Gravitational wave propagation over arbitrary background

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11th International Conference on New Frontiers in Physics
(ICNFP 2022)
Crete
30 August - 11 September, 2022

Introduction

Relic gravitational waves from the very early universe may give important info about inflationary stage of the universe evolution.

Gravitational wave (GW) propagation is considered mostly either over Minkowski or conformally flat Friedmann-Le'Maitre-Robertson-Walker (FLRW) spaces. (A few works are also dedicated to some Bianchi spaces.)
Here we derive equation of motion of gravitational waves in an arbitrary space-time metric. It is shown that there appear additional terms which are absent in the FLRW case.
These terms may make essential contribution at low frequencies in realistic cosmological situation.

Choice of gauge

We expand the total metric $\bar{g}_{\mu\nu}$ in terms of the small perturbations as

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \ \text{and} \ \bar{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu},$$

where, as we see in what follows, $h_{\mu\nu}$ is not necessarily $3D$ tensor but may contain $3D$ scalar part.
Under the coordinate transformation $\tilde{x}^\mu = x^\mu + \xi^{(1)\mu}$, where $\xi^{(1)\mu}$ is a small vector, $h_{\mu\nu}$ changes as:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - D_\mu \xi^{(1)}_\nu - D_\nu \xi^{(1)}_\mu.$$

Covariant derivative D_α is defined with respect to background metric $g_{\mu\nu}$.
Using freedom in the choice of four functions $\xi^{(1)\mu}$ we can impose the following four conditions:

$$\mathbf{D}_\mu \psi^{\boldsymbol{\mu}}_{\boldsymbol{\nu}} = \mathbf{0}, \tag{1}$$

where $\psi^\mu_\nu = h^\mu_\nu - \delta^\mu_\nu \, h/2$ and $h = h^\alpha_\alpha$.

Choice of gauge

Additional transformation $\tilde{h}^{\mu}_{\nu} = h^{\mu}_{\nu} - D^{\mu}\xi^{(2)}_{\nu} - D_{\nu}\xi^{(2)\mu}$ would not violate condition (1) if

$$D^2\xi^{(2)}_{\mu} + R^{\nu}_{\mu}\xi^{(2)}_{\nu} = 0. \tag{2}$$

We apply this freedom to demand

$$\mathbf{h_{t\alpha=0}} \tag{3}$$

for any α. In this case the condition $h = 0$ may be invalid. There is still some freedom to make the coordinate transformation with parameter $\xi^{(3)\mu}$, which, in addition to (2), satisfies the condition:

$$D_{\mu}\xi^{(3)\mu} = 0. \tag{4}$$

The transformation with functions $\xi^{(3)\mu}$ does not change the value of h.

Basic equations

We start from the exact Einstein equations:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi}{m_{Pl}^2}\,\overline{T}^{(phys)}_{\mu\nu} \equiv \overline{T}_{\mu\nu}. \tag{5}$$

Overline means that the corresponding exact (total) quantities are calculated in terms of the total metric $\bar{g}_{\mu\nu}$,

The total Ricci tensor and energy-momentum tensor expanded as a sum of the background and first order correction terms:

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + R^{(1)}_{\mu\nu}, \quad \overline{T}_{\mu\nu} = T_{\mu\nu} + T^{(1)}_{\mu\nu}, \tag{6}$$

All background quantities are taken in the background metric $g_{\mu\nu}$. Note, that lifting or lowering indices is not trivial e.g. $\boldsymbol{T^{(1)}_{\mu\nu} \neq g_{\mu\alpha} T_{\nu}^{(1)\alpha}}$. Indeed

$$\begin{aligned} \bar{T}_{\mu\nu} &= \bar{g}_{\alpha\nu}\bar{T}^{\alpha}_{\mu} = (g_{\alpha\nu} + h_{\alpha\nu})\left(T^{\alpha}_{\mu} + T^{(1)\alpha}_{\nu}\right), \\ T^{(1)}_{\mu\nu} &= (1/2)\left(h_{\mu\alpha}R^{\alpha}_{\nu} + h_{\nu\alpha}R^{\alpha}_{\mu} - h_{\mu\nu}R\right) + g_{\mu\alpha}T^{(1)\alpha}_{\nu}. \end{aligned}$$

Einstein equations for the background field are used.

First order expansion

$$\bar{R} = \left(g^{\alpha\beta} - h^{\alpha\beta}\right)\left(R_{\alpha\beta} + R^{(1)}_{\alpha\beta}\right) = g^{\alpha\beta}R_{\alpha\beta} - h^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}R^{(1)}_{\alpha\beta};$$
$$R^{(1)} = g^{\alpha\beta}R^{(1)}_{\alpha\beta} - h^{\alpha\beta}R_{\alpha\beta}.$$

According to Landau-Lifshitz, vol. 2:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(D_\alpha D_\nu h^\alpha_\mu + D_\alpha D_\mu h^\alpha_\nu - D_\alpha D^\alpha h_{\mu\nu} - D_\mu D_\nu h\right),$$

Final equation

$$\boxed{D_\alpha D^\alpha h_{\mu\nu} - 2h^{\alpha\beta}R_{\alpha\mu\nu\beta} - g_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta} - \frac{1}{2}g_{\mu\nu}D^2 h = -2g_{\mu\alpha}T^{\alpha(1)}_\nu}$$

Usually it is assumed that $T^{(1)\alpha}_\nu = 0$, though it is not always true, e.g. due to $g \to \gamma$ conversion or anisotropic stresses (e.g. by neutrinos or photons). *The 3^d term may essentially change the character of solutions, since it does not vanish when $\omega \to 0$*

Equation for trace $h = g^{\mu\nu} h_{\mu\mu}$

Taking trace over μ and ν in the "final equation" we find

$$D_\alpha D^\alpha h + 2h^{\alpha\beta} R_{\alpha\beta} = 2T_\alpha^{\alpha(1)}.$$

Since in the general case $h^{\alpha\beta} R_{\alpha\beta} \neq 0$, one must conclude that $h \neq 0$.

In FLRW metric $h^{\alpha\beta} R_{\alpha\beta} = 0$ and the condition $h = 0$ can be imposed.

Analogy to longitudinal photons in plasma

GW in FRWL spacetime

Background metric tensor: $g_{\mu\nu} = diag[1, -a^2, -a^2, -a^2]$,

$$R_{\mu\nu} = 0, \ \ \text{if} \ \ \mu \neq \nu, \ \ R_{tt} = 0, \ \ R_{ii} \sim \ \ g_{ii}, \ \ g_{ii} = -a^2 \delta_{ii}$$

The accepted usual equation for FRWL background:

$$\begin{aligned} D_\alpha D^\alpha h_{\mu\nu} - 2R_{\alpha\mu\nu\beta} h^{\alpha\beta} &= 0, \\ \left(\partial_t^2 - \frac{\Delta}{a^2} - H\partial_t - \frac{2\ddot{a}}{a} \right) h_{\nu\mu} &= 0. \end{aligned}$$

Realistic space-times deviating from FLRW

1. Density perturbations → **deviation from FRWL metric**

   Cloud of isotropically distributed matter *E.V. Arbuzova, A.D. Dolgov, L. Reverberi, Jeans Instability in Classical and Modified Gravity, Phys. Lett. B 739, 279-284 (2014)[e-Print: 1406.7104]*

   Schwarzschild-like isotropic coordinates:

   $$ds^2 = A(t,r)\, dt^2 - B(t,r)\, \delta_{ij} dx^i dx^j$$

   $$R_{ij} = F_1(A,B)\, \delta_{ij} + \partial_i \partial_j F_2(A,B) \Rightarrow h_{\alpha\beta} R^{\alpha\beta} \neq 0$$

2. Collapse of dust-like matter (Tolman solution) $h_{\alpha\beta} R^{\alpha\beta} \neq 0$

3. Perturbations of the inflaton field A.A. Starobinsky, Spectrum of adiabatic perturbations in the universe when there are singularities in the inflation potential, JETP Lett. 55, 489-494 (1992); Pisma Zh.Eksp.Teor.Fiz. 55, 477-482 (1992) → **deviation from FRWL metric**

Cosmological GW production

L. Parker theorem

1968

There is no massless particle (with s ¿ 0) creation by conformally flat expanding universe

because the corresponding field equations are conformally invariant,
$trace(T_{\mu\nu}) = 0$

L. P. Grishchuk

1974

Gravitons <u>*can be produced*</u> by conformally-flat expanding universe

because linearized graviton equation, obtained from the Einstein gravitational field equations, is not conformally invariant,
$trace(T_{\mu\nu}) \neq 0$

Result

Equation for GW propagation over **arbitrary background metri**c is obtained

A new term (which vanishes in FRWL) may essentially change the character of solutions

$$\frac{\text{Cosmological inflation} \rightarrow \text{relic GW}}{\textit{no experimental confirmation}}$$

The limit on long GW intensity from the CMB polarization data doesn't necessary exclude the model of inflation induced by a scalar field (inflaton)