On the quantisation of electric charge

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September 9, 2022

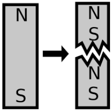
Overview

Motivation

Resolution

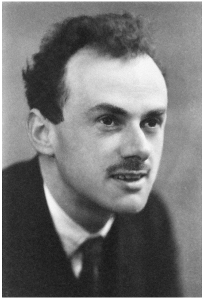
Background

Courtesy: CalTech Digital Archives



$$\vec{F} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}) + g(\vec{\mathcal{B}} - \frac{\vec{v}}{c^2} \times \vec{\mathcal{E}})$$

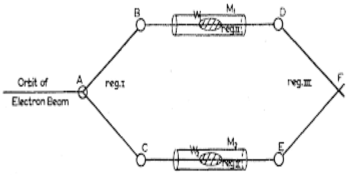
- 1911: Millikan – electric charge quantized as integral multiple of e *Phys. Rev. Vol.32(2), pp. 349-397*
- 1931: Dirac – magnetic monopoles. *RSPA, Vol. 133(821), pp.60-72*

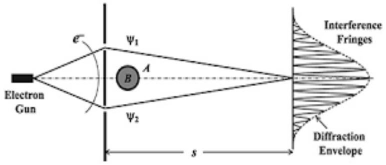


Courtesy: Nobel Foundation

MOEDAL experiment reported no observation of magnetic monopole (2021)

Aharonov & Bohm effect





- 1959: Aharonov & Bohm – Electric charges can be affected by EM field where the field is absent!
  Phys. Rev. 115 (3): pp.485–491
- 2020: ICNFP - resolution of nonlocality problem (semi-classical theory of conservative fields).
  Physica Scripta, Vol. 96 (8), 2021

– Fields can have wave functions
– As field strength → 0 these wavefunctions become non-normalizable and take the form of pure unitary phase.

* Aharonov-Casher effect
* Charge quantization: $q = \frac{n_1}{n_2} e$
* flux quantization $\phi = m\frac{h}{e}$

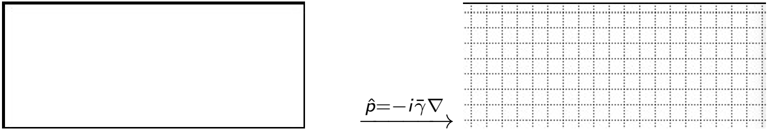
Wave-electrostatics§

$$\delta \int E\left(\frac{ds}{da}\right) da = 0$$

$$\mathcal{L}_{ES} = E(x,y,z)\sqrt{x'^2 + y'^2 + z'^2}$$

$$p_x = E_x, p_y = E_y, p_z = E_z$$

$$\mathcal{H}_{ES} = x' p_{x'} + y' p_{y'} + z' p_{z'} - \mathcal{L}_{ES} = 0$$



(a) (b)

$$\boxed{\hat{\vec{p}}\psi_E = -i\frac{\gamma}{2\pi}\nabla\psi_E = \vec{E}\psi_E}$$

Source-free region

$$\boxed{\bar{\gamma}^2\nabla^2\psi_E + \mathcal{E}^2\psi_E = 0} \xrightarrow[\text{state}]{\text{basis}} \boxed{\psi_E = \mathbf{e}^{[i\Phi/\bar{\gamma}]}} \xrightarrow[\text{normalized}]{} \boxed{\Psi_E = \int c(E)\mathbf{e}^{[i\Phi/\bar{\gamma}]}dE}$$

§K Bhattacharya, **ICNFP 2020**, Physica Scripta Vol. 96(8), 2021.

Physical example

- Change in action for charge q passing through a potential Φ over time t: $S = -q\Phi t \implies \bar{\gamma} = -\hbar/(q \cdot t)$.
- Consider the case of electron attachment in a stable Hydrogen atom:

$$\frac{e\Phi t}{\hbar} = 9 \cdot 10^{9} \frac{1.6 \cdot 10^{-19} \cdot 1.6 \cdot 10^{-19} t}{0.53 \times 10^{-10} \cdot 6.626 \cdot 10^{-34}} = 6.4 \cdot 10^{15} t$$

- For $t \sim 1 \text{ s} \gg 10^{-15}$ s, quantum effect of electrostatics is not visible.
- For $t \to 10^{-15}$ s, Bohr's model cannot hold true.
- Nascent states of atoms & chemical reactions happen over sub-femtoseconds scale (Nobel prize in Chemistry (1999). Prof. Ahmed Zewail, for developing World's fastest camera).

$\bar{\gamma}$ for electrostatic problems

- Initial interaction between a source charge distribution and a test body is essentially electrodynamic.
- Re-arrangement of charge etc. happens over relaxation time scale τ.
- Usually $\tau = RC$ for electric appliances. C is of the form $\epsilon_0 \times L$. [L is a geometric factor (e.g. $C = \frac{\epsilon_0 A}{d}$ for parallel plate capacitor)].
- Initial communication between the bodies happens via a transient displacement current through the vacuum of resistance $R_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$
- Electrostatic limit $t >> \tau = \sqrt{\frac{\mu_0}{\epsilon_0}}\epsilon_0 \times L \sim \frac{L}{c}$.
- Change in action S of a charge q over time τ, due to 'switching on' $\Phi(\mathbf{r})$ is $\Delta S = -\int_0^\tau q\Phi(\mathbf{r})dt = -q\Phi\tau$.
- Define $\bar{\gamma}$ as the minimum possible electrostatic potential in nature that corresponds to minimum action $\Delta S = \hbar \implies \bar{\gamma} = -\frac{\hbar}{q\tau}$.

In presence of source

$$\oint \mathcal{E}_x \; dx = 0 \qquad \oint \mathcal{E}_y \; dy = 0 \qquad \oint \mathcal{E}_z \; dz = 0$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\oint \mathcal{E}_x \; dx = n_x \bar{\gamma} \qquad \oint \mathcal{E}_y \; dy = n_y \bar{\gamma} \qquad \oint \mathcal{E}_z \; dz = n_z \bar{\gamma}$$

$$-i\bar{\gamma}\frac{\partial \psi_x}{\partial x} = \mathcal{E}_x \psi_x \qquad -i\bar{\gamma}\frac{\partial \psi_y}{\partial y} = \mathcal{E}_y \psi_y \qquad -i\bar{\gamma}\frac{\partial \psi_z}{\partial z} = \mathcal{E}_z \psi_z$$

$$\left[-\frac{\bar{\gamma}^2}{2\left(\frac{1}{\epsilon_0}\right)} \frac{\partial^2}{\partial x^2} - \frac{\epsilon_0 \mathcal{E}_x^2}{2} \right] \psi_x = i \left(-\frac{1}{2} \rho_x \bar{\gamma} \right) \psi_x$$

Normalizable solutions are:

$$\boxed{\Psi_{E_x} = \int u(\mathcal{E}_x) \mathbf{e}^{-i\frac{\Phi_{E_x}}{\bar{\gamma}_{E_x}}} d\mathcal{E}_x = \int u(\mathcal{E}_x) \mathbf{e}^{-i\frac{\Phi}{\bar{\gamma}}} d\mathcal{E}_x}$$

Observations

$$\boxed{\left[-\frac{\bar{\gamma}^2}{2\left(\frac{1}{\epsilon_0}\right)}\frac{\partial^2}{\partial x^2}-\frac{\epsilon_0\mathcal{E}_x^2}{2}\right]\psi_x = i\left(-\frac{1}{2}\rho_x\bar{\gamma}\right)\psi_x}$$

- Has the form of the time-independent Schrödinger's equation.
- 1^{st} and 2^{nd} terms on the left-hand side denote the kinetic and the potential energy density terms.
- Inverse of the permittivity plays the role of mass of ψ_E field.
- Presence of anti-Hermitian operator.

Rational multiple of e

- Classical field remains the same if we change $\Phi \to \Phi + \Phi_0$.
- Argue: ψ_{E_x} must remain the same if $\Phi_{E_x} \to \Phi_{E_x} + \Phi_0$

$$\int u(-\nabla\Phi)\mathbf{e}^{i\frac{q\Phi\tau}{\hbar}}\, d\vec{\mathcal{E}} = \int u(-\nabla(\Phi+\Phi_0))\mathbf{e}^{i\frac{q(\Phi+\Phi_0)\tau}{\hbar}}\, d\vec{\mathcal{E}}$$

$$\implies \mathbf{e}^{i\frac{q\Phi\tau}{\hbar}} = \mathbf{e}^{i\frac{q(\Phi+\Phi_0)\tau}{\hbar}}$$

$$\implies \frac{q(\Phi+\Phi_0)\tau}{\hbar} = \frac{q\Phi\tau}{\hbar} + 2n\pi ......[n \in \mathbb{N}]$$

$$\implies q\Phi_0\tau = 2n\pi\hbar = n\ 2\pi e\frac{\gamma}{2\pi}\tau$$

$$\implies q = ne\frac{\gamma}{\Phi_0} = \frac{n}{N}e$$

Proof of electric charge quantization

- Consider charged capacitor, axis along x direction; plates at $[0, a]$.

$$-\frac{\hbar^2}{2\left(\frac{e^2\tau^2}{\epsilon_0}\right)}\frac{d^2\psi_{E_x}}{dx^2}-\frac{\epsilon_0}{2}\mathcal{E}_x^2\psi_{E_x}=-i\bar{\gamma}\frac{\sigma}{2}\delta(x-a)\psi_{E_x} \qquad (1)$$

- Between the plates $(0 < x < a)$, $\mathcal{E}_x \neq 0$ & $\Psi_{E_x} = \int a(\mathcal{E}_x)\mathbf{e}^{-i\frac{e\mathcal{E}_x a\tau}{\hbar}} d\mathcal{E}_x$
- Exactly at $x = a$, $\psi_{E_x} = 0$, otherwise the RHS of Eq.(1) will diverge.
- $\implies \sin\left(\frac{e\mathcal{E}_x a\tau}{\hbar}\right) = 0$ (or $\cos\left(\frac{e\mathcal{E}_x a\tau}{\hbar}\right) = 0$)

$$\left(\frac{e\mathcal{E}_x a\tau}{\hbar}\right) = n\pi$$

$$\implies \mathcal{E}_x = n\frac{\pi\hbar}{ea\tau} = n\pi\frac{\bar{\gamma}}{a}\left(=\frac{\sigma}{\epsilon_0}\right)$$

Summary

- The eigenvalue at the right hand side of Eq.(1) can be quantized according to appropriate boundary conditions, exactly in the same way the energy eigenvalues of a particle in a potential well are quantized.
- Boundary condition requires the quantisation of the charge in the source distribution.
- Fundamentally, it is not quantized. In several situations, it can be expressed as a rational multiple of e.
- This description, therefore, removes the need for magnetic monopoles due to the quantisation of electric charge.

Thank You

Questions, comments welcome...

The presentation of this work in ICNFP 2022 is supported by the HBCSE (DAE) Contingency grant.