The Colour Glass Condensate

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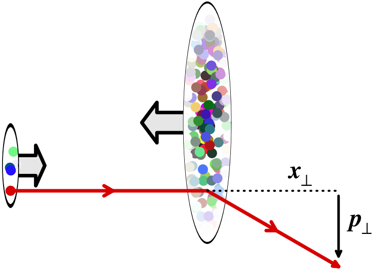

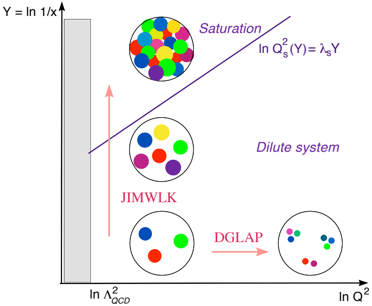

Acknowledgements

- Before we proceed, I would like to thank the organisers for inviting me to give these lectures ...
  - it took me less than one month to prepare them !
- ... and all the previous speakers who introduced/explained related concepts (and made my life easier) ...
  - Raju Venugopalan, Tuomas Lappi, Jani Penttala, Greg Chachamis, Francesco Celiberto, Vadim Guzey
- ... and for (priceless) phenomenology support to
  - Helen Caines, Paul Newman and Thomas Ullrich

Motivation: Some big questions

- What is the structure of a hadron in the high energy limit ?
- Or that of a very large nucleus with mass number $A \gg 1$?
- How to compute QCD scattering when s and/or A are large ?
- Can one rely on perturbation theory ?
- Does QCD approach the unitarity limit at high energy, and how ?
- Are these asymptotic limits relevant for the phenomenology ?

Motivation: ... and some answers

- A high-energy and/or large-A hadron is mostly made with gluons
  - in a special state: the Color Glass Condensate
  - very small longitudinal momentum fractions $x \leq 0.01$
  - large occupation numbers $n \sim 1/\alpha_s$
- This form of matter is weakly-coupled, due to its high density
- It controls the hadronic interactions at high energies, due to the large number of its constituents — the small-x gluons
- QCD scattering unitarises due to the phenomenon of gluon saturation
- Promising phenomenology at HERA, RHIC, and the LHC
- ... and even stronger expectations for the EIC: the smoking gun ?

Outline: Lecture 1

- Why small x gluons ?
- Experimental motivation from particle production at RHIC and the LHC
- ... and from DIS structure functions at HERA

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  - typical scales (and their separation)
  - uncertainty principle
  - oversimplified formulae

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- Focus on the physical picture:
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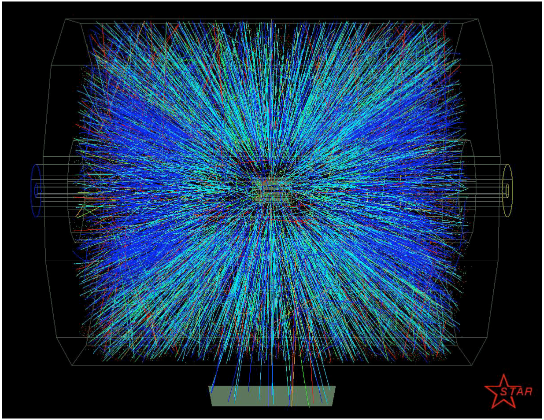
  - uncertainty principle
  - oversimplified formulae
  - ... and lots of suggestive (?) cartoons !

Pedagogical references

- General introductions to heavy ion collisions
  - *QCD in heavy ion collisions*, by E. Iancu, arXiv:1205.0579
  - *Small x physics and RHIC data*, by T. Lappi, arXiv:1003.1852
  - *Some Aspects of the Theory of Heavy Ion Collisions*, by F. Gelis, arXiv:2102.07604
- Review papers & lecture notes on the CGC (not exhaustive):
  - *The Color Glass Condensate and High Energy Scattering in QCD*, by E. Iancu and R. Venugopalan, arXiv:hep-ph/0303204
  - *The Color Glass Condensate*, by F. Gelis, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, arXiv:1002.0333
  - *Color Glass Condensate and Glasma*, by F. Gelis, arXiv:1211.3327
  - *Initial state and thermalization in the Color Glass Condensate framework*, by F. Gelis, arXiv:1508.07974
- A book (more advanced): *Quantum chromodynamics at high energy*, by Yuri V. Kovchegov and Eugene Levin, 2012, 349 pp. (Cambridge Univ Press)

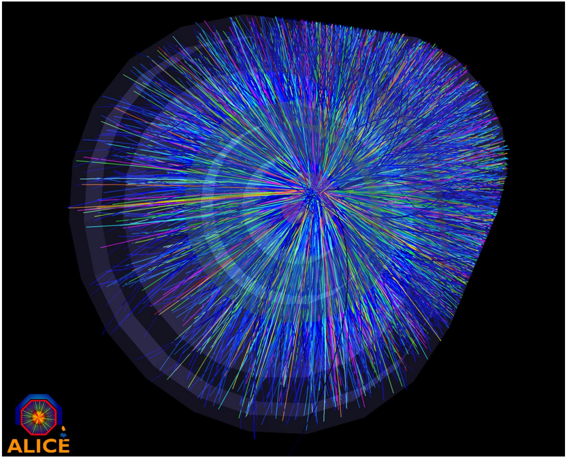
Heavy Ion Collisions @ RHIC & the LHC

Au+Au collisions at RHIC

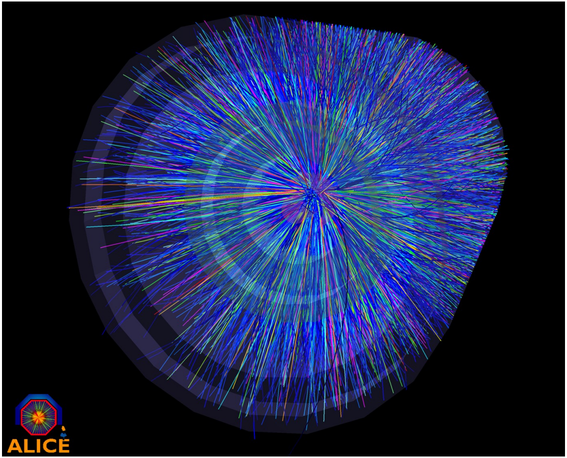


- Au+Au collision at STAR: longitudinal projection
- $\sim$ 7000 produced particles streaming into the detector
- Collision energy (COM frame) : $\sqrt{s} = 200$ GeV/nucleon

Pb+Pb collisions at the LHC

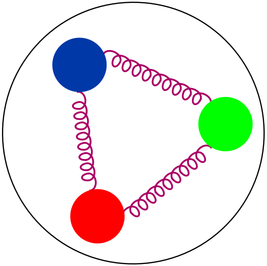


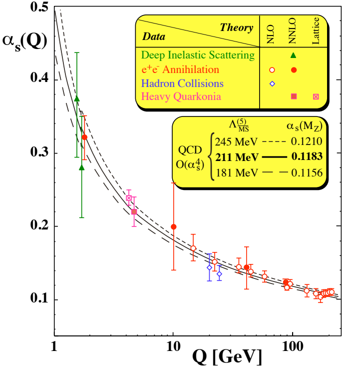
- Pb+Pb collision recorded by ALICE: $\sqrt{s} = 2760$ GeV/nucleon
- About 20,000 hadrons in the detectors
- Compare to $2A \simeq 400$ protons and neutrons in the incoming nuclei



- Where are all these hadrons coming from ?
- A brief reminder of the parton picture ...
- ... and of the kinematics of high energy collisions

- A proton is a bound state made with 3 valence quarks ...



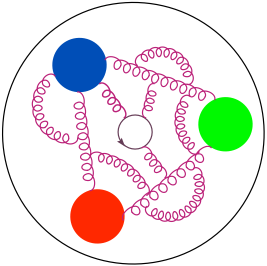


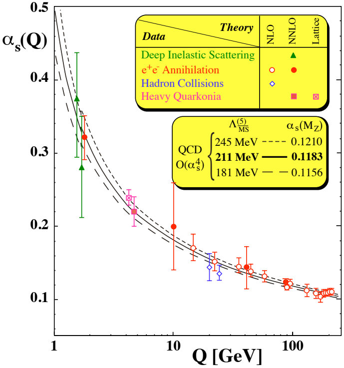
- ... which interact by exchanging gluons
- The coupling is weak at large transferred momenta, or short distances :

$$Q \sim 1/R \gg \Lambda_{\rm QCD} \simeq 200 \text{ MeV} \Longrightarrow \text{perturbative approaches}$$

- ... but it becomes of order 1 at $Q \sim \Lambda_{\rm QCD}$ or $R \sim 1$ fm (proton radius)

- A proton is a bound state made with 3 valence quarks ...



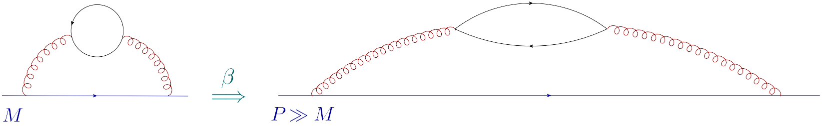


- Virtual fluctuations with typical energies and momenta of order $\Lambda_{\rm QCD}$
- Typical lifetimes (duration) $\Delta t \sim 1/\Lambda_{\rm QCD} \sim 1$ fm
- No meaningful concept of "parton"
  - non-perturbative, off-shell, mixing with vacuum fluctuations...

Infinite momentum frame

- Consider the proton in a boosted frame with large velocity $\beta \simeq 1$

$$P^\mu = (E, 0, 0, P) \quad \text{with} \quad E = \frac{M}{\sqrt{1-\beta^2}} = \sqrt{P^2 + M^2} \simeq P \gg M$$



- The lifetime of the fluctuations is amplified by Lorentz time dilation:

$$\Delta t_{\rm IMF} = \gamma\, \Delta t_{\rm RF} \sim \frac{\gamma}{\Lambda} \gg \frac{1}{\Lambda}, \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

- They last much longer than the vacuum fluctuations $\Delta t_{\rm vac} \sim 1/\Lambda$ (the vacuum is boost invariant!)

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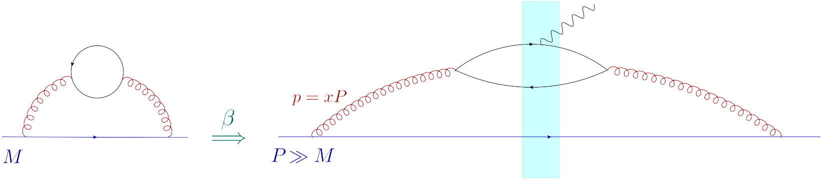
$$P^\mu = (E, 0, 0, P) \quad \text{with} \quad E = \frac{M}{\sqrt{1-\beta^2}} = \sqrt{P^2 + M^2} \simeq P \gg M$$



- The lifetime of the fluctuations is amplified by Lorentz time dilation:

$$\Delta t_{\rm IMF} = \gamma\, \Delta t_{\rm RF} \sim \frac{\gamma}{\Lambda} \gg \frac{1}{\Lambda}, \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

- Also much longer than a collision with a projectile: $\Delta t_{\rm IMF} \gg \Delta t_{\rm coll}$

- Long-lifetime fluctuations are nearly on-shell: partons

Parton picture

- Parton energies (p_0) and longitudinal momenta (p_z) are boosted
- Transverse momenta $(\boldsymbol{p}_\perp)$ and virtualities $(p^2 \equiv p^\mu p_\mu)$ are boost invariant

$$p^2 = p_0^2 - p_z^2 - p_\perp^2 \sim \Lambda^2 \quad \text{and} \quad p_\perp \sim \Lambda \implies p_0 \simeq p_z \gg p_\perp$$

- $p^\mu \simeq (xP, 0, 0, xP) = xP^\mu$: partons on-shell & collinear with the proton
  - x : longitudinal momentum fraction
  - negligible intrinsic transverse momentum $p_\perp$
- Proton wavefunction $\approx$ a Fock state built with partons
- Parton distributions: $f_i(x, Q^2)$, $i =$ quark (q), antiquark $(\bar{q})$, or gluon (g)
  - also depend upon resolution scale Q^2, via quantum evolution (DGLAP)
- Collinear factorisation:
  - hadronic cross-sections = PDFs $\otimes$ partonic cross-sections

Intrinsic $p_\perp$

- Intrinsic $p_\perp$: the transverse momentum of a parton from the hadron
- "Non-perturbative but small and negligible" in the parton model: $p_\perp \sim \Lambda$
  - the basis of the collinear factorisation
- ... but this can change ! : quantum fluctuations & high density effects
  - DGLAP and CSS evolutions (Collins, Soper, Sterman)

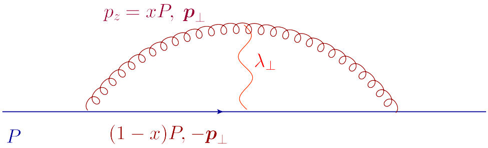
  ▷ particle production with net transverse momentum

  ▷ transverse-momentum dependent (TMD) distributions & factorisation
  - high energy evolution, large nucleus, gluon saturation

  ▷ dipole picture, k_T–factorisation, hybride factorisation, CGC
- The intrinsic $p_\perp$ enters many physical/technical arguments underlying the modern-day parton picture: the parton model + its quantum evolution

The lifetime of a fluctuation *(Ioffe time)*

- Even if boosted, parton fluctuations do still have a finite lifetime
- Δt: the lifetime of a quark-gluon fluctuation of a quark inside a hadron
- Maximal transverse separation $\sim$ gluon transverse wavelength $\lambda_\perp \sim 1/p_\perp$
  - if $\Delta x_\perp > \lambda_\perp \Longrightarrow$ quantum decoherence: the gluon can be emitted



$$\Delta x_\perp \sim \frac{p_\perp}{p_z}\,\Delta t \lesssim \lambda_\perp \sim \frac{2}{p_\perp}$$

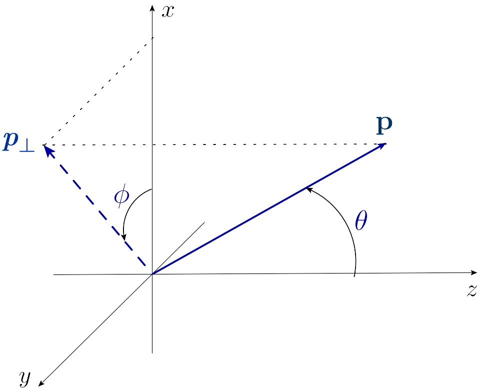
$$\Delta t \simeq \frac{2p_z}{p_\perp^2} = \frac{2xP}{p_\perp^2}$$

- Yet another argument: the uncertainty principle: $\Delta t = \frac{1}{\Delta E}$

$$\Delta E \equiv \sqrt{(xp_z)^2 + p_\perp^2} + \sqrt{((1-x)p_z)^2 + p_\perp^2} - p_z \simeq \frac{p_\perp^2}{2x(1-x)p_z}$$

Kinematics: Pseudo-rapidity

- Consider a particle with 3-momentum $\mathbf{p}$; take z as the collision axis
- Detectors: transverse momentum $\boldsymbol{p}_\perp$ & polar angle θ (or pseudo-rapidity η)



$$\mathbf{p} = (p_x, p_y, p_z) = (\boldsymbol{p}_\perp, p_z)$$

$$p_z = p\cos\theta,\ \ p_\perp = p\sin\theta$$

$$\eta \equiv \frac{1}{2}\ln\frac{p+p_z}{p-p_z} = -\ln\tan\frac{\theta}{2}$$

- $\theta \to 0 \ \Rightarrow\ \eta \to \infty$: forward
- $\theta = \frac{\pi}{2} \ \Rightarrow\ \eta = 0$: central
- $\theta \to \pi \ \Rightarrow\ \eta \to -\infty$: backward

- Exercice (easy): demonstrate that

$$p = m_\perp \cosh\eta\,, \quad p_z = m_\perp \sinh\eta\,, \quad \text{with} \quad m_\perp \equiv \sqrt{m^2 + p_\perp^2}$$

Kinematics: Momentum rapidity

- The momentum (or proper) rapidity y is also useful: boost-covariant

$$y \equiv \frac{1}{2}\ln\frac{E+p_z}{E-p_z} \quad \text{with} \quad E=\sqrt{m^2+p_\perp^2+p_z^2}$$

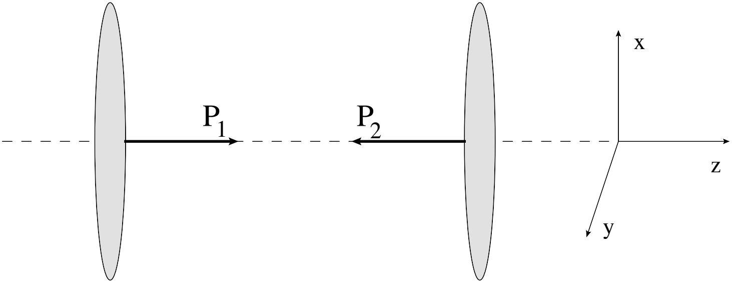
- y transforms via a shift under a Lorentz boost along z :

$$E \to \gamma(E+\beta p_z), \quad p_z \to \gamma(p_z+\beta E) \implies y \to y+\frac{1}{2}\ln\frac{1+\beta}{1-\beta}$$

- $y=\eta$ for massless (or ultrarelativistic) particles: $E \simeq p$
  - experimentally, it is more convenient to measure angles: θ, η
  - conceptually, y turns out to be more useful: $\Delta y =$ boost invariant
- These lectures: particles will always be ultrarelativistic : $y=\eta$
- Exercice: demonstrate that

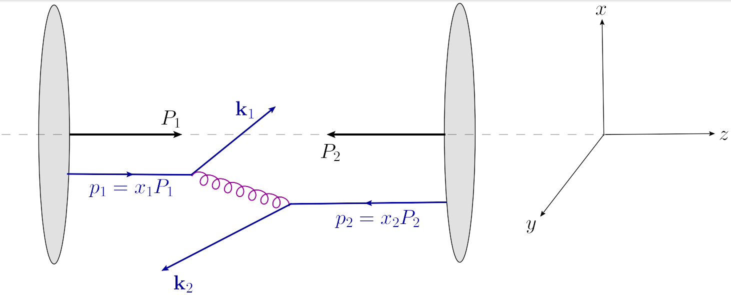
$$E=p_\perp \cosh y\,, \qquad p_z=p_\perp \sinh y$$

Kinematics: A hadron-hadron collision



- pp or nucleon–nucleon (NN) pair from a pA or AA collision
- z : longitudinal (or 'beam') axis; $\boldsymbol{x}_\perp = (x, y)$: transverse plane
- Center-of-mass frame : $P_1^\mu = (P, 0, 0, P), \quad P_2^\mu = (P, 0, 0, -P)$
  - high energy: particle masses are negligible: $E = \sqrt{P^2 + M^2} \simeq P$
  - huge boost factor $\gamma = E/M \sim 1000$ at the LHC: Lorentz contraction
- Center-of-mass energy squared : $s = (P_1 + P_2)^2 = 2P_1 \cdot P_2 = 4P^2$

A partonic subcollision



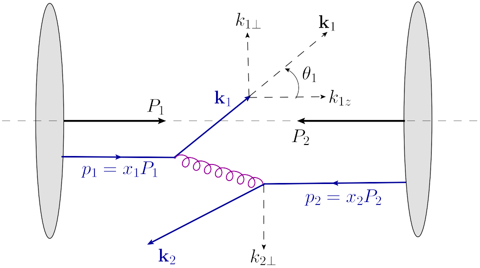
- To lowest order in perturbative QCD: a $2 \to 2$ subcollision
  - e.g. $q(p_1) + q(p_2) \;\to\; q(k_1) + q(k_2)$
- Initial partons assumed to be collinear with the incoming hadrons

$$p_1^\mu = x_1(P, 0, 0, P)\,, \qquad p_2^\mu = x_2(P, 0, 0, -P)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2 k_{1\perp}\mathrm{d}^2 k_{2\perp}\mathrm{d}y_1\mathrm{d}y_2} = \sum_{ij}\int \mathrm{d}x_1\mathrm{d}x_2 f_{i/1}(x_1,\mu^2) f_{j/2}(x_2,\mu^2)\,\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}^2 k_{1\perp}\mathrm{d}^2 k_{2\perp}\mathrm{d}y_1\mathrm{d}y_2}$$

Initial parton kinematics

- The longitudinal fractions x_1 and x_2 of the incoming partons are fixed by the kinematics of the final state, via energy-momentum conservation



$$k_i^\mu = (E_i, \boldsymbol{k}_{i\perp}, k_{iz}), \;\; i = 1, 2$$

$$\boldsymbol{k}_{1\perp} + \boldsymbol{k}_{2\perp} = 0$$

$$E_1 + E_2 = (x_1 + x_2)P$$

$$k_{1z} + k_{2z} = (x_1 - x_2)P$$

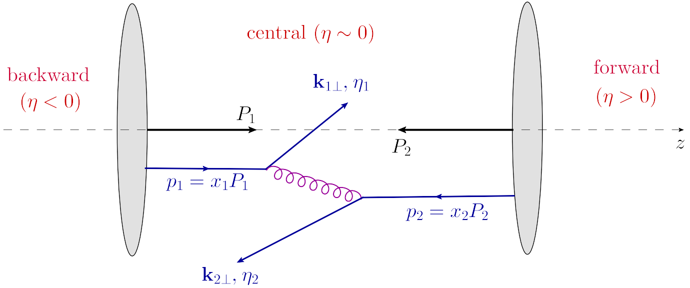
$$E_i = k_\perp \cosh \eta_i, \;\; k_{iz} = k_\perp \sinh \eta_i$$

$$x_1 \; = \; \frac{k_\perp}{\sqrt{s}} \left(\mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}\right), \quad x_2 \; = \; \frac{k_\perp}{\sqrt{s}} \left(\mathrm{e}^{-\eta_1} + \mathrm{e}^{-\eta_2}\right)$$

- Forward rapidities, $\eta_1, \, \eta_2 \gtrsim 1$, probe large x_1, but small x_2

- Vice versa for the backward rapidities

Collinear factorization

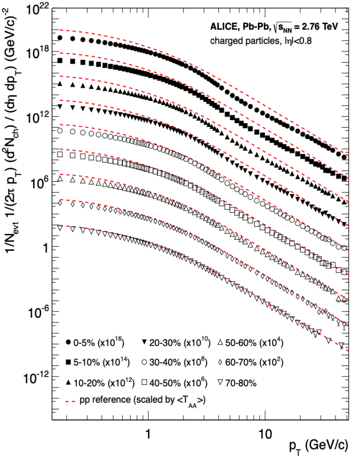


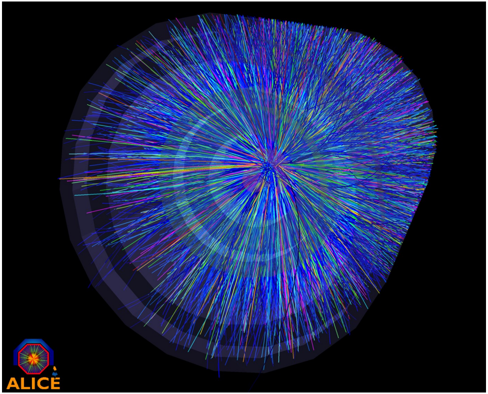
$$\frac{d\sigma}{d^2k_\perp d\eta_1 d\eta_2} = \sum_{ij} x_1 f_i(x_1, \mu^2)\, x_2 f_j(x_2, \mu^2)\, \frac{d\hat{\sigma}_{ij}}{dk_\perp^2}$$

- μ^2 : factorization scale (of order $k_\perp^2 \gg \Lambda_{\rm QCD}^2$)
- Leading-order pQCD: $\frac{d\hat{\sigma}_{ij}}{dk_\perp^2} \sim \frac{\alpha_s^2}{k_\perp^4} \Longrightarrow$ favours particles with low $k_\perp$
- The total cross-section (integrated over $k_\perp$) appears to be divergent

Charged hadron multiplicity in AA collisions

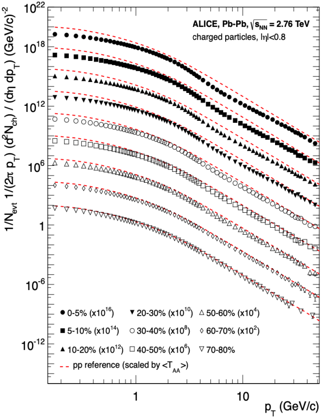
- p_T spectrum of charged particles produced in Pb+Pb at midrapidities and in different bins of centrality (arXiv:12082711, ALICE)





The importance of small-x partons

- Pb+Pb @ the LHC: p_T spectrum at midrapidities (arXiv:12082711, ALICE)
  - 99% of the total multiplicity lies at low momenta, below $p_\perp = 2$ GeV



$$\left.\frac{\mathrm{d}N_{\rm ch}}{\mathrm{d}^2 p_\perp \mathrm{d}\eta}\right|_{\eta=0}$$

- Take $p_\perp = 1$ GeV and $\eta = 0$
- $x \sim 5 \times 10^{-3}$ at RHIC ($\sqrt{s} = 200$ GeV)
- $x \sim 2 \times 10^{-4}$ at the LHC ($\sqrt{s} = 5$ TeV)
- Nuclei liberate thousands of partons with small $x \ll 1$
- Consistent with energy conservation

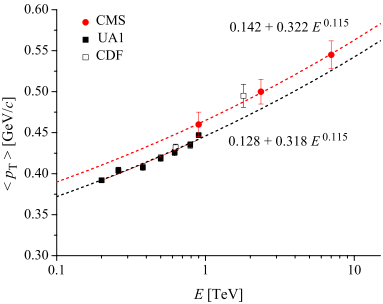
- Also consistent with parton distributions measured in DIS at HERA

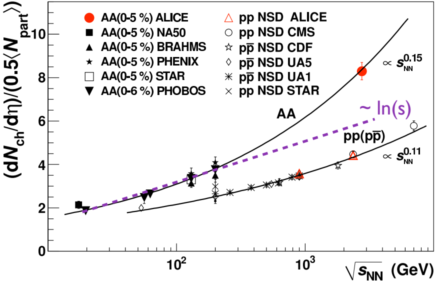
LHC: average $p_\perp$ and multiplicity

- Multiplicity: divergent in collinear fact. ... but measured in the experiments

$$\frac{\mathrm{d}N_{\mathrm{ch}}}{\mathrm{d}\eta} = \int \mathrm{d}^2p_\perp \, \frac{\mathrm{d}N_{\mathrm{ch}}}{\mathrm{d}^2p_\perp \mathrm{d}\eta}$$

- "Controlled by $p_\perp \sim \Lambda_{\mathrm{QCD}} \sim 200$ MeV" ?? Not true, according to the data!





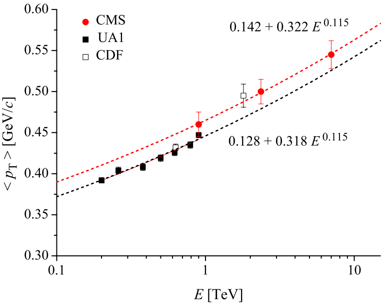
- Data suggests intrinsic $p_\perp$ which grows like a power of the energy

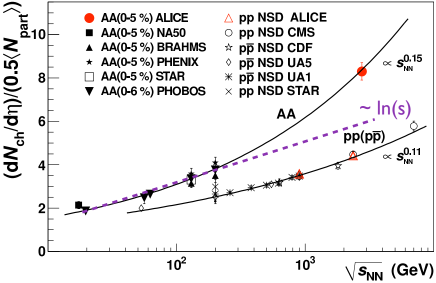
LHC: average $p_\perp$ and multiplicity

- Multiplicity: divergent in collinear fact. ... but measured in the experiments

$$\frac{\mathrm{d}N_{\mathrm{ch}}}{\mathrm{d}\eta} = \int \mathrm{d}^2 p_\perp \, \frac{\mathrm{d}N_{\mathrm{ch}}}{\mathrm{d}^2 p_\perp \mathrm{d}\eta}$$

- Average $p_\perp$ in pp collisions: $\langle p_\perp \rangle \sim E^{0.115}$ *(McLerran, Praszalowicz, 2010)*



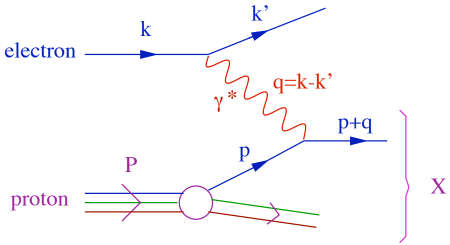


- Charged particle multiplicity in pp and AA: power law increase with s_{NN}

Deep Inelastic Scattering @ HERA & the EIC

Deep inelastic scattering

- Electron-proton (ep) collisions mediated by a virtual photon (γ^*)



$$q^2 = (k-k')^2 = -2EE'(1-\cos\alpha) \; < \; 0$$

$$\hat{s} \equiv (P+q)^2 = M^2 - Q^2 + 2P\cdot q$$

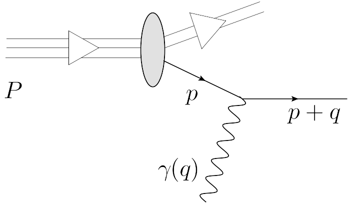
- 2 useful invariants:

$$Q^2 \equiv -q^2, \quad x_{\rm Bj} \equiv \frac{Q^2}{2P\cdot q}$$

- These 2 invariants control the resolution(s) of the virtual photon:
  - γ^* couples to quarks with longitudinal momentum fraction $x = x_{\rm Bj}$ and with transverse momenta $k_\perp^2 \le Q^2$
- A fine probe of the parton distribution functions (PDFs) in the proton
- Electron-Ion Collider ($\sim$ 2035): similar measurements of the nuclear PDFs

Longitudinal resolution: small x

- Work in a target infinite momentum frame: $P^\mu = (P, 0, 0, P)$
- Struck quark (roughly) collinear with the proton: $p^\mu = xP^\mu$
- The quark is on shell both before and after absorbing the photon



$$p^2 = (p+q)^2 = 0$$

$$0 = 2p\cdot q + q^2 = 2xP\cdot q - Q^2$$

$$x = \frac{Q^2}{2P\cdot q} = x_{\rm Bj}$$

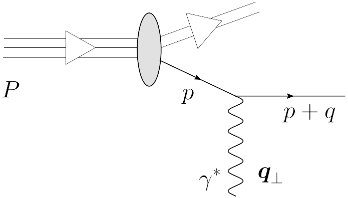
- High DIS energy: $\hat{s} \simeq 2P\cdot q \gg Q^2 \gtrsim M^2 \iff$ small $x \ll 1$

Transverse resolution: Bjorken frame

- One needs to more carefully specify the kinematics & and the frame
- Parton intrinsic $p_\perp$ plays a role: $p^\mu = (xP, \boldsymbol{p}_\perp, xP)$
- A target infinite momentum frame $P^\mu = (P, 0, 0, P)$ ... but which one ?
- Bjorken frame: the photon carries "only" transverse momentum $\boldsymbol{q}_\perp$

$$q^\mu = (q_0 \simeq 0,\, \boldsymbol{q}_\perp, q_z = 0), \quad Q^2 \simeq q_\perp^2$$

- The photon is absorbed over a transverse distance $\Delta x_\perp \sim 1/Q$
- The quark transverse size $\lambda_\perp \sim 1/p_\perp$ must be larger



$$\Delta x_\perp \sim \frac{1}{Q} \lesssim \lambda_\perp \sim \frac{1}{p_\perp}$$

- γ^* couples to quarks having

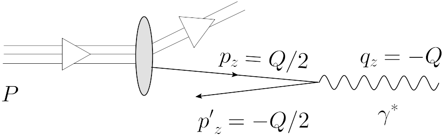
$$p_\perp^2 \lesssim Q^2$$

Transverse resolution: Breit frame

- Parton intrinsic $p_\perp$ plays a role: $p^\mu = (xP, \boldsymbol{p}_\perp, xP)$
- Breit frame: the photon carries only longitudinal momentum q_z

$$q^\mu = (0,0,0,q_z=-Q), \quad x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{2PQ} \;\Rightarrow\; xP = \frac{Q}{2}$$

- The photon is absorbed over a time $\Delta t_{\rm coll} \sim 1/Q$
- The lifetime Δt_q of the quark fluctuation must be larger



$$\Delta t_q \sim \frac{2xP}{p_\perp^2} \gtrsim \Delta t_{\rm coll} \sim \frac{1}{Q}$$

- γ^* couples to quarks having

$$p_\perp^2 \lesssim Q^2$$

- After collision, the struck quark propagates in the photon direction:

$$p'_z = p_z + q_z = \frac{Q}{2} - Q = -\frac{Q}{2}$$

Parton distributions

- DIS allows us to measure the parton distributions: $f_i(x,Q^2)$, $i=q,\bar{q},g$
- The number of partons with longitudinal momentum fraction x and any transverse momentum $p_\perp \lesssim Q$:

$$x f_i(x,Q^2) = \int^Q \mathrm{d}^2 p_\perp \, x \frac{\mathrm{d}N_i}{\mathrm{d}x\mathrm{d}^2 p_\perp} = \begin{cases} xq(x,Q^2) & \text{for } i=q, \\ xG(x,Q^2) & \text{for } i=g\,. \end{cases}$$

- The cross-section for virtual photon absorption by the proton:

$$\sigma_{\gamma^* p}(x,Q^2) = \frac{4\pi^2 \alpha_{\rm em}}{Q^2} \underbrace{\sum_{f=u,d,s\ldots} e_f^2 \left[x q_f(x,Q^2) + x\bar{q}_f(x,Q^2)\right]}_{F_2(x,Q^2)}$$

- Valid to leading order in $1/Q^2$ for $Q^2 \gg \Lambda^2$ ("leading twist")
- Gluons are indirectly measured, via their effect on quark distribution

Parton evolution in QCD

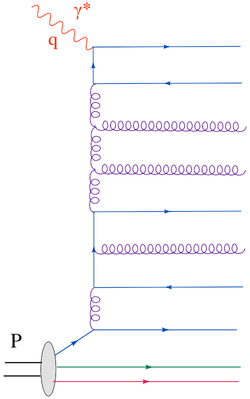
- Quantum evolution: change in the partonic content when changing the resolution scales x and Q^2, due to additional radiation
- Perturbative QCD applicable for sufficiently large $Q^2 \gg \Lambda^2$
- Higher-order corrections, but enhanced by large kinematical logarithms

  - transverse logarithms: DGLAP

    powers of $\alpha_s \ln \frac{Q^2}{\Lambda^2}$

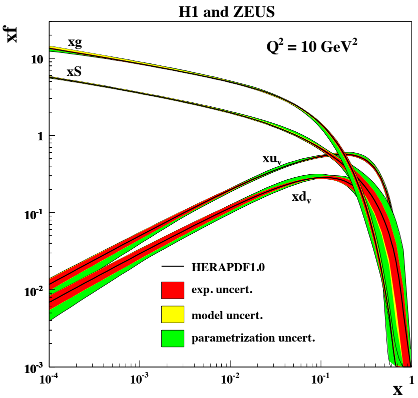
  - small-x: BFKL → BK/JIMWLK

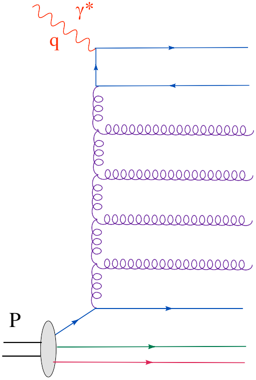
    powers of $\alpha_s \ln \frac{1}{x}$



Parton distributions at HERA

- Fits to $F_2(x, Q^2)$ using the DGLAP evolution

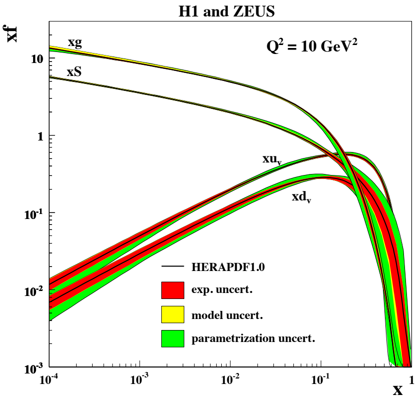


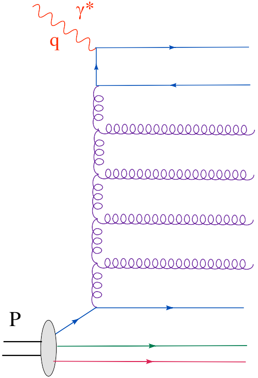


- For $x \leq 0.01$ the hadron wavefunction contains mostly gluons !
  - the virtual photon is absorbed by a sea quark $(g \to q\bar{q})$

Parton distributions at HERA

- Fits to $F_2(x, Q^2)$ using the DGLAP evolution





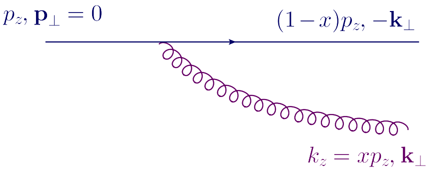
- Gluon distribution measured at HERA rises roughly like a power of $1/x$:

$$xG(x, Q^2) \propto \frac{1}{x^\lambda} \quad \text{with} \quad \lambda \simeq 0.20 \div 0.25$$

- Can one understand this rise from (perturbative) QCD ?

Bremsstrahlung

- A quark (e.g. a valence quark from the proton) emits a gluon with longitudinal momentum fraction $x \leq 1$, and transverse momentum $k_\perp$



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s}{2\pi} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} P_{q\to g}(x)\mathrm{d}x$$

$$P_{q\to g}(x) \equiv C_F \frac{1+(1-x)^2}{x}$$

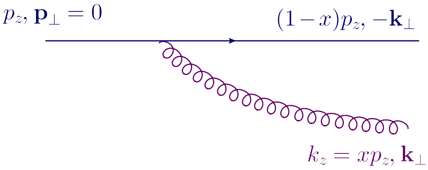
- Logarithmic enhancement for large-$k_\perp$ emissions $(p^2 \sim \Lambda^2 < k_\perp^2 < Q^2)$:

$$\int_{\Lambda^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \ln \frac{Q^2}{\Lambda^2}$$

- ... and also for soft/low-energy $(x \to 0)$ gluons: $P_{q\to g}(x) \simeq 2C_F/x$

$$\int_{x_0}^{1} \frac{\mathrm{d}x}{x} = \ln \frac{1}{x_0} \equiv Y_0$$

- A quark (e.g. a valence quark from the proton) emits a gluon with longitudinal momentum fraction $x \leq 1$, and transverse momentum $k_\perp$



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s}{2\pi}\frac{\mathrm{d}k_\perp^2}{k_\perp^2} P_{q\to g}(x)\mathrm{d}x$$

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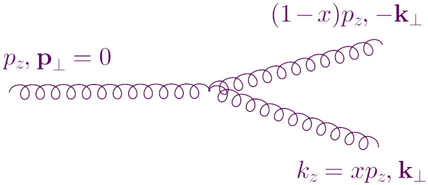
$$\int_{\Lambda^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \ln\frac{Q^2}{\Lambda^2}$$

- Emissions of soft quarks are not enhanced: $\xi \equiv 1 - x \ll 1$

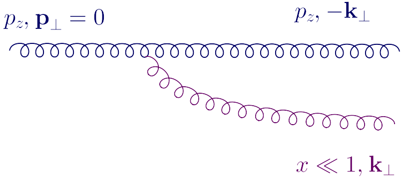
$$P_{q\to q}(\xi) = P_{q\to g}(x = 1-\xi) = C_F \frac{1+\xi^2}{1-\xi}$$



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s}{2\pi}\,\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\,P_{g\to g}(x)\mathrm{d}x$$

$$P_{g\to g}(x) \equiv 2N_c\,\frac{[1-x(1-x)]^2}{x(1-x)}$$

- Logarithmic singularities for both $x \to 0$ and $x \to 1$
  - symmetry: $x \leftrightarrow 1-x \;\Rightarrow$ choose $x \ll 1$ and multiply by 2

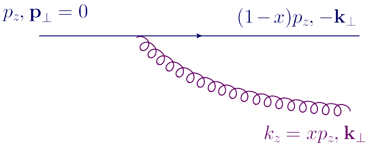


$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s N_c}{\pi}\,\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\,\frac{\mathrm{d}x}{x}$$

- Soft gluons can act as sources for even softer ones: high-energy evolution

$$xG(x,Q^2) = \int^Q \mathrm{d}^2\boldsymbol{k}_\perp \, x \frac{\mathrm{d}N_{\rm gluon}}{\mathrm{d}x\mathrm{d}^2k_\perp}$$

- To leading order in α_s: single (soft) gluon emission by a quark



$$\frac{\mathrm{d}N^{(0)}_{\rm gluon}}{\mathrm{d}x\mathrm{d}^2k_\perp} = \frac{\alpha_s C_F}{\pi}\frac{1}{x}\frac{1}{k_\perp^2}$$

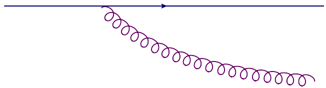
- "unintegrated gluon distribution"

  - number of gluons with fixed values for both the longitudinal momentum (xP) and the transverse momentum ($k_\perp$)

Gluon distribution at small x

$$xG^{(0)}(x,Q^2) = \frac{\alpha_s C_F}{\pi}\int_{\Lambda^2}^{Q^2}\frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \frac{\alpha_s C_F}{\pi}\ln\frac{Q^2}{\Lambda^2}$$

- To leading order in α_s: single (soft) gluon emission by a quark



$$\frac{\mathrm{d}N^{(0)}_{\rm gluon}}{\mathrm{d}x\mathrm{d}^2k_\perp} = \frac{\alpha_s C_F}{\pi}\,\frac{1}{x}\,\frac{1}{k_\perp^2}$$

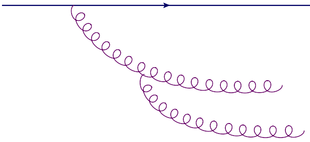
- "unintegrated gluon distribution"

  - number of gluons with fixed values for both the longitudinal momentum (xP) and the transverse momentum ($k_\perp$)

- The leading-order gluon distribution $xG^{(0)}(x,Q^2)$:
  - the first 'transverse' logarithm of the DGLAP evolution
  - independent of x

BFKL evolution *(Balitsky, Fadin, Kuraev, Lipatov, 1974-78)*

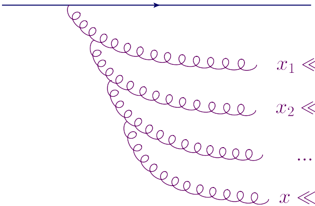
- The x-dependence enters via the quantum evolution
- Two successive gluon emissions, strongly ordered in x



- the "price" of the additional gluon

$$\alpha_s \int_x^1 \frac{dx_1}{x_1} = \alpha_s \ln \frac{1}{x}$$

- $Y \equiv \ln(1/x)$: rapidity difference between parent quark and final gluon
- When $\alpha_s Y \sim \mathcal{O}(1) \Rightarrow$ need for resummation: arbitrary many emissions



- a n–gluon cascade:

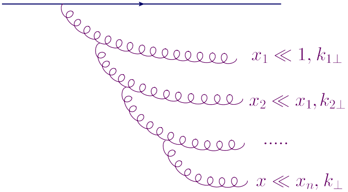
$$\frac{1}{n!}\left(\bar{\alpha} Y\right)^n, \quad \bar{\alpha} \equiv \frac{\alpha_s N_c}{\pi}$$

- sum over $n \Rightarrow$ an exponential:

$$xG(x, Q^2) \propto e^{\lambda \bar{\alpha} Y}$$

Coherence in the parton cascades

- The parton cascades in pQCD feature quantum coherence
  - partons overlap in time and possibly also in space
- Successive parton emissions are strongly ordered in lifetimes



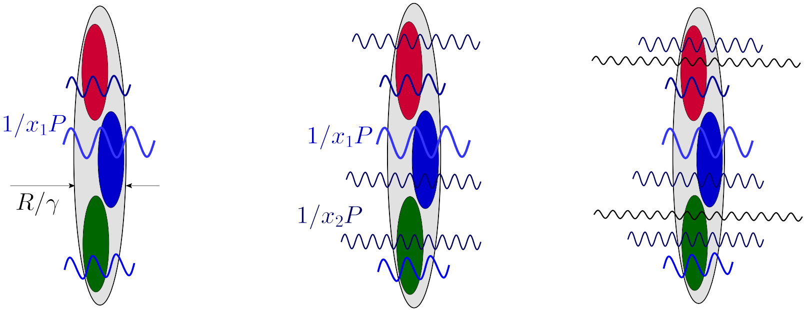
$$\Delta t_n \sim \frac{2x_n P}{k_{n\perp}^2}$$

$$\Delta t_n \ll \Delta t_{n-1}$$

- BFKL cascades: $x_n \ll x_{n-1}$, but no ordering in $k_\perp$
- DGLAP cascades: $k_{n\perp}^2 \gg k_{(n-1)\perp}^2$, but no strong ordering in x
- Yet, the physical consequences are very different in the two cases !

Overlapping gluons

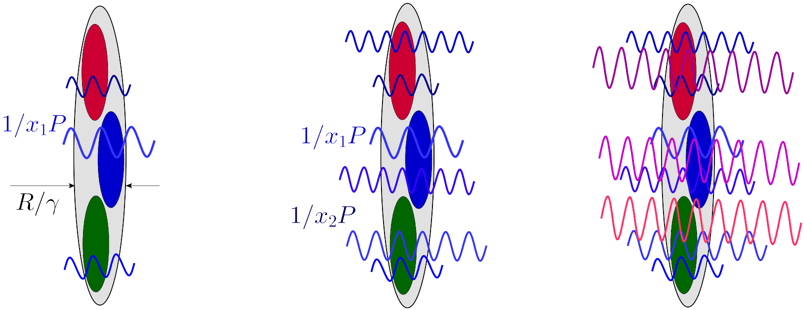
- A gluon w/ transverse momentum $\boldsymbol{k}_\perp$ and longitudinal momentum $k_z = xP$
  - occupies a transverse area $\Delta x_\perp^2 \sim 1/k_\perp^2$
  - and has a longitudinal extent $\Delta z \sim 1/(xP)$



- DGLAP evolution maintains a dilute system of partons
  - rapid decrease in their transverse sizes $\Longrightarrow$ no possible overlap

Overlapping gluons

- A gluon w/ transverse momentum $\boldsymbol{k}_\perp$ and longitudinal momentum $k_z = xP$
  - occupies a transverse area $\Delta x_\perp^2 \sim 1/k_\perp^2$
  - and has a longitudinal extent $\Delta z \sim 1/(xP)$



- BFKL evolution leads to an increasing density
  - gluons which overlap can interact

Gluon occupancy

- Gluons mutual interactions are measured by their occupation number

$$n(x, \boldsymbol{k}_\perp, \boldsymbol{x}_\perp) \equiv \frac{(2\pi)^3}{2(N_c^2-1)}\, x \frac{\mathrm{d}N_{\rm gluon}}{\mathrm{d}x\, \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{x}_\perp}$$

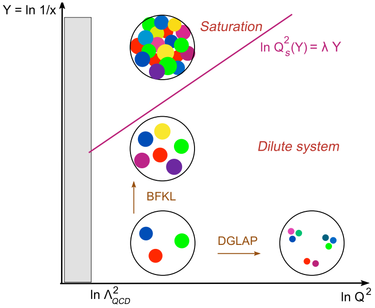
  - the unintegrated gluon distribution per unit transverse area

  - a simple estimate

$$n(x, Q^2) \simeq \frac{1}{Q^2} \times \frac{xG(x, Q^2)}{\pi R^2}$$

  - HERA data suggest

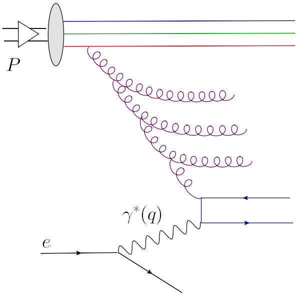
$$n(x, Q^2) \sim \frac{1}{x^\lambda} \text{ with } \lambda \simeq 0.2$$



- When $n \gtrsim 1$, gluons overlap, but their interactions are still suppressed by α_s
- Interactions become of $\mathcal{O}(1)$ when $n \sim 1/\alpha_s$

Non-linear evolution towards saturation

- When $n \ll 1/\alpha_s$, interactions are weak $\Longrightarrow$ linear evolution



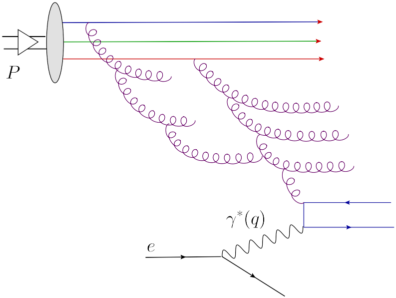
- BFKL: $Y \equiv \ln(1/x)$

$$\frac{\partial n}{\partial Y} = \omega\alpha_s n \quad \Longrightarrow \quad n \propto \mathrm{e}^{\omega\alpha_s Y}$$

- Rapid gluon multiplication: $g \to gg$

Non-linear evolution towards saturation

- When $n \ll 1/\alpha_s$, interactions are weak $\Longrightarrow$ linear evolution



- Non-linear evolution: BK-JIMWLK

$$\frac{\partial n}{\partial Y} = \omega \alpha_s n - \alpha_s^2 n^2$$

- Non-linear term taming the growth

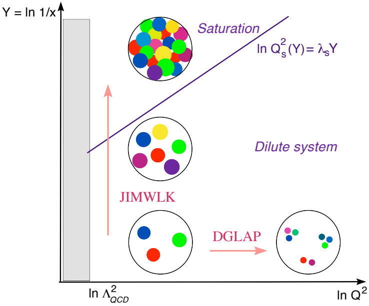
- When $n \sim 1/\alpha_s$: non-linear effects like gluon recombination $gg \to g$
- A functional equation $\Longleftrightarrow$ an infinite hierarchy for multi-gluon correlations

(Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, and Kovner, 97–00)

  - mean field approximation $\Rightarrow$ a closed equation: *Balitsky-Kovchegov*

Non-linear evolution towards saturation

- When $n \ll 1/\alpha_s$, interactions are weak $\Longrightarrow$ linear evolution



- Non-linear evolution: BK-JIMWLK

$$\frac{\partial n}{\partial Y} = \omega \alpha_s n - \alpha_s^2 n^2$$

- Saturation fixed point:

$$\frac{\partial n}{\partial Y} \sim 0 \quad \text{when } n \sim \frac{1}{\alpha_s}$$

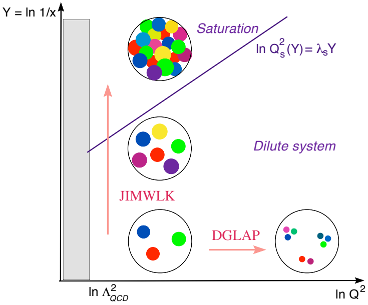
- Saturation momentum: $n(x, Q^2) \sim 1/\alpha_s$ when $Q^2 = Q_s^2(x)$

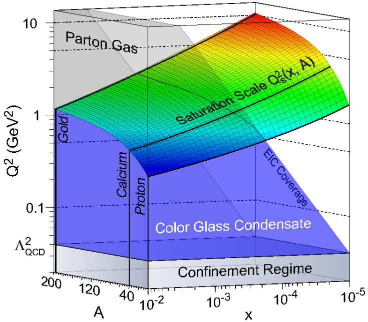
$$Q_s^2(x) \simeq \alpha_s \frac{xG(x, Q_s^2)}{\pi R^2} \sim \frac{1}{x^{\lambda_s}} \quad \text{with} \quad \lambda_s \simeq 0.2 \div 0.3$$

- Saturation exponent $\lambda_s \sim 0.2$: confirmed by NLO studies of BK equation

- For a large nucleus ($A \gg 1$): additional enhancement $\sim A^{1/3}$

$$Q_s^2(x,A) \simeq \alpha_s \frac{xG_A(x,Q_s^2)}{\pi R_A^2} \sim \frac{A^{1/3}}{x^{\lambda_s}}$$



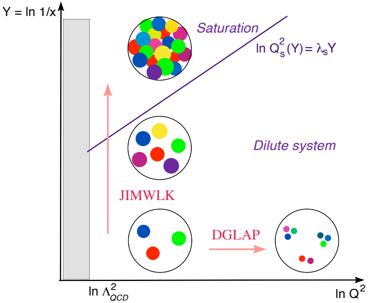


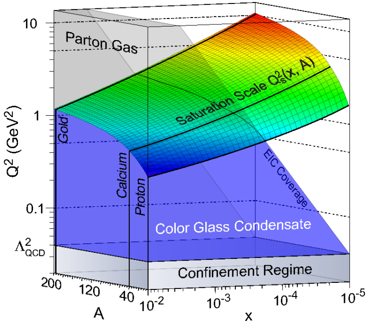
- $x \sim 10^{-3}$ (EIC): $Q_s^2 \sim 2$ GeV^2 for Pb or Au
- $x \sim 10^{-5}$ (LHC): $Q_s^2 \sim 10$ GeV^2 for Pb and ~ 1 GeV^2 for a proton

Saturation momentum

- For a large nucleus ($A \gg 1$): additional enhancement $\sim A^{1/3}$

$$Q_s^2(x,A) \simeq \alpha_s \frac{xG_A(x,Q_s^2)}{\pi R_A^2} \sim \frac{A^{1/3}}{x^{\lambda_s}}$$



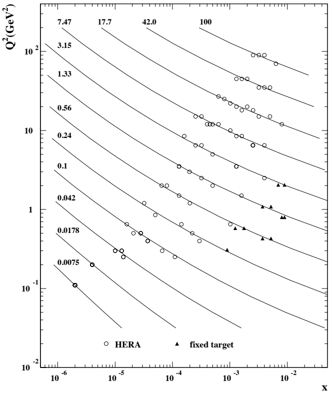


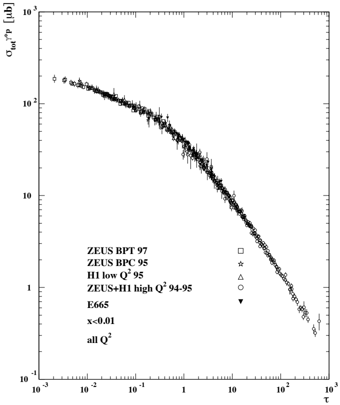
- $Q_s(x,A)$: typical transverse momentum for the gluons with a given $x \ll 1$
- One can (at least, marginally) study gluon saturation in pQCD

Geometric scaling at HERA

- DIS cross-section $\sigma(x, Q^2)$ is *a priori* a function of 2 variables
- At small x, proton structure involves one intrinsic scale $Q_s(x)$

  $\Longrightarrow$ physics should depend upon the ratio $Q^2/Q_s^2(x)$: geometric scaling

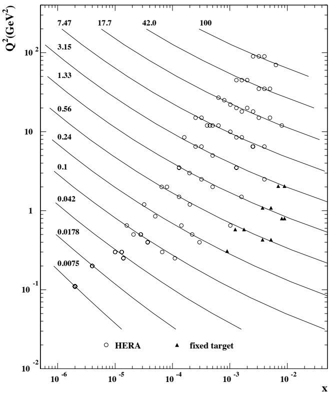


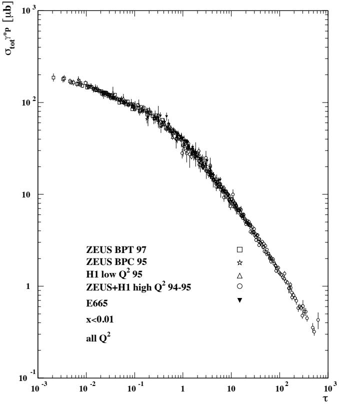


Geometric scaling at HERA

- DIS cross-section $\sigma(x, Q^2)$ is *a priori* a function of 2 variables
- DIS cross–section at HERA *(Staśto, Golec-Biernat, Kwieciński, 2000)*

  $\sigma(x, Q^2)$ vs. $\tau \equiv Q^2/Q_s^2(x) \propto Q^2 x^{0.3}$: $x \le 0.01,\quad Q^2 \le 450$ GeV^2

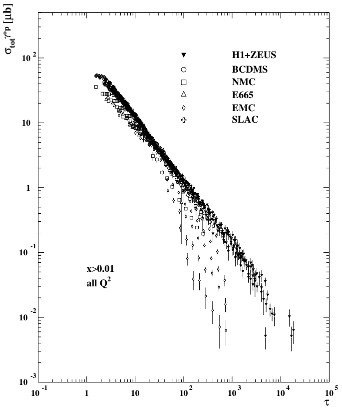


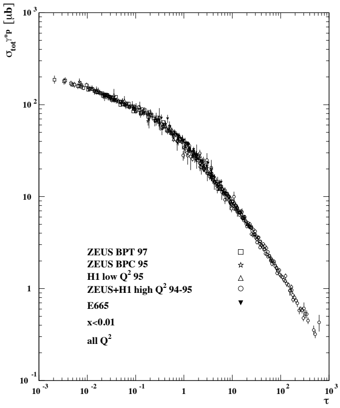


- Left: data in (x, Q^2) plane. Right: cross-section as a function of τ

Geometric scaling at HERA

- DIS cross-section $\sigma(x, Q^2)$ is *a priori* a function of 2 variables





- No scaling for the HERA data corresponding to larger values $x > 0.01$

- Theory (CGC): *E.I., Itakura, McLerran, '02; Mueller, Triantafyllopoulos, '02*

Multiplicity : pp, pA, AA

- Particle multiplicity $\mathrm{d}N/\mathrm{d}\eta$: number of hadrons per unit rapidity near $\eta = 0$
- Saturated gluons which are released by the collision and hadronise

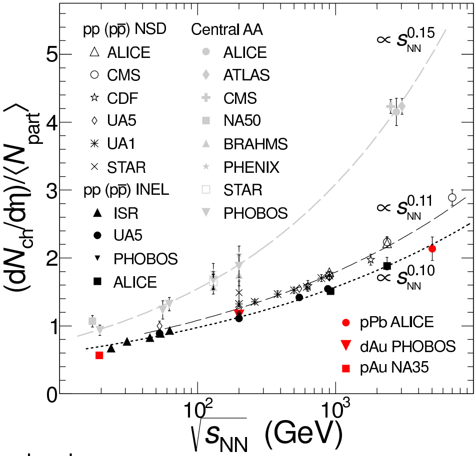
$$\frac{\mathrm{d}N}{\mathrm{d}\eta} \propto xG(x, Q_s^2) \propto Q_s^2(x)$$

▷ which value for x ?

$$x \simeq \frac{k_\perp}{\sqrt{s}} \quad \& \quad k_\perp \sim Q_s$$

$$Q_s^2(x) \propto \frac{1}{x^{\lambda_s}} \sim s^{\frac{\lambda_s}{2+\lambda_s}}$$

$$\lambda_s \simeq 0.2 \div 0.3$$



- Qualitatively consistent with the data