Pedagogical implications of semi-classical description of electrostatic fields

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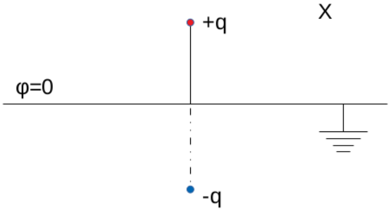
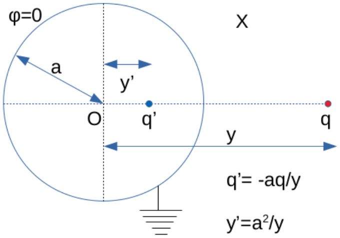
Overview

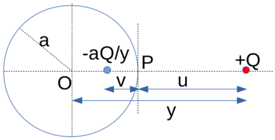
Background

Resolution


+q
X
φ=0
-q

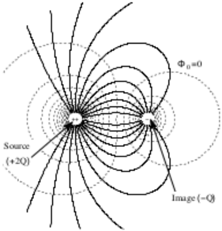

φ=0
X
a
y'
O
q'
q
y
q'= -aq/y
y'=a²/y

Analogy with mirror optics*



$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

$$\frac{q'}{q} = -\frac{|v|}{|u|} = -\frac{a}{y}$$



Conjugate focii

Reversibility of "electric field lines"?

Lorrain & Corson

EM Fields & Waves

Longitudinal magnification

$$\frac{dv}{du} = -\frac{v^2}{u^2}$$

Thoughts...

Huygen's wavelets?

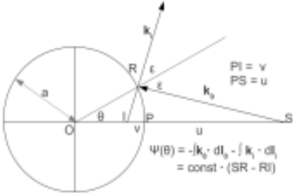
Fermat's principle?

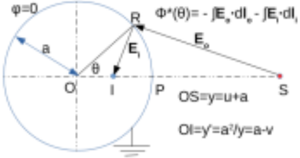
Anything else???

*K. Bhattacharya, Eur. J. Phys. 32 (2011)

$\delta \int Eds = 0$ in electrostatics[†],[‡]

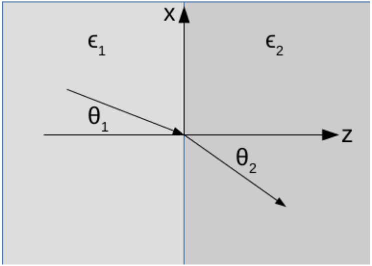
- Irrespective of path, the line integral remains the same
  - The Euler-Lagrange equation: $\nabla E = \frac{d}{ds}\left(E\frac{d\mathbf{r}}{ds}\right)$
    - This is only possible if $\nabla \times \mathbf{E} = \mathbf{0}$
  - $d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz\ldots \;\ldots\; \ldots(1)$
- $\boxed{\delta \int Vds = 0 \text{ whenever } \nabla \times \mathbf{V} = \mathbf{0} \text{ (or } \oint \mathbf{V}\cdot d\mathbf{s} = 0)}$

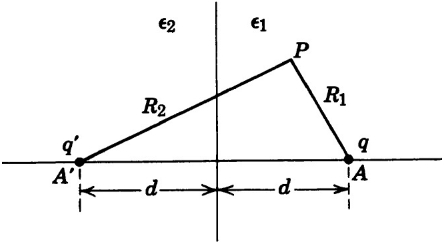




[†] K Bhattacharya, Journal of Electrostatics, Vol 71(5) 2013.

[‡] **K Bhattacharya**, D Syam, American Journal of Physics, Vol. 90(3), 2022.





$$\boxed{\frac{\epsilon_1}{\epsilon_2} = \frac{\tan\theta_1}{\tan\theta_2}}$$

$$q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} q$$

$$q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} q$$

Can we develop a semi-classical or quantum model of electrostatics?

Wave-electrostatics§

$$\delta \int E \left(\frac{ds}{da} \right) da = 0$$

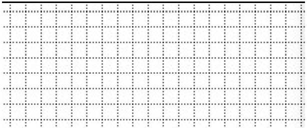
$$\mathcal{L}_{ES} = E(x,y,z)\sqrt{x'^2 + y'^2 + z'^2}$$

$$p_x = E_x, p_y = E_y, p_z = E_z$$

$$\mathcal{H}_{ES} = x' p_{x'} + y' p_{y'} + z' p_{z'} - \mathcal{L}_{ES} = 0$$

(a)

$\xrightarrow{\hat{p} = -i\bar{\gamma}\nabla}$



(b)

$$\boxed{\hat{\vec{p}}\psi_E = -i\frac{\gamma}{2\pi}\nabla\psi_E = \vec{E}\psi_E}$$

Source-free region

$$\boxed{\bar{\gamma}^2\nabla^2\psi_E + \mathcal{E}^2\psi_E = 0} \xrightarrow[\text{state}]{\text{basis}} \boxed{\psi_E = \mathbf{e}^{[i\Phi/\bar{\gamma}]}} \xrightarrow[\text{normalized}]{} \boxed{\Psi_E = \int c(E)\mathbf{e}^{[i\Phi/\bar{\gamma}]} dE}$$

§K Bhattacharya, **ICNFP 2020**, Physica Scripta Vol. 96(8), 2021.

In presence of source

$$\oint \mathcal{E}_x \ dx = 0 \qquad \oint \mathcal{E}_y \ dy = 0 \qquad \oint \mathcal{E}_z \ dz = 0$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\oint \mathcal{E}_x \ dx = n_x \bar{\gamma} \qquad \oint \mathcal{E}_y \ dy = n_y \bar{\gamma} \qquad \oint \mathcal{E}_z \ dz = n_z \bar{\gamma}$$

$$-i\bar{\gamma}\frac{\partial \psi_x}{\partial x} = \mathcal{E}_x \psi_x \qquad -i\bar{\gamma}\frac{\partial \psi_y}{\partial y} = \mathcal{E}_y \psi_y \qquad -i\bar{\gamma}\frac{\partial \psi_z}{\partial z} = \mathcal{E}_z \psi_z$$

$$\left[-\frac{\bar{\gamma}^2}{2\left(\frac{1}{\epsilon_0}\right)} \frac{\partial^2}{\partial x^2} - \frac{\epsilon_0 \mathcal{E}_x^2}{2} \right] \psi_x = i \left(-\frac{1}{2} \rho_x \bar{\gamma} \right) \psi_x$$

Normalizable solutions are:

$$\boxed{\Psi_{E_x} = \int u(\mathcal{E}_x) \mathbf{e}^{-i \frac{\Phi_{E_x}}{\bar{\gamma}_{E_x}}} d\mathcal{E}_x = \int u(\mathcal{E}_x) \mathbf{e}^{-i \frac{\Phi}{\bar{\gamma}}} d\mathcal{E}_x}$$

Observations

$$\boxed{\left[-\frac{\bar{\gamma}^2}{2\left(\frac{1}{\epsilon_0}\right)}\frac{\partial^2}{\partial x^2}-\frac{\epsilon_0\mathcal{E}_x^2}{2}\right]\psi_x=i\left(-\frac{1}{2}\rho_x\bar{\gamma}\right)\psi_x}$$

- Has the form of the time-independent Schrödinger's equation.
- 1^{st} and 2^{nd} terms on the left-hand side denote the kinetic and the potential energy density terms.
- Inverse of the permittivity plays the role of mass of ψ_E field.
- Presence of anti-Hermitian operator.

Wave-equation for electric displacement $\vec{\mathcal{D}}$

$$\nabla \times \vec{\mathcal{D}} = \nabla \times \vec{\mathcal{P}}$$

Consider cases when $\nabla \times \vec{\mathcal{P}} = \mathbf{0} \implies \oint \mathcal{P} \cdot d\mathbf{s} = 0$

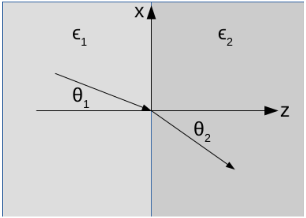
$$\oint \mathcal{D}_x \, dx = 0 \qquad \oint \mathcal{D}_y \, dy = 0 \qquad \oint \mathcal{D}_z \, dz = 0$$

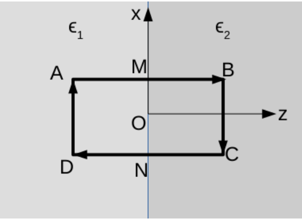
$$-i\bar{\gamma}_D \frac{\partial \psi_x}{\partial x} = \mathcal{D}_x \psi_x \qquad -i\bar{\gamma}_D \frac{\partial \psi_y}{\partial y} = \mathcal{D}_y \psi_y \qquad -i\bar{\gamma}_D \frac{\partial \psi_z}{\partial z} = \mathcal{D}_z \psi_z$$

$$\left[-\frac{\bar{\gamma}^2}{2\left(\frac{1}{\epsilon}\right)} \frac{\partial^2}{\partial x^2} - \frac{\mathcal{D}_x \cdot \mathcal{E}_x}{2} \right] \psi_{D_x} = i \left(-\frac{1}{2} \rho_f \bar{\gamma} \right) \psi_{D_x}$$

Normalizable solutions are:

$$\Psi_{D_x} = \int v(\mathcal{D}_x) \mathbf{e}^{-i\frac{\Phi_D}{\bar{\gamma}_D}} d\mathcal{D}_x = \int v(\mathcal{D}_x) \mathbf{e}^{-i\frac{\Phi}{\bar{\gamma}}} d\mathcal{D}_x$$





$$\bar{\gamma}^2 \frac{d^2}{dx^2}\psi_{E_x} + \mathcal{E}_x^2 \psi_{E_x} = 0$$

$$\bar{\gamma}^2 \frac{d^2}{dy^2}\psi_{E_y} + \mathcal{E}_y^2 \psi_{E_y} = 0$$

$$\bar{\gamma}^2 \frac{d^2}{dz^2}\psi_{E_z} + \mathcal{E}_z^2 \psi_{E_z} = i\bar{\gamma}\frac{\sigma_b}{\epsilon_0}\delta(z)\psi_{E_z}$$

$$\boxed{\mathcal{E}_{1_{x,y}} = \mathcal{E}_{2_{x,y}}}$$

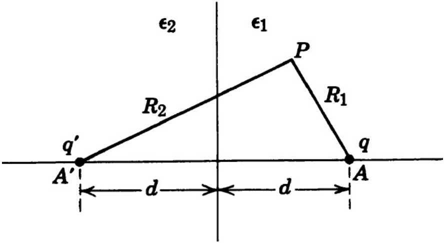
- $\oint \mathcal{D}_z dl = 0$ (or $\oint \mathcal{P}_z dl = 0$)
- $\oint \mathcal{D}_x dl \neq 0$ (or $\oint \mathcal{P}_x \ dl = 0$)
- Conceive: $\delta \int_{\mathcal{C}} \mathcal{D}_z ds = 0$

$$\bar{\gamma}_D^2 \frac{d^2}{dz^2}\psi_{D_z} + \mathcal{D}_z^2 \psi_{D_z} = 0$$

$$\implies \boxed{D_{1_z} = D_{2_z}}$$

$$\boxed{\frac{\epsilon_1}{\epsilon_2} = \frac{\tan\theta_1}{\tan\theta_2}}$$

Dielectric half-plane image problem



$$\psi_D^1(z) := \mathbf{e}^{i\frac{\mathcal{E}_{1z}z}{\bar{\gamma}}} + r\mathbf{e}^{-i\frac{\mathcal{E}_{1z}z}{\bar{\gamma}}}$$

$$\psi_D^2(z) := t\mathbf{e}^{i\frac{\mathcal{E}_{2z}z}{\bar{\gamma}}}$$

$$\left(\psi_D^1\right)_{(z=0)} = \left(\psi_D^2\right)_{(z=0)} \qquad 1 + r = t \qquad r = -\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}$$

$$\left(\frac{\partial \psi_D^1}{\partial z}\right)_{z=0} = \left(\frac{\partial \psi_D^2}{\partial z}\right)_{z=0} \qquad \mathcal{E}_{1z} - r \cdot \mathcal{E}_{1z} = t\mathcal{E}_{2z} \qquad t = 1 + r = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}$$

$$\mathcal{E}_{1z}^{q'} = -r\mathcal{E}_{1z}^{q} \implies q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}q \qquad \mathcal{E}_{1z}^{q''} = t\mathcal{E}_{2z}^{q} \implies q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}q$$

Wrapping up...

$$\left[-\frac{\bar{\gamma}^2}{2\left(\frac{1}{\epsilon_0}\right)}\frac{\partial^2}{\partial x^2} - \frac{\epsilon_0 \mathcal{E}_x^2}{2}\right]\psi_x = i\left(-\frac{1}{2}\rho_x\bar{\gamma}\right)\psi_x \tag{1}$$

Take the complex conjugate of Eq.(1) and call it Eq.(2). Multiplying Eq.(1) with ψ_x^* and Eq.(2) by ψ_x, and adding the the resulting equations, we find:

$$\nabla^2|\psi_x|^2 = 0$$

If boundary conditions on $|\psi_x|^2$ and electrostatic potential Φ are identical, then these two functions are identical.
Example: standard problems on the method of images.

Thank You

Questions welcome...

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