

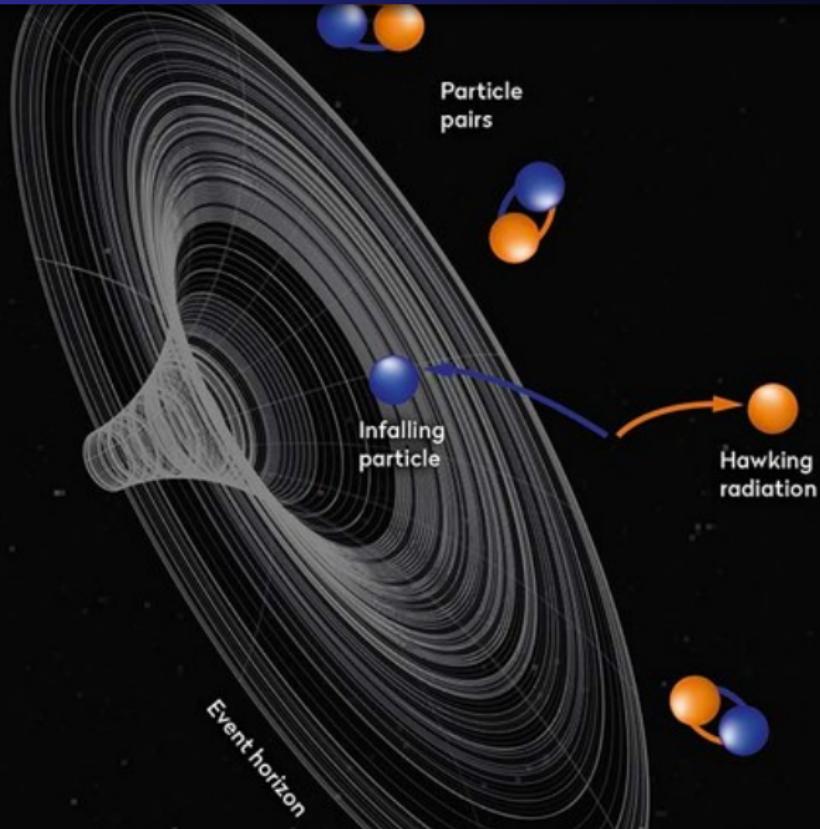
Entropy of an Unruh source and Schwarzschild black hole

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Hawking radiation



Black hole entropy

Bekenstein-Hawking entropy H_{BH}

A Schwarzschild black hole of radius r

$$H_{\text{BH}} = A/4, \quad A = 4\pi r^2$$

Example

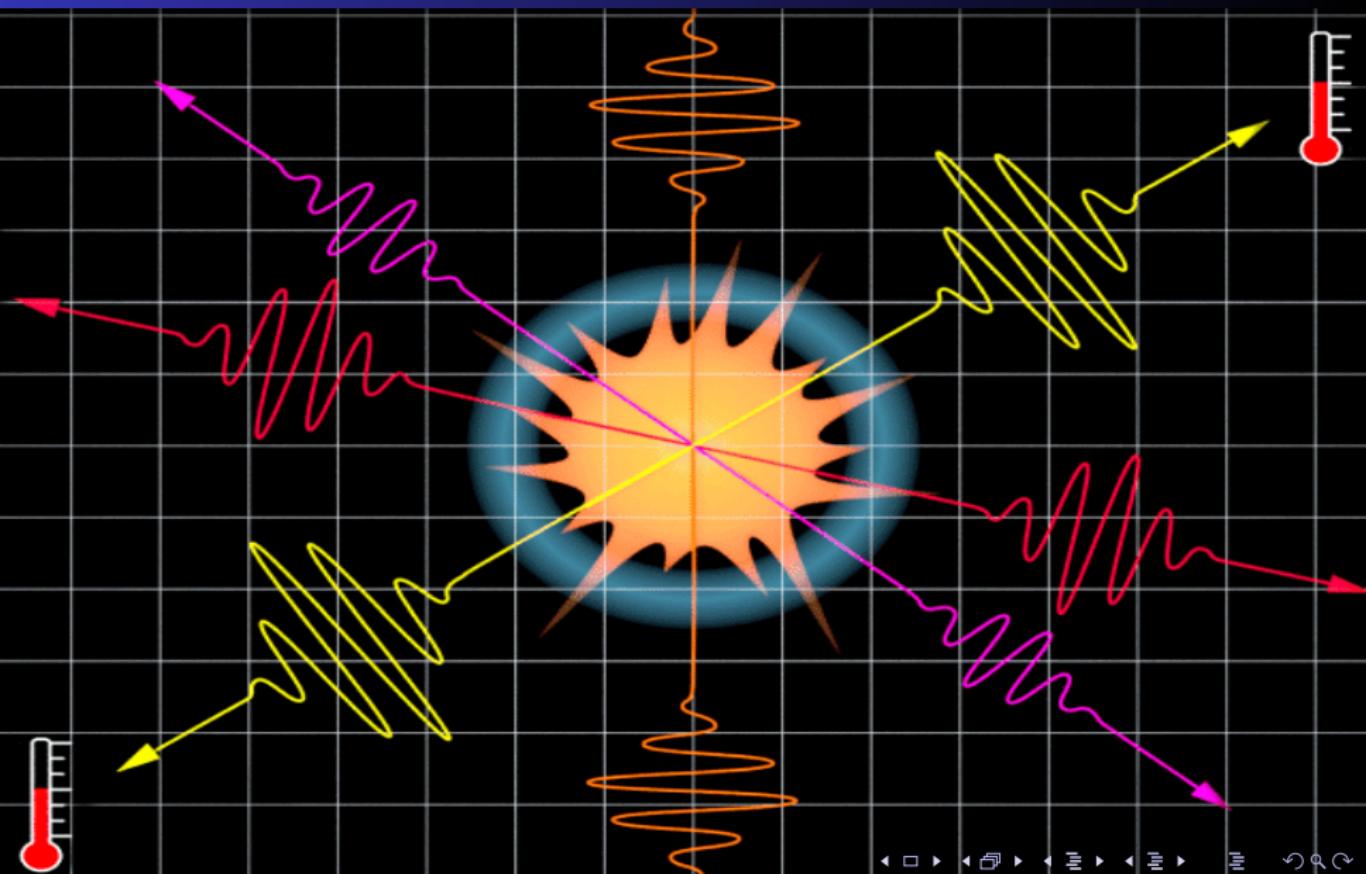
For the solar mass BH entropy is large:

$$H_{\text{BH}\odot} \sim 10^{77}$$

Problems

- In equilibrium $N_{DOF} = e^H$
⇒ What are the DOFs providing such large entropy?
- Information paradox: do BHs destroy information
(⇒ non-unitarian gravity)?

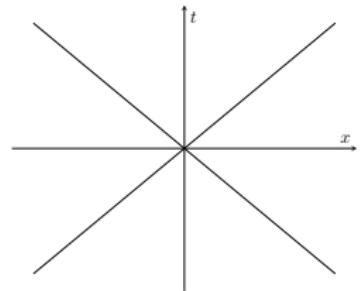
Unruh effect



Reference frames

Inertial reference frame (IRF)

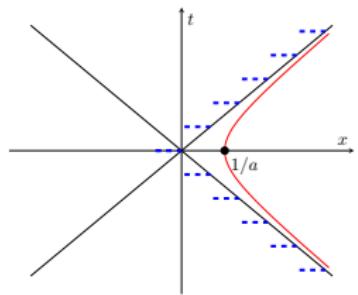
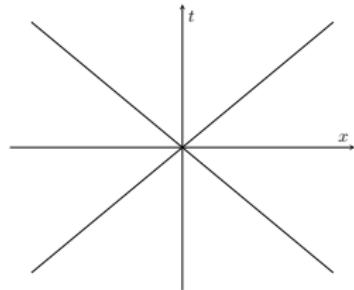
Minkowski vacuum $|0\rangle_M$



Reference frames

Inertial reference frame (IRF)

Minkowski vacuum $|0\rangle_M$



Non-inertial reference frame (NRF)

Acceleration a , horizon thermal radiation
at temperature $T = a/2\pi$ [Unruh, 1976]
Another basis is required
 \Rightarrow Rindler modes $|n\rangle_{\text{in}} |n\rangle_{\text{out}}$

NRF density matrix

IRF

$|0\rangle_M$ is a pure state

NRF

Only $|n\rangle_{\text{out}}$ modes are detectable \Rightarrow mixed state

$$\begin{aligned}\rho_{\text{out}} &= \text{Tr}_{\text{in}} |0\rangle_M \langle 0|_M \\ &= \frac{1 - e^{-E/T}}{1 - e^{-NE/T}} \sum_{n=0}^{N-1} e^{-nE/T} |n\rangle_{\text{out}} \langle n|_{\text{out}}\end{aligned}$$

Eigenvalues of ρ_{out} : probability to find n particles at energy E and temperature T (**conditional distribution**)

Distribution & Shannon entropy H

Distribution $\{X\}$

Find x with probability

$$p(x): \sum_x p(x) = 1$$

$$H(X) = - \sum_x p(x) \ln p(x)$$



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Joint distribution $\{X, Y\}$

Find x & y with probability

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H is information we need to describe our system
 \equiv how much we do not know

Conditional distribution

Conditional distribution $\{X|Y\}$

Probability to find x being given y

$$p(x|y) = \frac{p(x,y)}{p(y)}, \quad p(y) = \sum_x p(x,y)$$



Entropy

$$H(X|y) = - \sum_x p(x|y) \ln p(x|y)$$

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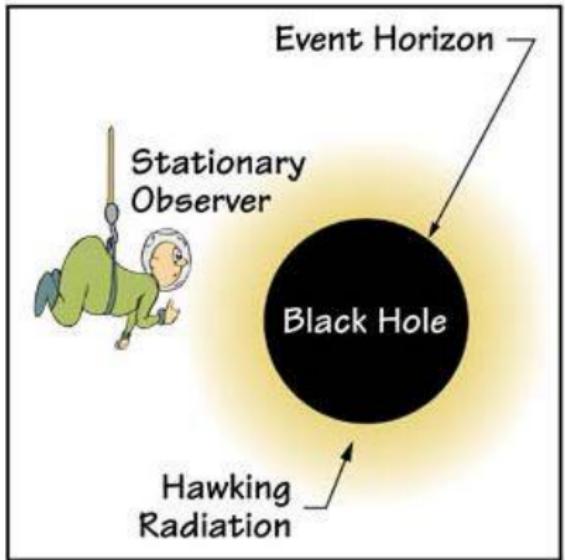


Entropy

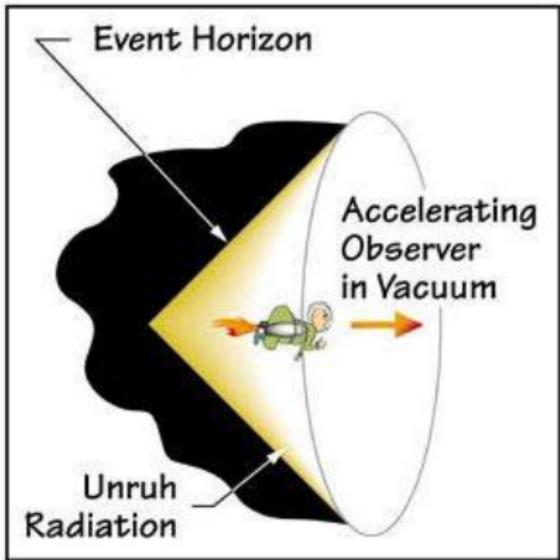
$$H(X|y) = - \sum_x p(x|y) \ln p(x|y)$$

$$\begin{aligned} H(X, Y) &= H(Y) + \sum_y p(y) H(X|y) \\ &= H(Y) + \langle H(X|y) \rangle_y \end{aligned}$$

Hawking radiation and Unruh effect

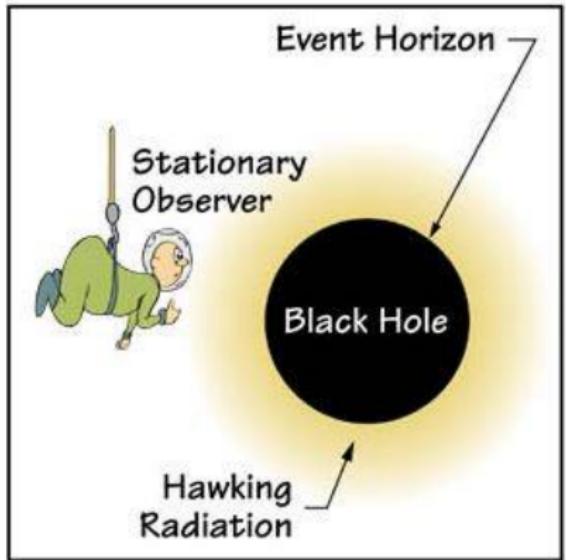


A stationary observer outside the black hole would see the thermal Hawking radiation.

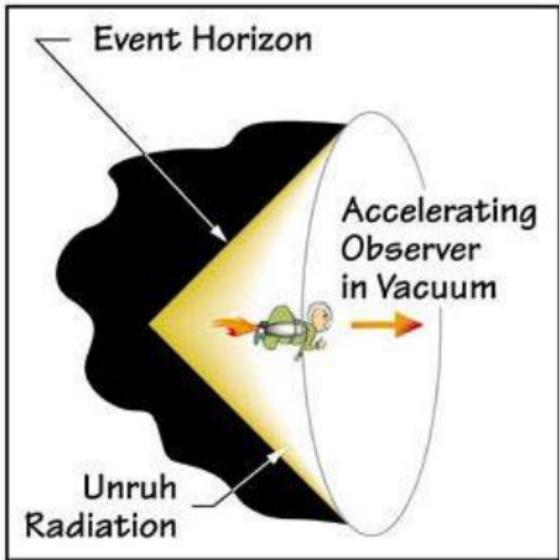


An accelerating observer in vacuum would see a similar Hawking-like radiation called Unruh radiation.

Hawking radiation and Unruh effect



A stationary observer outside the black hole would see the thermal Hawking radiation.



An accelerating observer in vacuum would see a similar Hawking-like radiation called Unruh radiation.

$$H(\rho_{\text{out}}) = H(n|N, E/T)$$

Model assumptions

1D → 3D

Unruh effect is 1D effect

But Schwarzschild black hole is 3D

⇒ need to take angular DOF into account

$$H \rightarrow \sum_{l=0}^{L} \sum_{-l}^l H = (L+1)^2 H$$

$$\sqrt{L(L+1)} = r\sqrt{E^2 - m^2}$$



Energy

- Distribution $\{E\}$ is homogeneous, $p(E) = \text{const}$
- $m \leq E \leq E_{\max}$

Notations

Series decomposition:

$$(1+x)^\alpha = \sum_{q=0}^{\infty} \binom{\alpha}{q} x^q, \quad |x| < 1$$

Lower incomplete gamma function $\gamma(\nu, x)$:

$$\gamma(\nu, x) = \int_0^x t^{\nu-1} e^{-t} dt = (\nu - 1)! \left(1 - e^{-x} \sum_{j=0}^{\nu-1} \frac{x^j}{j!} \right)$$

$$\begin{aligned}\sigma(qE/T) &= \frac{qE/T}{e^{qE/T} - 1} - \ln \left(1 - e^{-qE/T} \right) \\ &= \sum_{k=1}^{\infty} (qE/T + 1/k) e^{-kqE/T}\end{aligned}$$

Notations

$$H(\rho_{\text{out}}) = \sigma(E/T) - \sigma(NE/T)$$

$$\int_m^1 \sigma(qE/T) E^\nu dE = \frac{T^{\nu+1}}{q^{\nu+1}} \sum_{k=1}^{\infty} \frac{\gamma(\nu+1, x) + \gamma(\nu+2, x)}{k^{\nu+2}} \Big|_{x=kqm/T}^{x=kq/T}$$

$$f(\zeta) \Big|_{\zeta=a}^{\zeta=b} = f(b) - f(a)$$

Υ_{U} : exact expression

$$\begin{aligned}\Upsilon_{\text{U}} = & \left[(8\pi^2 T^2 - m^2) \sum_{k=1}^{\infty} \frac{\gamma(1, x) + \gamma(2, x)}{k^2} \left(\Big|_{x=km/T}^{x=k/T} - \frac{1}{N} \Big|_{x=kNm/T}^{x=kN/T} \right) \right. \\ & + T^2 \sum_{k=1}^{\infty} \frac{\gamma(3, x) + \gamma(4, x)}{k^4} \left(\Big|_{x=km/T}^{x=k/T} - \frac{1}{N^3} \Big|_{x=kNm/T}^{x=kN/T} \right) \\ & + 8\pi^2 T^2 \sum_{n=0}^{\infty} \binom{1/2}{n} \sum_{q=0}^{\infty} \frac{m^{2q}}{T^{2q}} (-1)^q \\ & \times \left. \sum_{k=1}^{\infty} \begin{cases} A_{nqk}(N, m, T), & 2\pi T > \sqrt{E^2 - m^2} \\ B_{nqk}(N, m, T), & 2\pi T < \sqrt{E^2 - m^2} \end{cases} \right] \frac{T}{\pi(1-m)}\end{aligned}$$

A_{nqk}, B_{nqk} : exact expressions

$$A_{nqk}(N, m, T) = \binom{n}{q} \frac{k^{2q-2n-2}}{(2\pi)^{2n}} \\ \times [\gamma(1+2n-2q, x) + \gamma(2+2n-2q, x)] \\ \times \left(\left| \frac{x=k/T}{x=km/T} - \frac{1}{N^{1+2n-2q}} \right|_{x=kNm/T}^{x=kN/T} \right)$$

$$B_{nqk}(N, m, T) = \binom{1/2-n}{q} \frac{(2\pi)^{2n-1}}{k^{2q+2n-3}} \\ \times [\gamma(2-2n-2q, x) + \gamma(3-2n-2q, x)] \\ \times \left(\left| \frac{x=k/T}{x=km/T} - \frac{1}{N^{2-2n-2q}} \right|_{x=kNm/T}^{x=kN/T} \right)$$

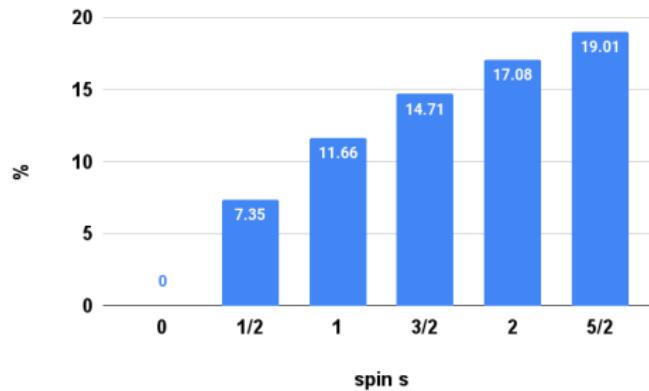
Υ_s : exact expression

$$\begin{aligned}\Upsilon_s &= 4\pi T^2 \frac{\ln(2s+1)}{1-m} \int_m^1 \left(\sqrt{\frac{E^2 - m^2}{4\pi^2 T^2} + 1} + 1 \right)^2 dE \\ &= \left(\frac{1+m-2m^2}{3\pi} + 2T \frac{\sqrt{1+4\pi^2 T^2 - m^2}}{1-m} + 4\pi T^2 \frac{1-2m}{1-m} \right. \\ &\quad \left. + 2T \frac{4\pi^2 T^2 - m^2}{1-m} \ln \frac{1+\sqrt{1+4\pi^2 T^2 - m^2}}{2\pi T + m} \right) \ln(2s+1)\end{aligned}$$

Results

Spin s

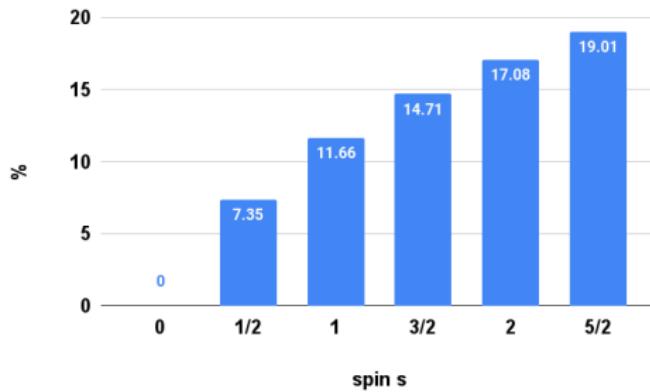
$$\frac{H(n, E|T, s)}{H_{\text{BH}}} \approx 16\pi T^2 H(E) + \gamma_U + \gamma_s \geq \frac{\ln(2s+1)}{3\pi}$$



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$$\frac{H(n, E|T, s)}{H_{\text{BH}}} \approx 16\pi T^2 H(E) + \gamma_U + \gamma_s \geq \frac{\ln(2s+1)}{3\pi}$$



$s = 0$

$$\frac{H(n, E|T)}{H_{\text{BH}}} < 1\%$$

[BT, CQG (2009)]

[TT, CQG (2013)]

Conclusions

- Schwarzschild black hole is represented as the set of Unruh horizons. Therefore, the problem of black hole entropy reduces to the geometrical one
- Entropy H of the Unruh radiation emitted from the event horizon is calculated analytically
- Spin dependence of H is taken into account
- H is lower-bounded; the boundary increases with spin
- Spins of different particles may be correlated due to conservation laws $\Rightarrow H_{\text{total}} \neq \sum_s H$