Instantons: thick-wall approximation

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Based on

JHEP 07 (2022) 147, 2206.13994 [hep-th]
Phys.Lett. B827 (2022) 136951, 2111.13928 [hep-th]
JCAP 10 (2021) 049, 2105.01996 [hep-th]
JCAP 10 (2021) 066, 2104.12661 [hep-th]

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Citations iNSPIRE hep before October 14, 2021

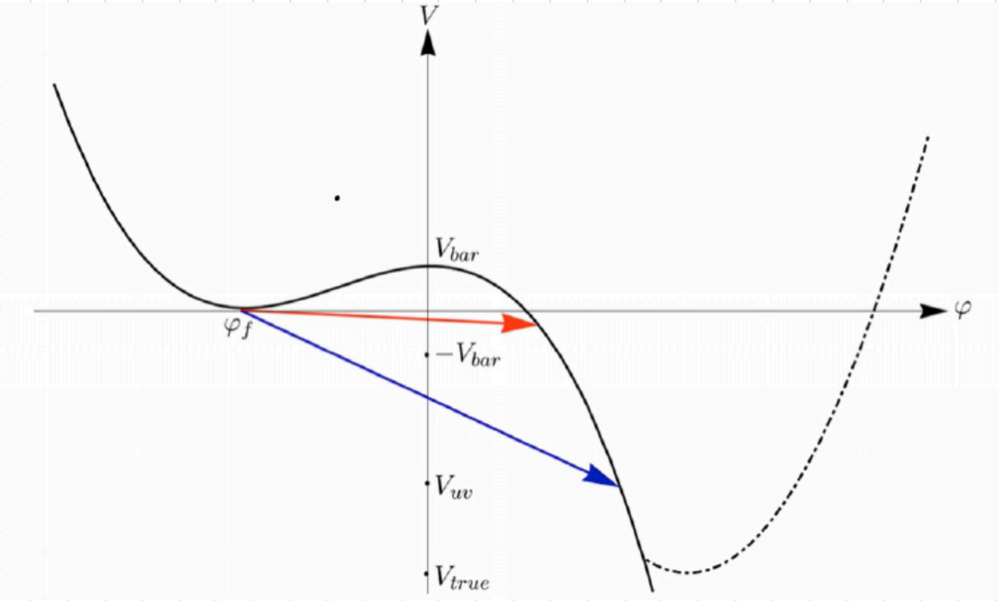
Bubbles in Metastable Vacuum
I. Yu. Kobzarev, L. B. Okun, M. B. Voloshin
Sov.J.Nucl.Phys. 20 (1975) 644-646, *Yad.Fiz.* 20 (1974) 1229-1234
414 citations (iNSPIRE hep)

The Fate of the False Vacuum. 1. Semiclassical Theory
Sidney R. Coleman
Phys.Rev.D 15 (1977) 2929-2936, *Phys.Rev.D* 16 (1977) 1248 (erratum)
2181 citations (iNSPIRE hep)

The Fate of the False Vacuum. 2. First Quantum Corrections
Curtis G. Callan, Jr., Sidney R. Coleman
Phys.Rev.D 16 (1977) 1762-1768
1406 citations (iNSPIRE hep)

Action Minima Among Solutions to a Class of Euclidean Scalar Field Equations
Sidney R. Coleman, V. Glaser, Andre Martin
Commun.Math.Phys. 58 (1978) 211-221
283 citations (iNSPIRE hep)

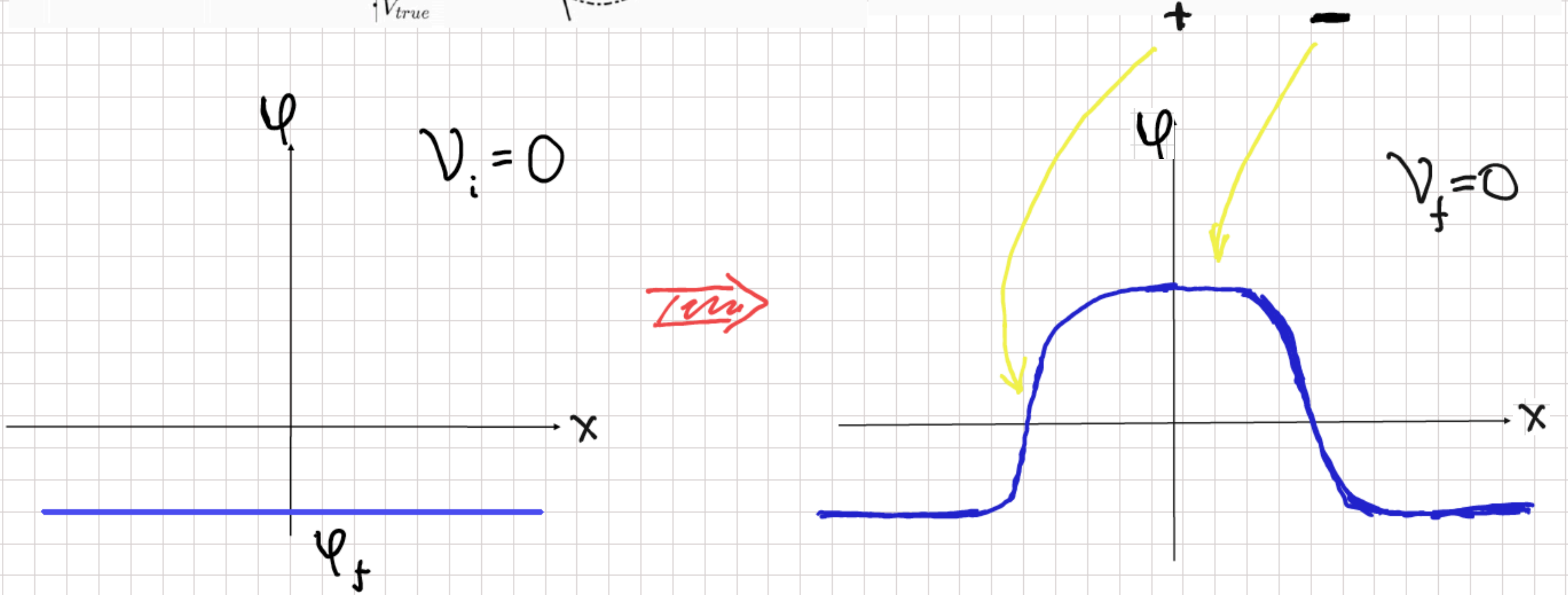
The false vacuum decay



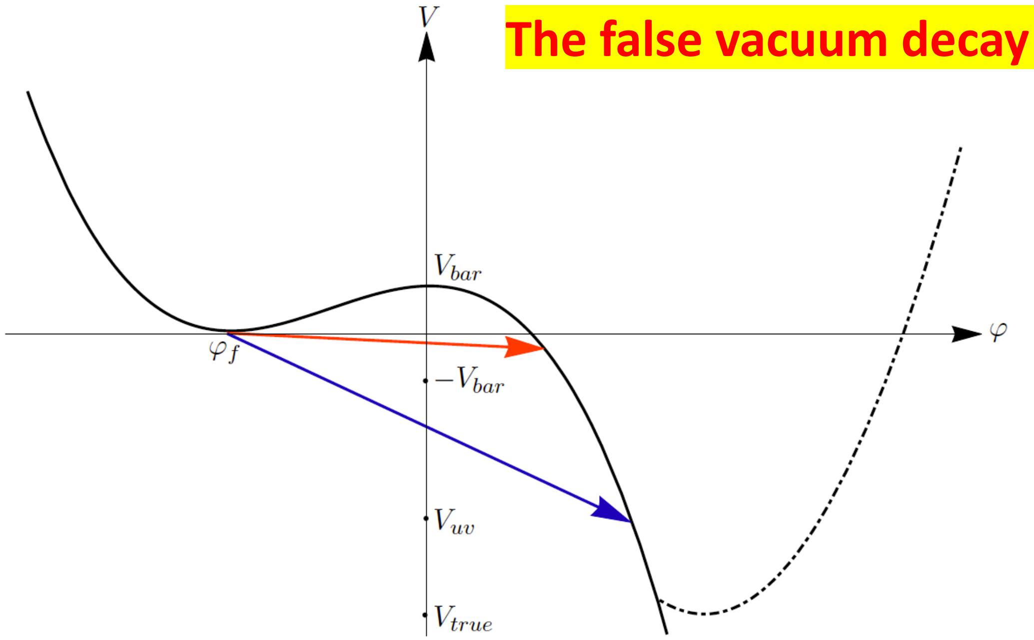
$$S = \int (\mathcal{K} - \mathcal{V})dt\,,$$

$$\mathcal{K} \equiv \frac{1}{2}\int (\partial_t \varphi_{\mathbf{x}})\, d^3x\,,$$

$$\mathcal{V} \equiv \int \left(\frac{1}{2}(\partial_i \varphi_{\mathbf{x}})^2 + V(\varphi_{\mathbf{x}})\right) d^3x\,.$$



The false vacuum decay (quasiclassical approximation)



$$S = \int (\mathcal{K} - \mathcal{V})dt\,, \qquad \mathcal{K} \equiv \frac{1}{2}\int (\partial_t \varphi_{\mathbf{x}})^2 d^3x\,, \qquad \mathcal{V} \equiv \int \left(\frac{1}{2}(\partial_i \varphi_{\mathbf{x}})^2 + V(\varphi_{\mathbf{x}})\right) d^3x$$

$$\tau = it$$

$$S_E = \int \left(\frac{1}{2}(\partial_\tau \varphi)^2 + \frac{1}{2}(\partial_i \varphi)^2 + V(\varphi)\right) d^3x d\tau$$

$$\Gamma \simeq \exp(iS) = \exp(-S_E) \;\rightarrow\; \partial_\tau^2 \varphi + \Delta\varphi - V' = 0\,,$$

we must find field configurations with $\varphi(\tau \to -\infty, \mathbf{x}) = \varphi_f$,
matching the classically allowed state $\varphi(\tau = 0, \mathbf{x})$ with $\partial\varphi/\partial\tau = 0$ and $\mathcal{V}(\varphi(\mathbf{x})) = 0$

The Coleman instanton *(Phys.Rev.D* 15 (1977) 2929)

Coleman proposed to consider $O(4)$ - invariant solutions for which φ depends only on $\varrho = \sqrt{\tau^2 + \mathbf{x}^2}$,

$$\varphi(\tau, \mathbf{x}) = \varphi(\varrho) \rightarrow \ddot{\varphi}(\varrho) + \frac{3}{\varrho}\dot{\varphi}(\varrho) - V' = 0$$

with the two boundary conditions:

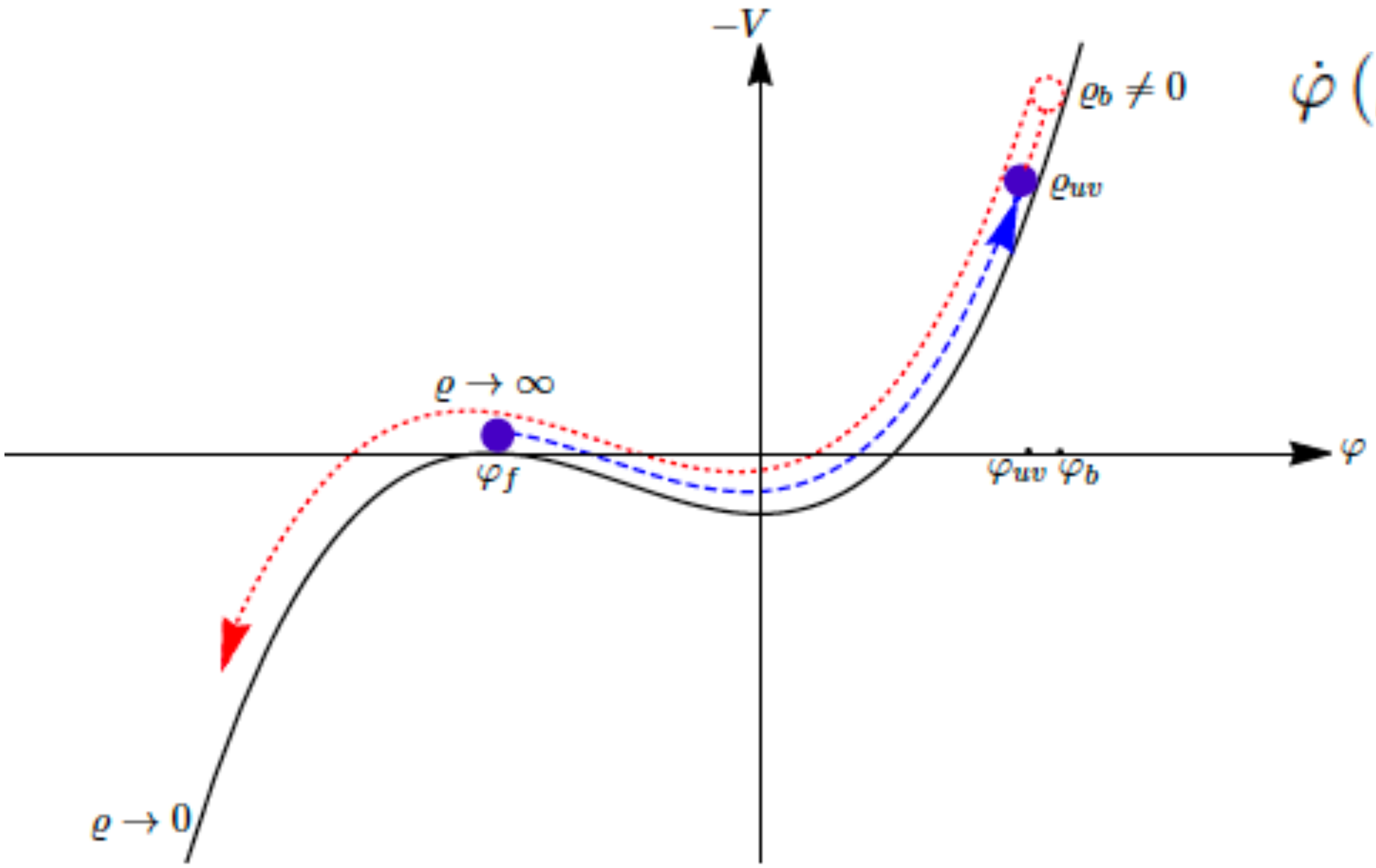
$$\varphi(\varrho \to \infty) = \varphi_f$$
$$\dot{\varphi}(\varrho = 0) = 0$$

$$\rightarrow \quad S_E = 2\pi^2 \int_0^{+\infty} d\varrho\, \varrho^3 \left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \qquad \rightarrow \quad \Gamma \simeq \varrho_0^{-4} \exp(-S_E)$$

$$\ddot{\varphi}(\varrho) + \frac{3}{\varrho}\dot{\varphi}(\varrho) - V' = 0$$

$$\varphi\left(\varrho \to \infty\right) = \varphi_f$$

$$\dot{\varphi}(\varrho = 0) = 0$$

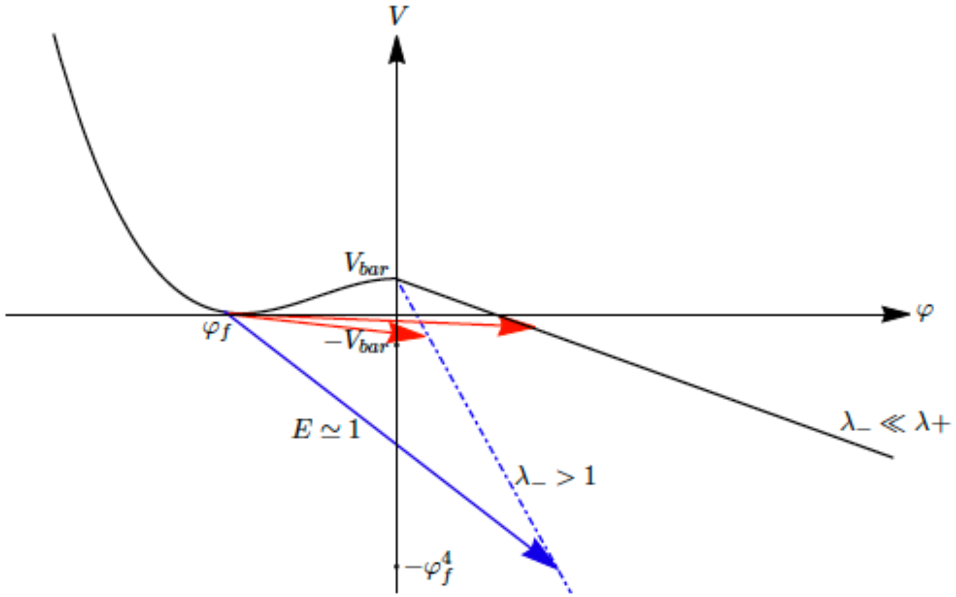


$$\dot{\varphi}\left(\varrho_b\right) = 0$$

The two puzzles with the Coleman instanton

1. The very fast false vacuum decay

V.F. Mukhanov, E. Rabinovici and A.S.S.,
Fortsch. Phys. 69 (2021) 2000100 [arXiv:2009.12445]



$\lambda_- \gg 1$ (*zero size instanton problem*)

$\varrho_0 << 1$, $S_E << 1$, $\Gamma >> 1$

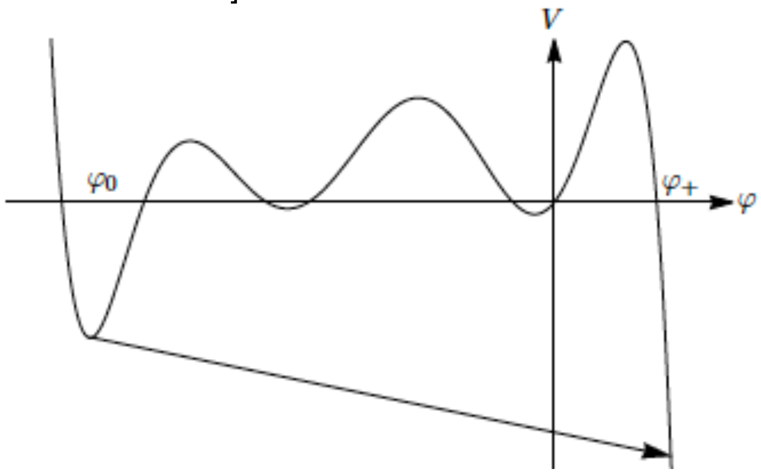
The quasiclassical approximation is not trustable!

2. There are no Coleman instanton solutions at all

V.F. Mukhanov, E. Rabinovici and A.S.S., Fortsch. Phys. (2021) [arXiv:2009.12445]
V.F. Mukhanov and A.S.S., JCAP (2022), [arXiv:2104.12661]

$V(\varphi) \equiv -\varphi^\alpha v_\alpha(\varphi)$, $\alpha \geq 4$

$\frac{d v_\alpha(\varphi)}{d\varphi} \geq 0$ at $\varphi > 0$



$$E(\alpha) = \varrho^{\frac{4}{\alpha-2}} \left(\frac{1}{2} \varrho^2 \dot{\varphi}^2 + \frac{2}{\alpha-2} \varrho \varphi \dot{\varphi} - \varrho^2 V - \frac{2(\alpha-4)}{(\alpha-2)^2} \varphi^2 \right)$$
$$+ \frac{2}{\alpha-2} \int_0^\varrho d\overline{\varrho}\, \overline{\varrho}^{\frac{6-\alpha}{\alpha-2}} \left[(\alpha-4) \left(\overline{\varrho} \dot{\varphi} + \frac{2}{\alpha-2} \varphi \right)^2 + \overline{\varrho}^2 (\alpha V - \varphi V') \right],$$

$$\frac{dE}{d\varrho} = \varrho^{\frac{\alpha+2}{\alpha-2}} \left(\ddot{\varphi}(\varrho) + \frac{3}{\varrho} \dot{\varphi}(\varrho) - V' \right) \left(\varrho \dot{\varphi} + \frac{2}{\alpha-2} \varphi \right),$$

Explicit examples of such potentials:

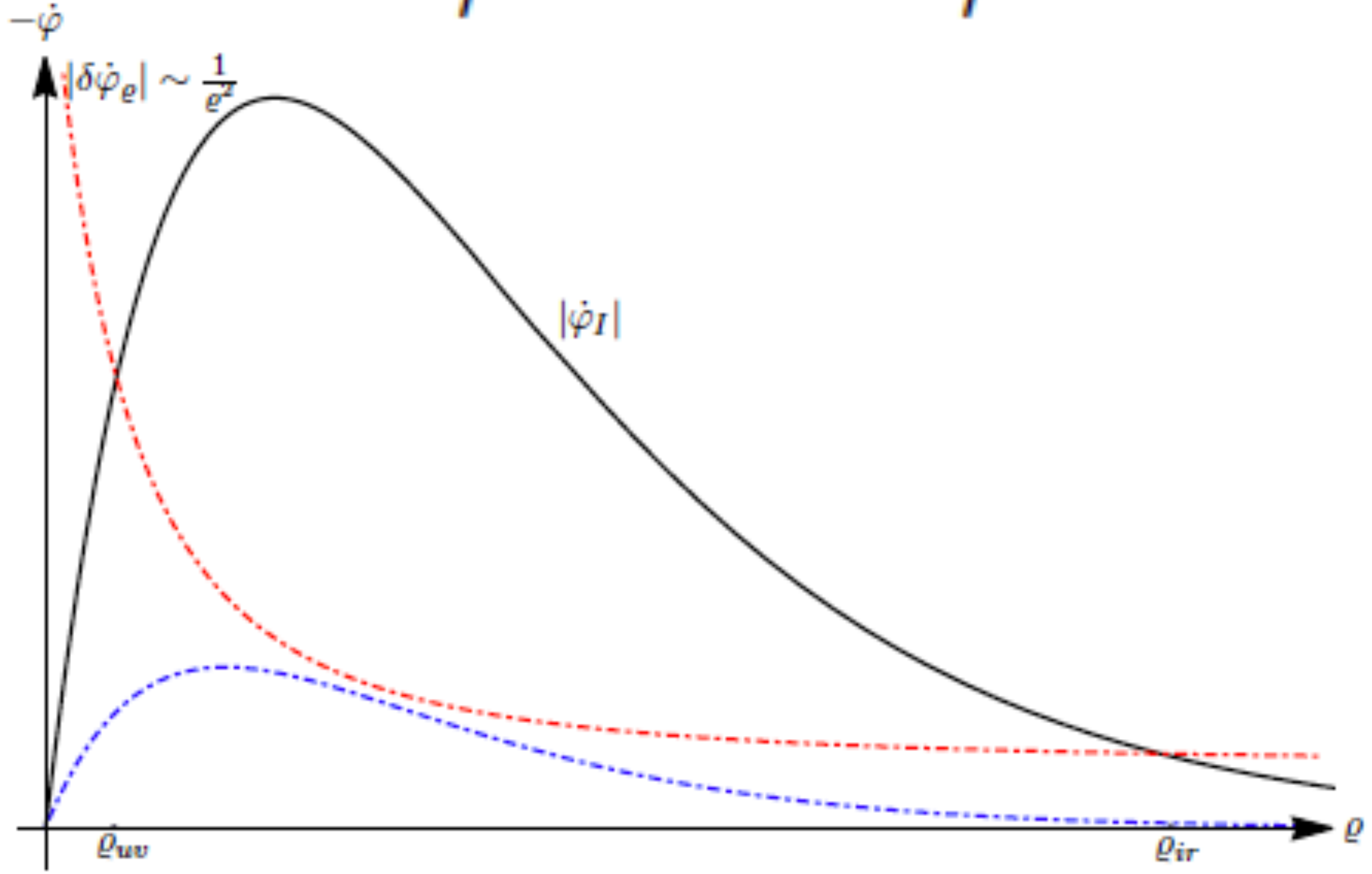
$$v'_\alpha = a \varphi^{-\alpha} \prod_{i=1}^{\alpha-2(m+1)} (\varphi - \lambda_i) \prod_{j=1}^{m} \left((\varphi + \beta_j)^2 + \gamma_j^2 \right),$$ where a>0, $\gamma_j, \lambda_i \leq 0$

$$V(\varphi) = \Lambda \varphi^4 + a \left(\varphi^3 + \beta \varphi^2 + \frac{\beta^2 + \gamma^2}{3} \varphi \right),$$ $\Lambda < 0$;

$$V = -\frac{\lambda}{4} \varphi^4 \ln \left(\frac{\varphi}{\mu} \right),$$ $\lambda > 0$

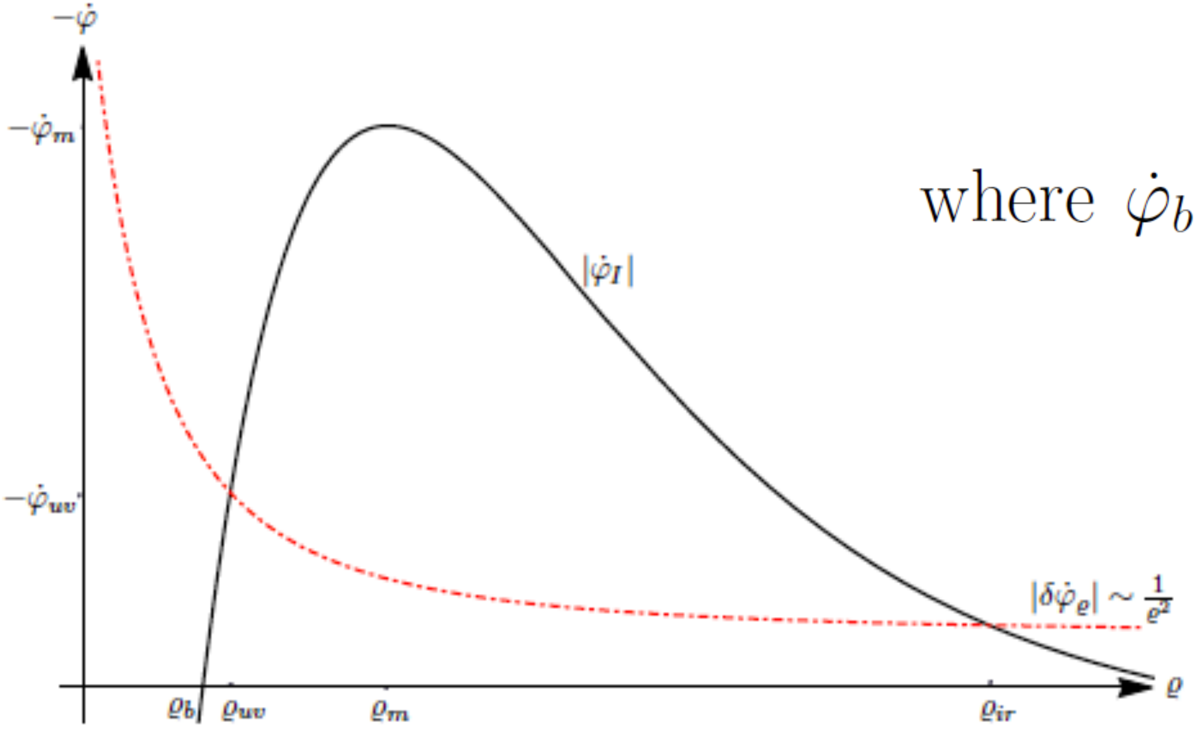
The resolution of the puzzles: quantum fluctuations

$$|\delta\varphi_r| \simeq \frac{\sigma}{r}, \quad |\delta\dot{\varphi}_r| \simeq \frac{\sigma}{r^2}$$



ϱ_{uv} and ϱ_{ir} are two different solutions of the equation: $\dot{\varphi}_I(\varrho) \simeq \frac{\sigma}{\varrho^2}$

Quantum fluctuations and new instantons



$$\mathcal{V}(0) = \frac{2\pi}{3}\dot\varphi_b^2\varrho_b^3 - \frac{4\pi}{3}V_b\varrho_b^3 + \mathcal{V}_{\varrho_b}(0) = 0,$$

where $\dot\varphi_b \equiv \dot\varphi(\varrho_b)$, $V_b \equiv V(\varphi(\varrho_b))$, $\mathcal{V}_{\varrho_b}(0) = \frac{4\pi}{3}V_b\varrho_b^3$

The instanton is trustable only in the range $\varrho_{uv} < \varrho < \varrho_{ir}$

$$\mathcal{V}(0) = \frac{2\pi}{3}\left(\dot\varphi_{uv}^2\varrho_{uv}^3 - \dot\varphi_{ir}^2\varrho_{ir}^3\right) \simeq \frac{2\pi\sigma}{3}\left(\frac{1}{\varrho_{uv}} - \frac{1}{\varrho_{ir}}\right)$$

With the precision allowed by the time-energy uncertainty relation $\varrho_{\rm uv}\mathcal{V} \simeq O(1)$ the potential energy vanishes and the bubble with the quantum core emerges from under the barrier.

New instantons

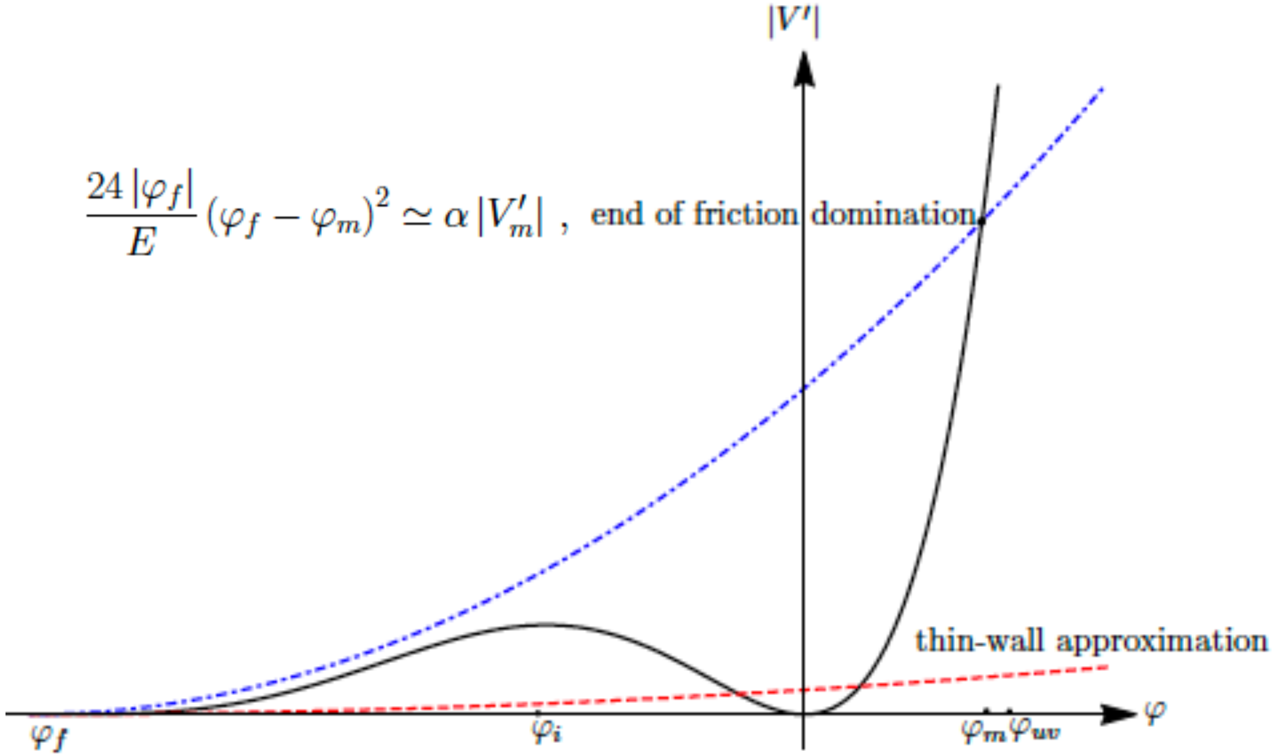
$\ddot\varphi(\varrho) + \frac{3}{\varrho}\dot\varphi(\varrho) - V' = 0$ with the boundary conditions: $\varphi(\varrho\to\infty) = \varphi_f$ and $\dot\varphi(\varrho_b) = 0$, ϱ_b is an arbitrary free parameter.

$$S_E = \frac{\pi^2}{2}\int_{\varrho_{uv}}^{\varrho_{ir}}\dot\varphi^2\varrho^3 d\varrho + \frac{\pi^2}{4}\left(2V_{ir}\varrho_{ir}^4 + \dot\varphi_{uv}^2\varrho_{uv}^4 - \dot\varphi_{ir}^2\varrho_{ir}^4\right) \simeq \frac{\pi^2}{2}\int_{\varrho_b}^{\infty}\dot\varphi^2\varrho^3 d\varrho$$

← Good approximation with the accuracy of one quantum: $\int_{\varrho_b}^{\varrho_{uv}}\dot\varphi^2\varrho^3 d\varrho < \dot\varphi_{uv}^2\varrho_{uv}^4 \simeq O(1)$

The false vacuum decay rate $\Gamma \simeq \varrho_0^{-4}\exp(-S_E)$, where ϱ_0 is the size of the bubble, i.e. $\varphi(\varrho_0) = 0$

The friction-dominated new instantons



$$\ddot{\varphi}(\varrho) + \frac{3}{\varrho}\dot{\varphi}(\varrho) - V' = 0$$

$$|V'| \ll \left|\frac{3}{\varrho}\dot{\varphi}\right| \;\rightarrow\; \ddot{\varphi}(\varrho) + \frac{3}{\varrho}\dot{\varphi}(\varrho) = 0$$

$$\varphi(\varrho) \simeq \varphi_f - \frac{E}{4\,\varphi_f\,\varrho^2}\,, \quad E = E(\varrho_6)$$

$$\downarrow$$

$$V_{\rm fr} \equiv \left|\frac{3}{\varrho}\dot{\varphi}\right| \simeq \frac{24\,|\varphi_f|}{E}(\varphi_f-\varphi)^2$$

$$\varrho_0^2 \simeq \frac{E}{4\,\varphi_f^2} \quad \rightarrow \quad \dot{\varphi}_0^2\,\varrho_0^4 \simeq \frac{E^2}{4\,\varphi_f^2\,\varrho_0^2} \simeq E$$ **>>1 (**number of quantum)

New instantons in D dimensions

$$\ddot{\varphi} + \frac{D-1}{\varrho}\,\dot{\varphi} - V' = 0\,, \qquad \begin{aligned}\varphi\,(\varrho \to \infty) &= \varphi_f\,,\\ \dot{\varphi}\,(\varrho = \varrho_b) &= 0\,.\end{aligned}$$

$$|\delta\varphi_q(\varrho)| \simeq \sigma\,\varrho^{\frac{2-D}{2}}\,, \quad |\delta\dot{\varphi}_q(\varrho)| \simeq \frac{\sigma\,(D-2)}{2}\,\varrho^{-\frac{D}{2}}\,, \quad |\dot{\varphi}_{\rm uv}| = \frac{\sigma\,(D-2)}{2}\;\varrho_{\rm uv}^{-\frac{D}{2}}\,, \quad |\dot{\varphi}_{\rm ir}| = \frac{\sigma\,(D-2)}{2}\;\varrho_{\rm ir}^{-\frac{D}{2}}$$

$$S_I = \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D+2}{2}\right)}\int_{\varphi_b}^{\varphi_f} d\varphi\,\varrho^{D-1}\dot{\varphi}\,, \qquad \Gamma \simeq \varrho_0^{-D}\,\exp\left(-S_I\right)$$

The friction-dominated new instantons

$$\ddot{\varphi} + \frac{D-1}{\varrho}\,\dot{\varphi} \simeq 0 \;\rightarrow\; \varphi(\varrho) = \varphi_f + \frac{E}{(D-2)^2\,|\varphi_f|\,\varrho^{D-2}} \;\rightarrow\; \varrho(\varphi) = \left(\frac{E}{(D-2)^2\,|\varphi_f|\,(\varphi-\varphi_f)}\right)^{\frac{1}{D-2}}$$

$$V_{\rm fr}(\varphi) \equiv \frac{D-1}{\varrho}|\dot{\varphi}| \equiv -\frac{1}{2}\,(D-2)^{\frac{2D}{D-2}}\left(\frac{|\varphi_f|}{E}\right)^{\frac{2}{D-2}}(\varphi-\varphi_f)^{\frac{2(D-1)}{D-2}} \leq 0$$

Generalization

$$\ddot{\varphi}(\varrho) + V'_{\rm fr}(\varphi) = 0\,, \quad \rightarrow \quad \frac{1}{2}\dot{\varphi}(\varrho)^2 + V_{\rm fr}(\varphi) = 0\,.$$

$$\ddot{\varphi}(\varrho) + \frac{3}{\varrho}\dot{\varphi}(\varrho) - V' = 0 \rightarrow \ddot{\varphi} + U'_{\rm eff}(\varphi) = 0\,, \rightarrow \frac{1}{2}\dot{\varphi}^2 + U_{\rm eff} = 0$$

$$U_{\rm eff} = V_{\rm fr} - V$$

The value of the scalar field at which its velocity vanishes satisftes: $U_{\rm eff}(\varphi) = 0 \rightarrow V_b \equiv V(\varphi_b) \simeq V_{\rm fr}(\varphi_b)$

If we assume that $\dot{\varphi}$ at the location of the maximum of the potential $V(\varphi = 0) = V_{\rm bar}$ is determined by the friction term, then

$$|V_{\rm fr}(\varphi = 0)| \gg V_{\rm bar}\,, \quad \rightarrow \quad 1 \ll E \ll (D-2)^D |\varphi_f|^D \, (2V_{\rm bar})^{\frac{2-D}{2}}$$

$$|V_{\rm fr}(\varphi_b)| \geq |V_{\rm fr}(0)| \quad \rightarrow \quad |V_b| \gg V_{\rm bar} \;!$$

Thus, the tunnelling depth is much larger than the height of the potential barrier, which corresponds to the thick-wall instantons.

The friction-dominated new instantons: the thick-wall approximation

$$S_I = \frac{\alpha\, \pi^{\frac{D}{2}} E\left(\varphi_b - \varphi_f\right)}{\Gamma\left(\frac{D+2}{2}\right)(D-2)\left|\varphi_f\right|} = \frac{\alpha\,(D-2)^{D-1} \pi^{\frac{D}{2}}}{\Gamma\left(\frac{D+2}{2}\right)} \frac{\left(\varphi_b + \left|\varphi_f\right|\right)^D}{\left|2V_b\right|^{\frac{D-2}{2}}}\,,$$

$$\varrho_0 = \left(\frac{E}{(D-2)^2\, \varphi_f^2}\right)^{\frac{1}{D-2}} = \frac{(D-2)}{\sqrt{\left|2V_b\right|}}\left(1 + \frac{\varphi_b}{\left|\varphi_f\right|}\right)^{\frac{D-1}{D-2}} \left|\varphi_f\right|$$

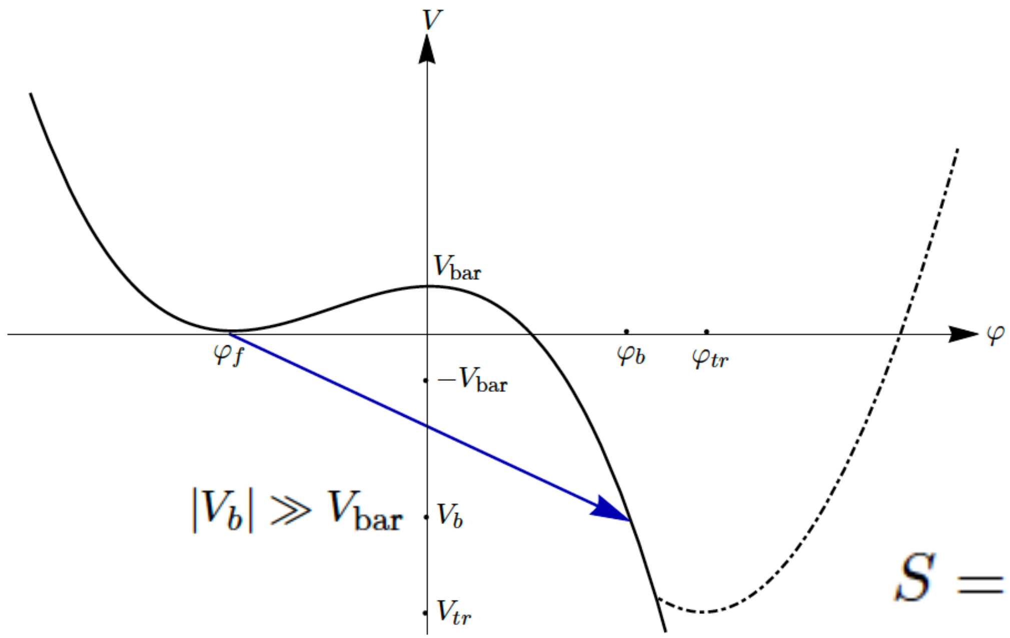
$$V\left(\varphi_b\right) = V_{\text{fr}}\left(\varphi_b\right)$$

$$V\left(\varphi_b^{\max}\right) \simeq -\frac{1}{2}(D-2)^{\frac{2D}{D-2}}\left(1 + \frac{\varphi_b^{\max}}{\left|\varphi_f\right|}\right)^{\frac{2(D-1)}{D-2}} \left|\varphi_f\right|^{\frac{2D}{D-2}}$$

The condition under which the thick-wall approximation is applicable:

$$\left(1 + \frac{\varphi_b}{\left|\varphi_f\right|}\right)^{\frac{2(D-1)}{D-2}} V_{\text{bar}} \ll \left|V_b\right| \ll \frac{1}{2}(D-2)^{\frac{2D}{D-2}}\left(1 + \frac{\varphi_b}{\left|\varphi_f\right|}\right)^{\frac{2(D-1)}{D-2}} \left|\varphi_f\right|^{\frac{2D}{D-2}}$$

Example for D=4



$$V(\varphi) = \begin{cases} \frac{\lambda_+}{4}(\varphi - \varphi_f)^4 & \text{for } \varphi < \beta\varphi_f \\ -\frac{\lambda_-}{n}\varphi_f^4\left(\frac{\varphi}{\varphi_f}\right)^n + V_{bar} & \text{for } \varphi > \beta\varphi_f \end{cases}$$

$$\beta \equiv \frac{\lambda_+^{\frac{1}{3}}}{\lambda_+^{\frac{1}{3}} + \lambda_-^{\frac{1}{3}}}, \quad V_{\rm bar} \equiv \frac{\lambda_-}{4}\beta^3\varphi_f^4$$

$$S = \frac{8\pi^2\alpha}{\lambda_-}\left(1 + \frac{|\varphi_f|}{\varphi_b}\right)^4, \; \varrho_0 = \sqrt{\frac{8}{\lambda_-}}\left(1 + \frac{\varphi_b}{|\varphi_f|}\right)^{3/2}\frac{|\varphi_f|}{\varphi_b^2}$$

α =1 for $\varphi_b \ll |\varphi_f|$ and α =1/3 for $\varphi_b >> |\varphi_f|$.

$$V\left(\varphi_b^{\rm max}\right) = \left(\frac{64}{\lambda_-}\right)^3 \varphi_f^4$$

$$\left(1 + \frac{\varphi_b}{|\varphi_f|}\right)^3 V_{\rm bar} \ll \frac{1}{4}\lambda_-\varphi_b^4 \ll 8\left(1 + \frac{\varphi_b}{|\varphi_f|}\right)^3 |\varphi_f|^4$$

or

$$1 \ll E \ll (D-2)^D|\varphi_f|^D\,(2V_{\rm bar})^{\frac{2-D}{2}}$$

$$V(\varphi_b) = V_{\rm fr}(\varphi_b) \quad \rightarrow \quad \left(1 + \frac{|\varphi_f|}{\varphi_b}\right)^3 \frac{|\varphi_f|}{\varphi_b} = \frac{\lambda_- E}{32}$$

Conclusions

When the depth of the true vacuum (or tunnelling at unbounded potential) significantly exceeds the height of the barrier the new instantons describing the tunnelling are dominated by the friction term in the instanton equation and the resulting bubbles of a true vacuum have thick walls.

Then one can replace the non-autonomous instanton equation by the autonomous completely integrable equation, which is a good approximation for the original one, and there exist the general formulas for the probability of the false-vacuum decay for arbitrary potentials in any number of dimensions.

Thank you for attention!