21 December 2022 to 2 January 2023
Conference venue: OAC conference center, Kolymbari, Crete, Greece. The conference will take place in Crete in physical form, however participation is also possible via internet
Egypt timezone

"Second MODE Workshop on Differentiable Programming for Experiment Design" following after ICNFP2022 (https://indico.cern.ch/event/1145124/)

Approaching the Einstein's everlasting dream; unification of general relativity and quantum mechanics

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14:00-14:20 Greece Time

Physical Problem

Possible unification of GR with QM requires quantization of GR and also gravitization of QM

General Relativity

(roughed at some scale)

- Spacetime discretization,
- Measurement uncertainty,
- Noncommutative relations,
- Generalized Riemann manifold

Quantum Mechanics

(smoothed at some scale)

- Gravitational field impacts,
- Generalized noncommutitavity,
- Relativity principle,
- Isotropy & Lorentz covariance

A minimum measurable length sets limits on space continuity
At low scales, GR coordinates would not be arbitrary smooth.

Quantizing GR allows corrections at low scale


Gravitating QM allows corrections at large scale


Non-Relativistic GUP

The quadratic momenta corrections to Heisenberg uncertainty principle as suggested by Kempf-Mangano-Mann reads

$$[x_i, p_j] = i\hbar\left(\delta_{ij} + \frac{\beta_0}{(M_P c)^2}\delta_{ij}p^2 + \frac{2\beta_0}{(M_P c)^2}p_i p_j\right)$$

where $p^2 := \vec{p}^2 = \sum_{j=1}^{3} p^j p_j$ and $M_P = \sqrt{\hbar c/G}$ is the Planck mass.

and

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\left(1 + \frac{\beta_0}{(M_P c)^2}\left((\Delta p)^2 + \langle p\rangle^2 + 2\Delta p_i^2 + 2\langle p_i\rangle^2\right)\right)$$

implying a minimum measurable length

$$\Delta x_{min} = \sqrt{3\beta_0}L_P, \text{ where } L_P = \sqrt{\hbar G/c^3}$$

- There are various GUP proposals, for example, Maggiore,

$$[x_i, p_j] = i\hbar\,\delta_{ij}\sqrt{1 + \frac{\gamma_0}{(M_P c)^2}(p^2 + m^2c^2)},$$

motivated by quantum deformation of the Pioncarre algebra implying a minimal length associated with as

$$\Delta x_{min} \simeq \sqrt{\gamma_0/2}L_P$$

Need for Relativistic GUP

The non-relativistic **3d-GUP** has no temporal dimension. Thus, in spacetime:

1. either commutators or uncertainties are NOT necessarily Lorentz covariant, *PPNL13(2016)59*
2. this means that Δx_{min} is frame dependent, and
3. this causes nonlinear additional law of momenta.

With the Lorentz transformation represented by the unitary operator

$$U(p^\nu, M^{\rho\sigma}) = e^{i\alpha_\nu p_\nu} e^{\frac{i}{2}\omega_{\rho\sigma} M^{\rho\sigma}}$$

$$M^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu = \left(1 + \beta p_0^\rho p_{0\rho}\right) \acute{M}^{\mu\nu}$$

generator of the Lorentz group

we get $\acute{x}^\mu = U\, x^\mu U^{-1}$, $\acute{p}^\mu = U\, p^\mu U^{-1}$, and find that

$$[\acute{x}^\mu, \acute{p}^\nu] = i\hbar\left(1 + \beta \acute{p}^\rho \acute{p}_\rho\right)\eta^{\mu\nu} + i\hbar\beta \acute{p}^\mu \acute{p}^\nu$$

which is <u>Lorentz covariant</u>

Need for Relativistic GUP

- Non-relativistic 3d-GUP generalizes the momentum operator $\hat{p}_i = \hat{p}_{0i}\left(1+\beta\, p^2\right)$ but not length operator $\hat{x}_i = \hat{x}_{0i}$,
- Accordingly, relativistic dispersion relation is deformed

  $E^2 = \left(m\, c^2\right)^2 + (pc)^2 + \mathcal{O}[p^4]$,

  and nonlinear additional law of momenta, like $\hat{p}_3 = \hat{p}_2 + \hat{p}_1$ appears.

- In Poincare algebra, the generator of the Lorentz group

$$M^{\mu\nu} = p^{\mu} x^{\nu} - p^{\nu} x^{\mu} = \left(1 + \beta p_0^{\rho} p_{0\rho}\right) \acute{M}^{\mu\nu}$$

  where $\acute{M}^{\mu\nu} = p_0^{\mu} x_0^{\nu} - p_0^{\nu} x_0^{\mu}$, $p_{\mu} p^{\mu}$ is Casimir operator of Lorentz algebra which commutes with p and p^2; $\left[M_{\mu\nu}, p^2\right] = \left[p^2, p_{\mu}\right] = 0$.

- The operator $p_{\mu} p^{\mu}$ also commutes with other Casimirs, like $W^{\mu} W_{\mu}$ where $W_{\mu} \coloneqq \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} p^{\lambda}$, so that $\left[p_{\mu}, W_{\mu}\right] = [p, W^{\nu} W_{\nu}] = 0$.

- This leads to $p_{\mu} p^{\mu} = -c^4 m^2$, squared dispersion relation and thus <u>fulfilling linearity of additional law of momenta</u>.

Need for Relativistic GUP

- To assure that Δx_{min} is Lorentz invariant, we start with the spacetime noncommutativity

$$[x^\mu, x^\nu] = -2i\hbar(x^\mu p^\nu - x^\nu p^\mu)$$

and the length-momentum noncommutativity

$$[x^\mu, p^\nu] = (x^\mu p^\nu - p^\nu x^\mu) = i\hbar(\eta^{\mu\nu} + 2\beta p^\mu p^\nu)$$

- Then,

$$[x^\mu, x^\nu] = 2\,\hbar^2(\eta^{\mu\nu} + 2\beta p^\mu p^\nu) - 2i\hbar M^{\nu\mu}$$

- This means that the spacetime coordinates
  i. are likely noncommutative,
  ii. have similarities with Snyder algebra but
  iii. not forming a closed algebra (as depending on p).

Relativistic GUP

- We assume that the physical position and momentum in terms of their auxiliary ***4-vectors*** x_0^μ and p_0^μ, read

$$\hat{x}^\mu = f(\hat{x}_0^\mu, \hat{p}_0^\mu)$$

$$\hat{p}^\mu = \hat{p}_0^\mu \left(1 + \beta p_0^\rho p_{0\rho}\right)$$

where $i \in \{1,2,3\}$ and $\mu, \nu \in \{0,1,2,3\}$, ρ is a dummy index,

$$x_0^0 = ct,$$

$$p_0^0 = E/c$$

are parameters, and $\hat{x}_0^\mu$, $\hat{p}_0^\mu$ are canonically conjugate variables, $\left[\hat{x}_0^\mu, \hat{p}_0^\nu\right] = i\hbar\eta^{\mu\nu}$. Both quantities are depending on $\hat{x}_0^\mu, \hat{p}_0^\nu$

- Under isotropic condition and with relativity principle, we ger that the relativistic generalized uncertainty principle is given as

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar\left[\left(1 + \beta \hat{p}_0^\rho \hat{p}_{0\rho}\right)\eta^{\mu\nu} + 2\beta \hat{p}^\mu \hat{p}^\nu\right]$$

- From Robertson uncertainty principle which follows from Schrödinger uncertainty principle,

$$\Delta x^\mu \Delta p^\nu \geq \frac{1}{2}\left|\langle[\hat{x}^\mu, \hat{p}^\nu]\rangle\right|$$

Relativistic GUP

- The relativistic generalized length-momentum uncertainties in curved spacetime, $\eta^{\mu\nu} \rightarrow g^{\mu\nu}$, are given as

$$\Delta x^{\mu} \Delta p^{\nu} \geq \frac{\hbar}{2}\left[g^{\mu\nu} + \beta(\Delta p)^2 + \beta\langle p\rangle^2 - \beta(\Delta p^{\mu})^2 - \beta(\Delta p^{\nu})^2\right]$$

In this regard, with RGUP in curved spacetime, Jacobi identity is well guaranteed.

- To have real roots for Δp^{ν}, it is required that

$$(\Delta x^{\mu})^2 \geq \hbar^2\left[\beta^2\left(\langle p\rangle^2 - (\Delta p^{\mu})^2 - (\Delta p^{\nu})^2\right) - \beta g^{\mu\nu}\right]$$

- This leads to a minimum measurable length, which is frame (coordinate) independent

$$\Delta x^{\mu}_{min} \geq \pm\sqrt{-g^{\mu\nu}}\,\hbar\sqrt{\beta} = \pm\sqrt{-|g|}\sqrt{\beta_0}\,\ell_p$$

Relativistic GUP

- Then, the minimum measurable length, which is frame (coordinate) independent

$$\Delta x^{\mu}_{min} \geq \pm\sqrt{-g^{\mu\nu}}\hbar\sqrt{\beta} = \pm\sqrt{-|g|}\sqrt{\beta_0}\ell_p$$

- The result that the minimum measurable length Δx^{μ}_{min} is given in terms of $\sqrt{-det\, g}$ is interesting, as in GR no physical dimensions are assigned to the coordinates. They are fundamentally arbitrary.

- Jacobian determinant J and $\sqrt{-det\, g}$ (sign) assure invariant transformation from one system of coordinates to another, so that

$$\sqrt{-det\, g}\,\Delta x^{\mu}_{min} = \Delta \acute{x}^{\mu}_{min}$$

is frame independent (Lorentz invariant).

Now, we can determine the uncertainties in GR:

- For length

$$x^{\mu} = x^{\mu}_0 - \Delta x^{\mu}_{min} = \left(\left(x^0_0 - |\Delta x_{min}|\right), \left(x^i_0 - |\Delta x_{min}|\right)\right)$$

then

$$\Delta x^{\mu} = \Delta x^{\mu}_0 - 2\sqrt{\beta_0}\ell_p$$

Relativistic GUP

- For momentum, which follows from roots of

$$\Delta x^{\mu}\Delta p^{\nu} \geq \frac{\hbar}{2}\left[g^{\mu\nu} + \beta(\Delta p)^2 + \beta\langle p\rangle^2 - \beta(\Delta p^{\mu})^2 - \beta(\Delta p^{\nu})^2\right]$$

Then

$$\Delta p^{\nu} \leq \frac{1}{\hbar\beta}\left[\Delta x^{\mu} \pm \left((\Delta x^{\mu})^2 - \beta\hbar^2 g^{\mu\nu}\right)^{1/2}\right]$$

$$\Delta p^{\nu} \leq \frac{1}{\hbar\beta}\left[\Delta x^{\mu} \pm \left(\Delta x^{\mu} + \frac{1}{2}\frac{\left(\Delta x^{\mu}_{min}\right)^2}{\Delta x^{\mu}}\right)\right]$$

- This suggests that

$$\frac{\Delta x^{\mu}}{\hbar\beta}\frac{1}{2}\left(\frac{\Delta x^{\mu}_{min}}{\Delta x^{\mu}}\right)^2 \leq \Delta p^{\nu} \leq 2\,\frac{\Delta x^{\mu}}{\hbar\beta}\left[1 + \frac{1}{4}\left(\frac{\Delta x^{\mu}_{min}}{\Delta x^{\mu}}\right)^2\right]$$

$$\Delta p^{\nu} \leq \frac{2}{\hbar\beta}\Delta x^{\mu}$$

$$\Delta p^{\nu}_0 \leq \frac{2}{\hbar\beta}\frac{\Delta x^{\mu}_0 - 2\sqrt{\beta_0}\ell_p}{1 + \beta g^{0\rho}\Delta|p_0|^2}$$

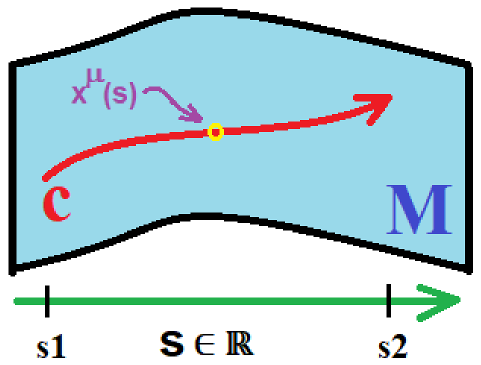
Relativistic GUP

If these results were correct, at relativistic energy scale, we conclude that

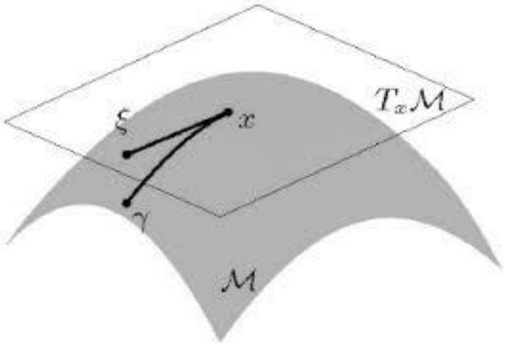
- **the spacetime is neither smooth nor continuous as an inaccessible spacetime element whose volume characterized by Δx^{μ}_{min} exists,**
- **not only coordinates and momenta are uncertain, but other physical quantities have noncommutative relations, as well,**
- **due uncertainties, events likely happen in jumps with nondeterministic outcomes,**
- **the measurements are likely neither precise nor noncoherent.**

Generalized Manifold (Finsler)

- Generalization of $g^{\mu\nu}$ would be possible on generalized Riemann manifold M:



- Riemann geometry (M,g): at point x:
  - metric tensor $g = g^{\mu\nu}(x)dx^{\mu} \otimes dx^{\nu}$,
  - length of curve **c**, $\int_c \sqrt{g^{\mu\nu}(x)dx^{\mu}dx^{\nu}}$ or $\int_{s_1}^{s_2} \sqrt{g^{\mu\nu}(s)d\dot{x}^{\mu}d\dot{x}^{\nu}}ds$, where $\dot{x} = \frac{dx}{ds}$



- Finsler geometry (M,F): at x on ***M***, the Finsler structure $F(x,\dot{x})$ is related to the generalized metric tensor

$$F = \sqrt{g^{\mu\nu}(x)d\dot{x}^{\mu}d\dot{x}^{\nu}} \text{ and } g := \frac{1}{2}\frac{\partial^2 F^2}{\partial\dot{x}^{\mu}\partial\dot{x}^{\nu}}.$$

- F is +ive for $\dot{x} \neq 0$ on tangent bundle ***TM*** & homogeneous of degree ***1*** in $\dot{x}$,
- therefore, on ***TM***, at local coordinates $(x,\dot{x})$,

$$F(x, lambda\, \dot{x}) = \lambda F(x,\dot{x}), \quad \forall \lambda \in \boldsymbol{R}^+$$

- the ratio of lengths of any two collinear vectors doesn't include metric functions.

Generalized Manifold (Finsler)

- **In Finsler geometry, the special case $F(x,\dot{x}) = \sqrt{g^{\mu\nu}(x) d\dot{x}^{\mu} d\dot{x}^{\nu}}$ distinguishes Finsler from Riemann geometry; a relaxation of quadratic restriction** *Notices Amer. Math. Soc. 43(1996) 95*.
- **The length of a curve *c* is given as $\int_c F(x,\dot{x})$ or $\int_{s1}^{s2} F\left(x^{\mu}(s), \dot{x}^{\mu}(s)\right) ds$.**
- **If Δx^{μ}_{min} sets limitations on the space and momentum coordinates in GR and determines their uncertainties, their measurements are likely no longer arbitrary precise or noncoherent. Thus, we assume that**

$$F\left(x, \Delta x^{\mu}_{min} \dot{x}\right) = \Delta x^{\mu}_{min} F(x,\dot{x}), \quad \forall\, \Delta x^{\mu}_{min} \geq 0.$$

- **On *TM*, the metric tensor given as $g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu}$ could be determined by the Hessian in the $(x,\dot{x})$-coordinates,**

$$g_{AB} = \frac{1}{2} \frac{\partial^2 F^2\left(x, \Delta x^{\mu}_{min} \dot{x}\right)}{\partial \dot{x}^{\alpha} \partial \dot{x}^{\beta}}$$

where each $g_{\mu\nu}$ is homogeneous of degree *0* in $\dot{x}$.

Quantized Metric Tensor

This leads to

$$\tilde{g}_{\mu\nu} = g_{AB}\frac{\partial x^A}{\partial \xi^\alpha}\frac{\partial x^B}{\partial \xi^\beta}$$

where $A, B \in \{0, 1, 2, \dots, 7\}$, and $\alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}$. Then,

$$\tilde{g}_{\mu\nu} = \left[1 + \left(-|g|\hbar^2\beta|\ddot{x}|^2\right)\right]g_{\mu\nu} = \left[1 + \left(-|g|\beta_0\ell_p^2|\ddot{x}|^2\right)\right]g_{\mu\nu}$$

where $|\ddot{x}|^2 = \ddot{x}^\lambda\ddot{x}_\lambda = g_{\gamma\delta}\ddot{x}^\delta\ddot{x}^\gamma$ and λ, γ, δ are dummy indices.

To summarize, for quadratic ***F***, $g_{\mu\nu}$ reduces to $g_{\mu\nu}(x)$ living on ***M***.

- On ***TM***, coordinates are 8d, $x^A = (x^\alpha,\ \Delta x^\mu_{min}\dot{x}^\alpha)$,
- On ***TM***, the line element is given as $d\tilde{s}^2 = g_{AB}dx^A dx^B$,
- On ***M***, 4d-manifold, the line element is $d\tilde{s}^2 = \tilde{g}_{\mu\nu}d\xi^\mu d\xi^\nu$.

Quantized Metric Tensor

- On the Riemann manifold ***M***,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \left(-|g|\beta_0 \ell_p^2\right)|\ddot{x}|^2 \, g_{\mu\nu} = \left(1 + \mathcal{T}|\ddot{x}|^2\right) g_{\mu\nu}$$

  where $\left(-|g|\beta_0 \ell_p^2\right)|\ddot{x}|^2 \, g_{\mu\nu} = \left(1 + \mathcal{T}|\ddot{x}|^2\right) g_{\mu\nu}$ could be seen as local perturbations to the curved spacetime.

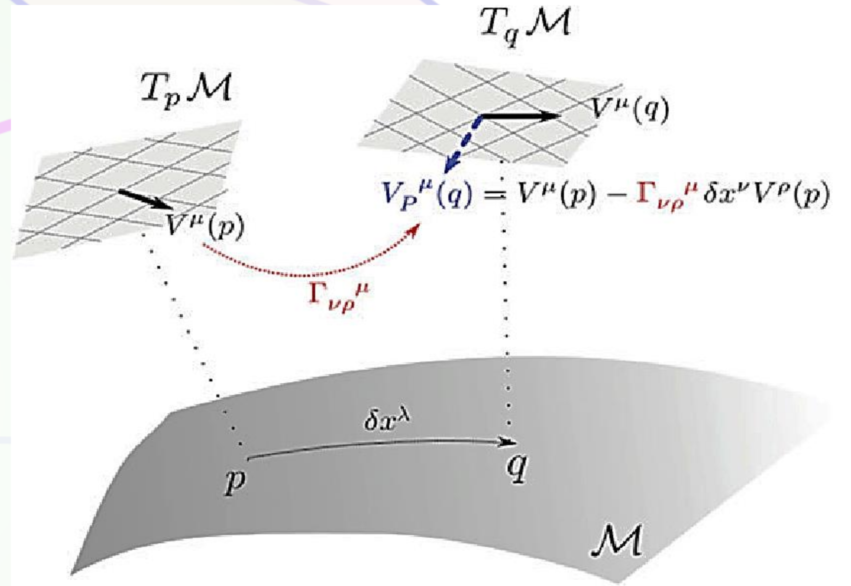
- Quantization of the local perturbations is possible through:
  - coordinate-independent discretization $\left(-|g|\beta_0 \ell_p^2\right)$, and
  - second-order derivatives of the coordinates $|\ddot{x}|^2$.

- $|\ddot{x}|^2 = \ddot{x}^\lambda \ddot{x}_\lambda = \frac{\partial \dot{x}^\lambda}{\partial \xi^\lambda} \frac{\partial \dot{x}_\lambda}{\partial \xi_\lambda}$ could be treated as
  - spacelike four-acceleration, or
  - local geodesic equation.

- To count for consequences of Δx^μ_{min}, tangent bundle ***TM*** and Finsler manifold with quadratic ***F*** are assumed, on which $g_{\mu\nu}$ reduces to the usual $g_{\mu\nu}(x)$ living on ***M***.

Affine Connection

- a geometric object connecting nearby tangent (curved) spaces, i.e., permitting differentiability of the tangent vector fields or assuring them restrict dependence on manifold in a fixed vector space,
- a function assigning to each tangent vector and each vector field a covariant derivative or a new tangent vector.



In differential geometry, the generic form of AC reads

$$\Gamma^{\mu}_{\lambda\nu} = \left\{{}^{\mu}_{\lambda\nu}\right\} + K^{\mu}_{\lambda\nu} + \frac{1}{2}\left(Q^{\mu}_{\lambda\nu.} + Q^{\mu}_{\nu\lambda.} - Q^{\mu}_{.\nu\lambda}\right)$$

Christoffel symbol

$$\left\{{}^{\mu}_{\lambda\nu}\right\}$$

Levi–Civita connection

$$K^{\mu}_{\lambda\nu} = \frac{1}{2}\left(T^{\mu}_{.\lambda\,\nu} - T^{\mu}_{\lambda.\nu} - T^{\mu}_{\nu.\lambda}\right)$$

Covariant derivative of metric

$$Q_{\mu\nu\lambda} = -D_{\mu}(\Gamma)g_{\nu\lambda}$$

$$T^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\lambda\nu} - \Gamma^{\mu}_{\nu\lambda} = 2\Gamma^{\mu}_{[\lambda\nu]}$$ Torsion

Affine Connection

GR assumes torsion-free and metric compatibility.
The latter implies linear independence of partial derivative tangent vectors and a flat space that can be found locally in a suitable frame (like Mikowski space),

$$\Gamma^{\mu}_{\lambda\nu} = \left\{{\mu \atop \lambda\nu}\right\} + K^{\mu}_{\lambda\nu} + \frac{1}{2}\left(Q^{\mu}_{\lambda\nu.} + Q^{\mu}_{\nu\lambda.} - Q^{\mu}_{.\nu\lambda}\right)$$

Christoffel symbol

$$\left\{{\mu \atop \lambda\nu}\right\}$$

Levi–Civita connection

$$K^{\mu}_{\lambda\nu} = \frac{1}{2}\left(T^{\mu}_{.\lambda\,\nu} - T^{\mu}_{\lambda.\nu} - T^{\mu}_{\nu.\lambda}\right)$$

Covariant derivative of metric

$$Q_{\mu\nu\lambda} = -D_{\mu}(\Gamma)g_{\nu\lambda}$$

Torsion

$$T^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\lambda\nu} - \Gamma^{\mu}_{\nu\lambda} = 2\Gamma^{\mu}_{[\lambda\nu]}$$

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right)$$

Affine Connection/Parallel Transport

With $\tilde{g}_{\mu\nu} = g_{\mu\nu} + (-|g|\beta_0 \ell_p^2)|\ddot{x}|^2\, g_{\mu\nu}$ or $\tilde{g}_{\mu\nu} \;=\; \left(1 + \mathscr{T}\,|\ddot{x}|^2\right)\, g_{\mu\nu}$

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \qquad \tilde{\Gamma}^{\gamma}_{\beta\mu} = \frac{1}{2}\tilde{g}^{\alpha\gamma}(\tilde{g}_{\alpha\beta,\mu} + \tilde{g}_{\alpha\mu,\beta} - \tilde{g}_{\beta\mu,\alpha})$$

$$\tilde{\Gamma}^{\gamma}_{\beta\mu} \;=\; \frac{1+2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}\,\frac{1}{2}\, g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right) = \frac{1+2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}\Gamma^{\gamma}_{\beta\mu}$$

Quantum-induced corrections are linearly factorized to classical AC.

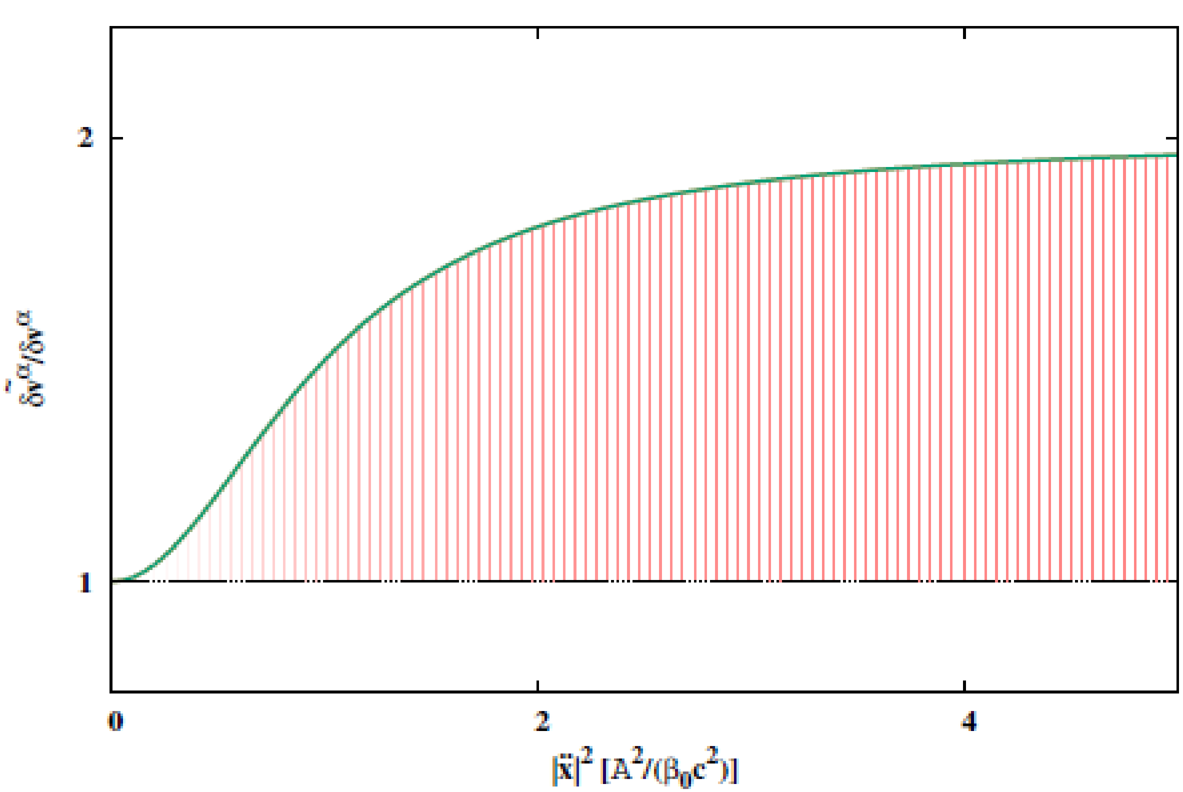
- **The covariant derivatives are rates of change of tangent vector fields with normal component subtracted, i.e., parallel transport.**
- **Vanishing covariant derivatives of a vector means that the vector is parallel transported, i.e., keeping it as constant as possible**

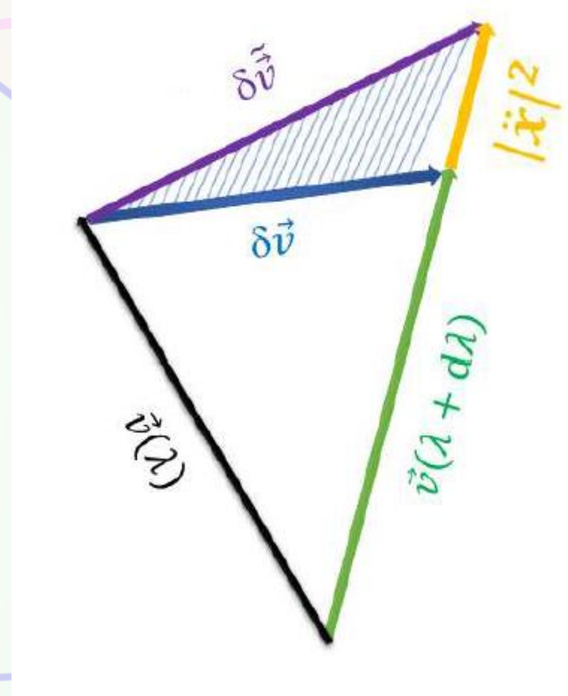
$$\frac{\partial}{\partial\lambda} v^{\alpha} \;=\; -v^{\rho}\,\Gamma^{\alpha}_{\sigma\rho} \qquad \frac{\partial}{\partial\lambda}\tilde{v}^{\alpha} \;=\; -\frac{1}{1+\mathscr{T}|\ddot{x}|^2}\, v^{\rho}\,\Gamma^{\alpha}_{\sigma\rho} - \frac{2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}\, v^{\rho}\,\Gamma^{\alpha}_{\sigma\rho}$$

$$\vec{v}(\lambda + d\lambda) \;\simeq\; \vec{v}(\lambda) - \left(\frac{1}{1+\mathscr{T}|\ddot{x}|^2} + \frac{2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}\right) v^{\rho}\,\Gamma^{\alpha}_{\sigma\rho}\, d\lambda\, e_{\alpha}$$

Quantum-induced corrections are linearly factorized to classical AC.

Affine Connection/Parallel Transport





Quantum-induced corrections increase with $|\ddot{x}|^2$

Riemann Curvature Tensor

The coefficients of the Riemann curvature tensor can be constructed from the Levi-Civita connections

$$R^{\gamma}_{\beta\mu\nu} = \Gamma^{\gamma}_{\beta\nu,\mu} - \Gamma^{\gamma}_{\beta\mu,\nu} + \Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

This expression holds for all connections regardless their metric compatibility or torsion-free property,

$$\tilde{R}^{\gamma}_{\beta\mu\nu} = \tilde{\Gamma}^{\gamma}_{\beta\nu,\mu} - \tilde{\Gamma}^{\gamma}_{\beta\mu,\nu} + \tilde{\Gamma}^{\gamma}_{\sigma\mu}\tilde{\Gamma}^{\sigma}_{\beta\nu} - \tilde{\Gamma}^{\gamma}_{\sigma\nu}\tilde{\Gamma}^{\sigma}_{\beta\mu}$$

$$\begin{aligned}
\tilde{\Gamma}^{\gamma}_{\sigma\mu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\beta|\ddot{x}|^2}\,\frac{g^{\gamma\alpha}}{2}\,(g_{\mu\alpha,\sigma}+g_{\sigma\alpha,\mu}-g_{\sigma\mu,\alpha}) = \frac{1+2\beta|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\,\Gamma^{\gamma}_{\sigma\mu},\\
\tilde{\Gamma}^{\gamma}_{\sigma\nu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\,\frac{g^{\gamma\alpha}}{2}\,(g_{\nu\alpha,\sigma}+g_{\sigma\alpha,\nu}-g_{\sigma\nu,\alpha}) = \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\,\Gamma^{\gamma}_{\sigma\nu},\\
\tilde{\Gamma}^{\sigma}_{\beta\mu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\,\frac{g^{\sigma\alpha}}{2}\,(g_{\mu\alpha,\beta}+g_{\beta\alpha,\mu}-g_{\beta\mu,\alpha}) = \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\,\Gamma^{\sigma}_{\beta\mu},\\
\tilde{\Gamma}^{\sigma}_{\beta\nu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\,\frac{g^{\sigma\alpha}}{2}\,(g_{\nu\alpha,\beta}+g_{\beta\alpha,\nu}-g_{\beta\nu,\alpha}) = \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\,\Gamma^{\sigma}_{\beta\nu}.
\end{aligned}$$

Riemann Curvature Tensor

Thus, we can straightforwardly derive

$$\begin{aligned}\tilde{\Gamma}^{\gamma}_{\beta\mu,\nu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\left[\frac{g^{\gamma\alpha}_{,\nu}}{2}\left(g_{\mu\alpha,\beta}+g_{\beta\alpha,\mu}-g_{\beta\mu,\alpha}\right)+\frac{g^{\gamma\alpha}}{2}\left(g_{\mu\alpha,\beta,\nu}+g_{\beta\alpha,\mu,\nu}-g_{\beta\mu,\alpha,\nu}\right)\right]\\ &+ \frac{2\left(1+\mathscr{T}\,|\ddot{x}|^2\right)-\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\mathscr{T}\,\left(g_{ij}\ddot{x}^i\ddot{x}^j\right)_{,\nu}\Gamma^{\gamma}_{\beta\mu},\\ &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\Gamma^{\gamma}_{\beta\mu,\nu}+\frac{\mathscr{T}\,g_{ij,\nu}\ddot{x}^i\ddot{x}^j}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\Gamma^{\gamma}_{\beta\mu},\end{aligned}$$

and similarly,

$$\begin{aligned}\tilde{\Gamma}^{\gamma}_{\beta\nu,\mu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\left[\frac{g^{\gamma\alpha}_{,\mu}}{2}\left(g_{\nu\alpha,\beta}+g_{\beta\alpha,\nu}-g_{\beta\nu,\alpha}\right)+\frac{g^{\gamma\alpha}}{2}\left(g_{\nu\alpha,\beta,\mu}+g_{\beta\alpha,\nu,\mu}-g_{\beta\nu,\alpha,\mu}\right)\right]\\ &+ \frac{2\left(1+\mathscr{T}\,|\ddot{x}|^2\right)-\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\mathscr{T}\,\left(g_{ij}\ddot{x}^i\ddot{x}^j\right)_{,\mu}\Gamma^{\gamma}_{\beta\nu},\\ &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\Gamma^{\gamma}_{\beta\nu,\mu}+\frac{\mathscr{T}\,g_{ij,\mu}\ddot{x}^i\ddot{x}^j}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\Gamma^{\gamma}_{\beta\nu}.\end{aligned}$$

$$\begin{aligned}\tilde{R}^{\gamma}_{\beta\mu\nu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}R^{\gamma}_{\beta\mu\nu}\\ &+ \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left(\Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}-\Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right)+\frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left[\Gamma^{\lambda}_{\lambda\mu}\Gamma^{\gamma}_{\beta\nu}-\Gamma^{\lambda}_{\lambda\nu}\Gamma^{\gamma}_{\beta\mu}\right]\end{aligned}$$

Riemann Curvature Tensor

The eminent ingredients added by the quantization $\mathscr{T}\,|\ddot{x}|^2$ can be illustrated in an example of 2–sphere with radius $r = 1$. The complementary geometric structure likely reveals supplementary insights to be discovered. The Cartesian coordinates could be expressed by polar coordinates; radius r, inclination u^1, and azimuth u^2 as $x = \cos(u^2)\sin(u^1)$, $y = \sin(u^2)\sin(u^1)$, and $z = \cos(u^1)$.

Therefore, we just need to determine the components of the unquantized metric tensor

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & (\sin(u^1))^2 \end{bmatrix},$$

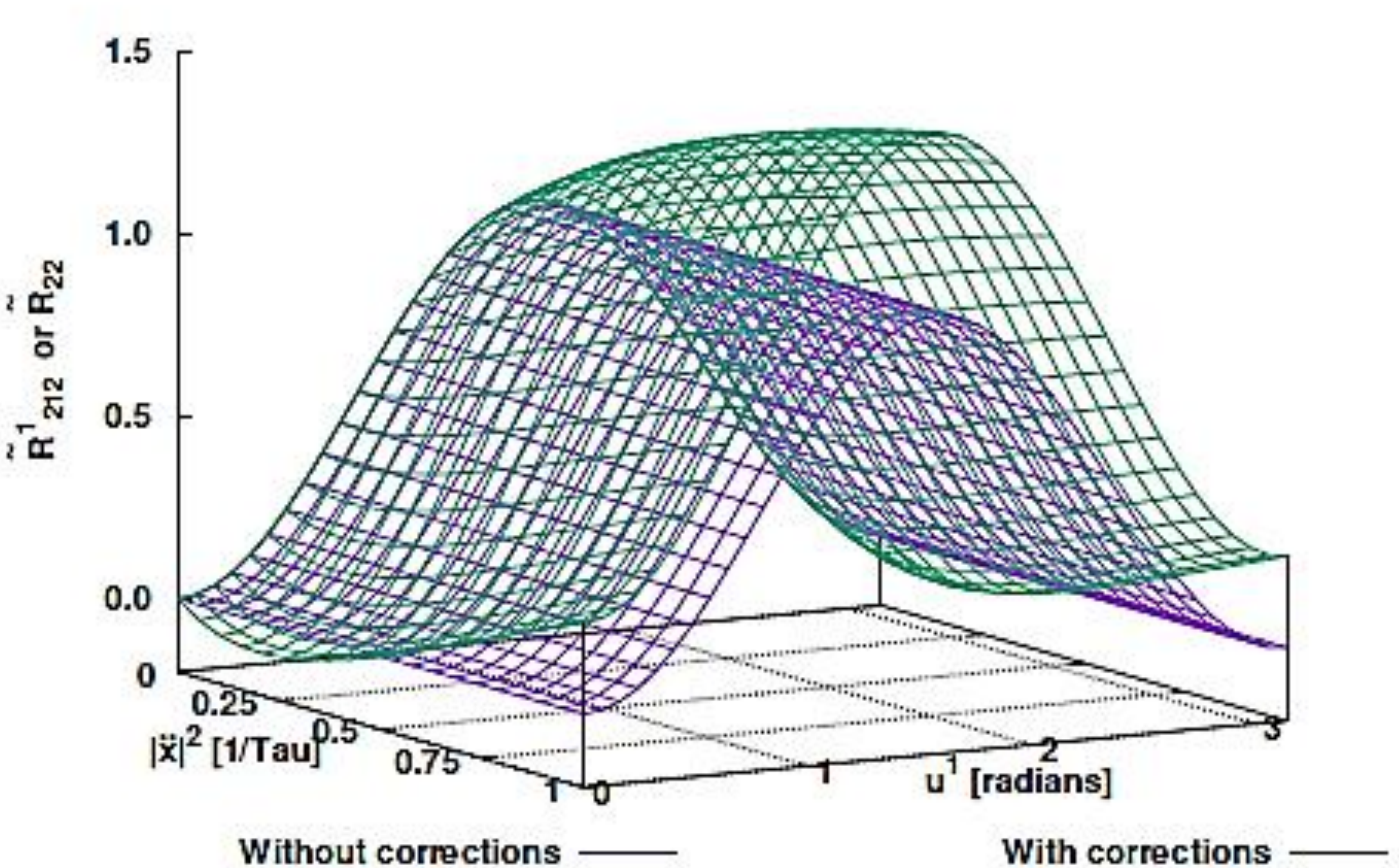
and coefficients of the unquantized c

$$\begin{aligned} \Gamma^1_{11} &= 0, & \Gamma^2_{11} &= 0, \\ \Gamma^1_{12} &= 0, & \Gamma^2_{12} &= \cot(u^1), \\ \Gamma^1_{21} &= 0, & \Gamma^2_{21} &= \cot(u^1), \\ \Gamma^1_{22} &= \frac{1}{2}\sin(2u^1), & \Gamma^2_{22} &= 0. \end{aligned}$$

Then, the corresponding coefficients of Riemann curvature tensor are given as

$$\tilde{R}^1_{212} = \frac{1 + 2\mathscr{T}\,|\ddot{x}|^2}{1 + \mathscr{T}\,|\ddot{x}|^2}\left[\sin(u^1)\right]^2 + \frac{\left(2\mathscr{T}\,|\ddot{x}|^2 - 1\right)\mathscr{T}\,|\ddot{x}|^2}{(1 + \mathscr{T}\,|\ddot{x}|^2)^2}\left[\cos(u^1)\right]^2$$

Riemann Curvature Tensor



Ricci Curvature Tensor

While the Riemann curvature tensor confirms whether a vector is twisted when it is parallel transported around a small loop in curved space, the Ricci curvature tensor, which is the only possible contraction of Riemann tensor, tracks the volume change along the geodesics, and therefore represents how quickly a volume is changing along the geodesics. Hence, Ricci curvature tensor represents gravity in the general theory of relativity.

$$g^{\mu}_{\gamma} R^{\gamma}_{\beta\mu\nu} = R^{\mu}_{\beta\mu\nu} = R_{\beta\nu}$$

$$\tilde{R}_{\beta\nu} = \tilde{\Gamma}^{\mu}_{\beta\nu,\mu} - \tilde{\Gamma}^{\mu}_{\beta\mu,\nu} + \tilde{\Gamma}^{\mu}_{\sigma\mu}\tilde{\Gamma}^{\sigma}_{\beta\nu} - \tilde{\Gamma}^{\mu}_{\sigma\nu}\tilde{\Gamma}^{\sigma}_{\beta\mu}$$

$$\begin{aligned} \tilde{R}_{\beta\nu} &= \frac{1 + 2\mathscr{T}\,|\ddot{x}|^2}{1 + \mathscr{T}\,|\ddot{x}|^2} R_{\beta\nu} \\ &+ \frac{\left(1 + 2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1 + \mathscr{T}\,|\ddot{x}|^2\right)^2} \left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right) + \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1 + \mathscr{T}\,|\ddot{x}|^2\right)^2} \left[\Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{\beta\nu} - \Gamma^{\lambda}_{\lambda\nu}\Gamma^{\mu}_{\beta\mu}\right] \end{aligned}$$

Ricci Scalar

The Ricci scalar gives how the volume in curved space deviates from its equivalent flat–space size and can be contracted from Ricci curvature tensor

$$g^{\beta\nu} g^{\gamma\mu} R_{\gamma\beta\mu\nu} \;=\; g^{\beta\nu} R_{\beta\nu} = R^{\nu}_{\nu} = R.$$

Therefore, the quantized Ricci scalar apparently reads

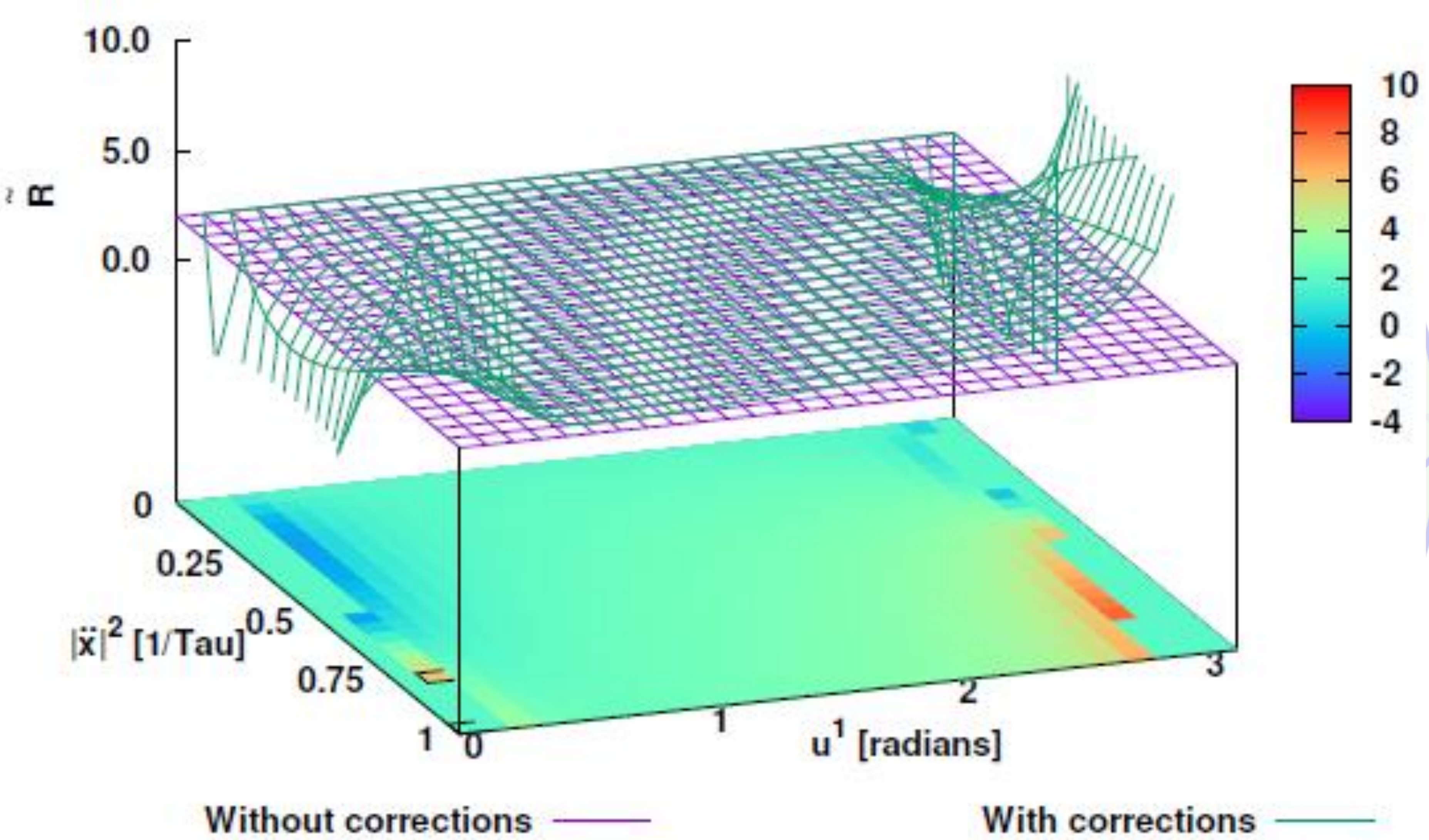
$$\tilde{R} \;=\; \tilde{g}^{\beta\nu}\, \tilde{R}_{\beta\nu}.$$

$$\begin{aligned}\tilde{R} \;=\; & \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2} R \\ & + \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{(1+\mathscr{T}\,|\ddot{x}|^2)^3} g^{\beta\nu}\left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right) + \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^3} g^{\beta\nu}\left[\Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{\beta\nu} - \Gamma^{\lambda}_{\lambda\nu}\Gamma^{\mu}_{\beta\mu}\right]\end{aligned}$$

For 2–sphere with unit radius, the quantized Ricci scalar is evaluated as

$$\begin{aligned}\tilde{R} \;=\; & 2\frac{1+2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2} \\ & + \frac{2\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left\{\left[\cot(u^1)\right]^2 - \cot(u^1)\right\} - \frac{4\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^3}\left[\cot(u^1)\right]^2\end{aligned}$$

Ricci Scalar



Einstein Tensor

The Einstein tensor is constructed as

$$G_{\beta\nu} \;=\; R_{\beta\nu} - \frac{1}{2}\, g_{\beta\nu}\, R.$$

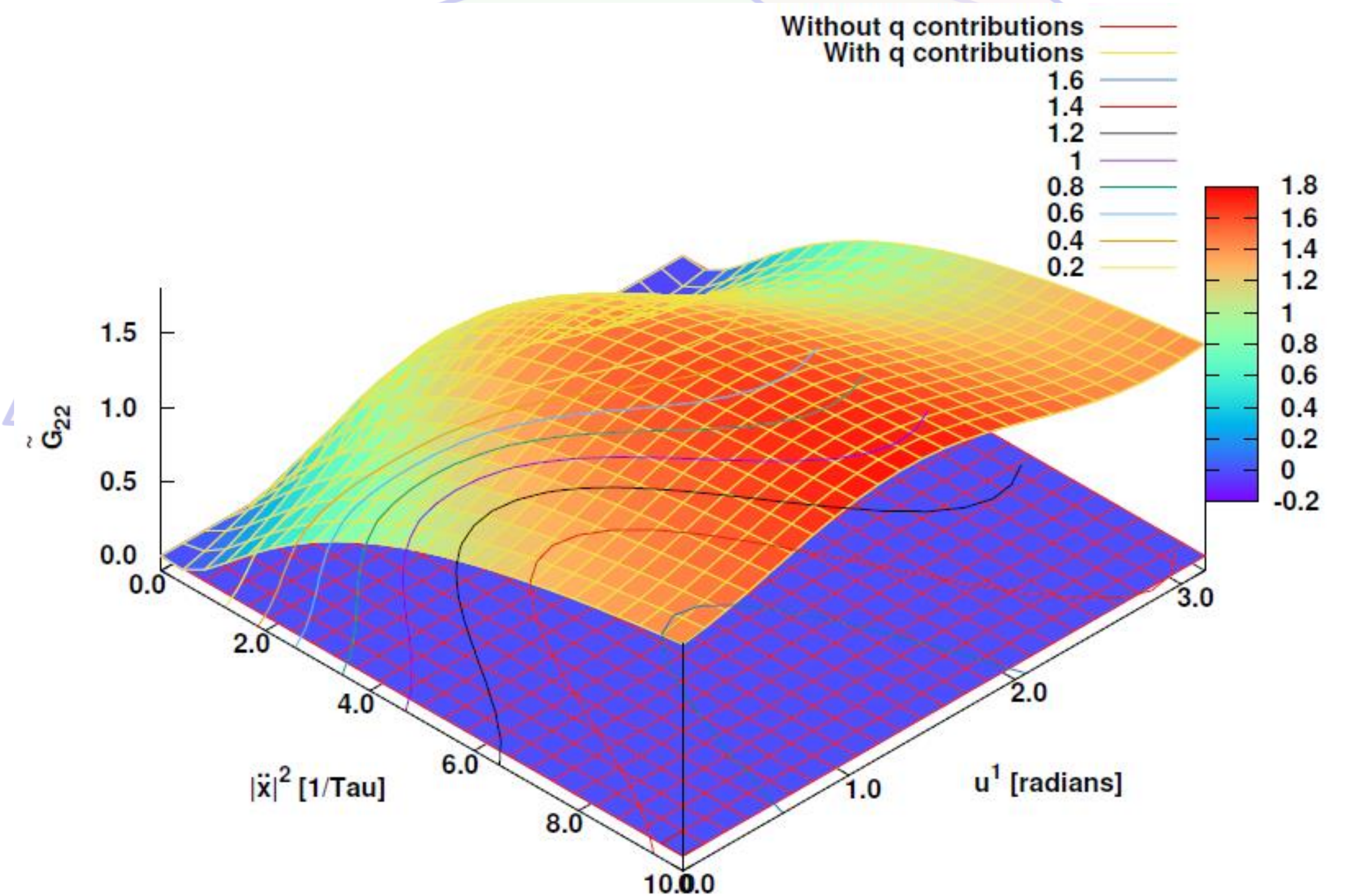
Accordingly, $\tilde{R}_{\mu\nu}$, $\tilde{g}_{\mu\nu}$, and $\tilde{R}$ result in a quantized version of the Einstein tensor

$$\tilde{G}_{\beta\nu} \;=\; \tilde{R}_{\beta\nu} - \frac{1}{2}\tilde{g}_{\beta\nu}\, \tilde{R}.$$

$$\begin{aligned}
\tilde{G}_{\beta\nu} \;&=\; \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2} G_{\beta\nu} \\
&+\; \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left[\left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right) - \frac{1}{2} g_{\beta\nu} g^{\alpha\rho}\left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\alpha\rho} - \Gamma^{\mu}_{\sigma\rho}\Gamma^{\sigma}_{\alpha\mu}\right)\right] \\
&+\; \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left[\left(\Gamma^{\mu}_{\lambda\mu}\Gamma^{\mu}_{\beta\nu} - \Gamma^{\lambda}_{\lambda\nu}\Gamma^{\mu}_{\beta\mu}\right) - \frac{1}{2} g_{\beta\nu} g^{\alpha\rho}\left(\Gamma^{\mu}_{\lambda\mu}\Gamma^{\mu}_{\alpha\rho} - \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\mu}_{\alpha\mu}\right)\right].
\end{aligned}$$

$$\begin{aligned}
\tilde{G}_{22} \;&=\; \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}\sin^2\left(u^1\right) - \frac{\left(1-2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)^2}\cos^2\left(u^1\right) - \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)^2}\sin^2\left(u^1\right) \\
&-\; \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)^3}\left[\cos^2\left(u^1\right) - \cos\left(u^1\right)\sin\left(u^1\right)\right] + \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)^3}\cos^2\left(u^1\right).
\end{aligned}$$

Einstein Tensor



Einstein Tensor

In freely falling frames, in which the covariant derivatives are the same for all observers, $\nabla^\beta G_{\beta\nu}$ and Levi–Civita connections $\Gamma^\alpha_{\mu\nu}$ and $\tilde{\Gamma}^\alpha_{\mu\nu}$ all vanish so that

$$\nabla^\beta \tilde{G}_{\beta\nu} = \nabla^\beta \left(\frac{1 + 2\mathscr{T}\, |\ddot{x}|^2}{1 + \mathscr{T}\, |\ddot{x}|^2} G_{\beta\nu} \right) = \left[\frac{\mathscr{T}\, \ddot{x}^i \ddot{x}^j}{(1 + \mathscr{T}\, |\ddot{x}|^2)^2} G_{\beta\nu} \right] \eta_{ij,\beta}.$$

Because $g_{ij,\beta} = \eta_{ij,\beta} = 0$, Eq. (54) vanishes as well. This leads to

$$\nabla^\beta \tilde{G}_{\beta\nu} = 0,$$

i.e., the divergence of both quantized and unquantized Einstein tensors vanish,

$$\nabla^\beta \tilde{G}_{\beta\nu} = \nabla^\beta G_{\beta\nu} = 0.$$

Summary

RGUP

$$\hat{x}^{\mu} = \hat{x}_0^{\mu} = \left(x_0^0, x_0^i\right) \quad \text{and} \quad \hat{p}^{\mu} = \hat{p}_0^{\mu}\left(1 + \beta p_0^{\rho} p_{0\rho}\right) = \left(p_0^0, p_0^i\right)\left(1 + \beta p_0^{\rho} p_{0\rho}\right),$$

$$[\hat{x}^{\mu}, \hat{p}^{\nu}] = i\hbar\left[\left(1 + \beta \hat{p}_0^{\rho} \hat{p}_{0\rho}\right)\eta^{\mu\nu} + 2\beta \hat{p}^{\mu} \hat{p}^{\nu}\right],$$

$$\Delta x^{\mu} \Delta p^{\nu} \geq \frac{\hbar}{2}\left[g^{\mu\nu} + \beta(\Delta p)^2 + \beta\langle p\rangle^2 - \beta(\Delta p^{\mu})^2 - \beta(\Delta p^{\nu})^2\right],$$

$$\Delta x_{min}^{\mu} \geq \pm\sqrt{-g^{\mu\nu}}\hbar\sqrt{\beta} = \pm\sqrt{|g|}\sqrt{\beta_0}\ell_p \quad \text{and} \quad \Delta x^{\mu} = \Delta x_0^{\mu} - 2\sqrt{\beta_0}\ell_p,$$

$$\Delta p^{\nu} \leq \frac{1}{\hbar\beta}\left[\Delta x^{\mu} \pm \left(\Delta x^{\mu} + \frac{1}{2}\frac{\left(\Delta x_{min}^{\mu}\right)^2}{\Delta x^{\mu}}\right)\right] \quad \text{or}$$

$$\Delta p^{\nu} \leq \frac{2}{\hbar\beta}\Delta x^{\mu} \quad \text{and} \quad \Delta p_0^{\nu} \leq \frac{2}{\hbar\beta}\frac{\Delta x_0^{\mu} - 2\sqrt{\beta_0}\ell_p}{1 + \beta g^{0\rho}\Delta|p_0|^2}.$$

Metric tensor

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \left(-|g|\beta_0\ell_p^2\right)|\ddot{x}|^2\, g_{\mu\nu} \qquad \text{where } |\ddot{x}|^2 = \ddot{x}^{\lambda}\ddot{x}_{\lambda} = \frac{\partial \dot{x}^{\lambda}}{\partial \xi^{\lambda}}\frac{\partial \dot{x}_{\lambda}}{\partial \xi_{\lambda}}$$

Quantum-induced corrections are linearly added so that vanishing corrections retrieve GR metric tensor $g_{\mu\nu}$.

Summary

Riemann Curvature Tensor

$$\tilde{R}^{\gamma}_{\beta\mu\nu} = \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2} R^{\gamma}_{\beta\mu\nu} + \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left(\Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right) + \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left[\Gamma^{\lambda}_{\lambda\mu}\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\lambda}_{\lambda\nu}\Gamma^{\gamma}_{\beta\mu}\right]$$

Ricci Curvature Tensor

$$\tilde{R}_{\beta\nu} = \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2} R_{\beta\nu} + \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right) + \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left[\Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{\beta\nu} - \Gamma^{\lambda}_{\lambda\nu}\Gamma^{\mu}_{\beta\mu}\right]$$

Ricci Scalar

$$\tilde{R} = \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2} R + \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^3} g^{\beta\nu}\left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right) + \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^3} g^{\beta\nu}\left[\Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{\beta\nu} - \Gamma^{\lambda}_{\lambda\nu}\Gamma^{\mu}_{\beta\mu}\right]$$

Summary

Einstein Tensor

$$
\begin{aligned}
\tilde{G}_{\beta\nu} &= \frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2} G_{\beta\nu} \\
&+ \frac{\left(1+2\mathscr{T}\,|\ddot{x}|^2\right)\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left[\left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}-\Gamma^{\mu}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}\right)-\frac{1}{2}g_{\beta\nu}g^{\alpha\rho}\left(\Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\alpha\rho}-\Gamma^{\mu}_{\sigma\rho}\Gamma^{\sigma}_{\alpha\mu}\right)\right] \\
&+ \frac{2\mathscr{T}\,|\ddot{x}|^2}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}\left[\left(\Gamma^{\mu}_{\lambda\mu}\Gamma^{\mu}_{\beta\nu}-\Gamma^{\lambda}_{\lambda\nu}\Gamma^{\mu}_{\beta\mu}\right)-\frac{1}{2}g_{\beta\nu}g^{\alpha\rho}\left(\Gamma^{\mu}_{\lambda\mu}\Gamma^{\mu}_{\alpha\rho}-\Gamma^{\lambda}_{\lambda\rho}\Gamma^{\mu}_{\alpha\mu}\right)\right].
\end{aligned}
$$

Covariant Derivative

$$
\nabla^{\beta}\tilde{G}_{\beta\nu} = \nabla^{\beta}\left(\frac{1+2\mathscr{T}\,|\ddot{x}|^2}{1+\mathscr{T}\,|\ddot{x}|^2}G_{\beta\nu}\right) = \left[\frac{\mathscr{T}\,\ddot{x}^i\ddot{x}^j}{\left(1+\mathscr{T}\,|\ddot{x}|^2\right)^2}G_{\beta\nu}\right]\eta_{ij,\beta} = 0,
$$

Thank you very much!