = \frac{i}{2}[\gamma^\rho, \gamma^\sigma]$

$$x^\mu \to x'^\mu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu)x^\nu \quad \text{with} \quad \omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

$$\psi(x) \to \psi'(x') = \left(1_4 + \frac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]\right)\psi(x)$$

Lorentz-invariant Lagrangian

$\mathcal{L} = \psi^\dagger M \,(i\gamma^\mu \partial_\mu - m)\,\psi$ is Lorentz-invariant iff $\gamma^0[\gamma^\nu, \gamma^\mu]\gamma^0 M + M[\gamma^\mu, \gamma^\nu] = 0$

$M = \gamma^0$ is a solution and it defines the Dirac Lagrangian. $\boxed{\bar\psi \equiv \psi^\dagger \gamma^0}$

Symmetries and invariants

SU(N)

the transformations among the components of a complex N-vector that leaves its norm invariant

$$|\phi|^2 = \phi_1^*\phi_1 + \dots \phi_N^*\phi_N \to |\phi'|^2 = |\phi|^2$$

SU(N,M)

the transformations among the components of a complex (N+M)-vector that leaves its (N,M) norm invariant

$$|\phi|^2 = \phi_1^*\phi_1 + \dots \phi_N^*\phi_N + \phi_{N+1}^*\phi_{N+1} - \dots - \phi_{N+M}^*\phi_{N+M} \to |\phi'|^2 = |\phi|^2$$

SO(N)

the transformations among the components of a real N-vector that leaves its norm invariant

$$|\phi|^2 = \phi_1^2 + \dots \phi_N^2 \to |\phi'|^2 = |\phi|^2$$

SO(N,M)

the transformations among the components of a real (N+M)-vector that leaves its (N,M) norm invariant

$$|\phi|^2 = \phi_1^2 + \dots \phi_N^2 + \phi_{N+1}^2 - \dots - \phi_{N+M}^2 \to |\phi'|^2 = |\phi|^2$$

The Lorentz group is thus SO(1,3)

Lorentz transformation

SO(1,3)

The elements of SO(1,3) satisfy $U^t \eta U = \eta$ where =diag(1,-1,-,1,-1)

The infinitesimal transformations are $U = e^{\theta^a T^a} \approx 1 + \theta^a T^a + \ldots$

The generators satisfy the constraints: $T^{a\,t}\eta + \eta T^a = 0$

One particular generator is $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

We obtain $e^{\theta T} = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

We indeed recover the usual Lorentz transformation with the identification

$$\gamma = \cosh\theta \quad \text{and} \quad \beta\gamma = \sinh\theta$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \Leftrightarrow \quad \cosh^2\theta - \sinh^2\theta = 1$$