Fundação para a Ciência e a Tecnologia

LABORATÓRIO DE INSTRUMENTAÇÃO E FÍSICA EXPERIMENTAL DE PARTÍCULAS

BFKL and low-x physics Lecture 2

(26-06-2024)

Grigorios Chachamis, LIP Lisbon

Midsummer School in QCD 2024
Saariselkä, 24 June to 6 July 2024, Finland

In the QCD era

- Let us leave the old Regge theory for now, we keep in our discussion though terms like "**Regge Limit**", "**trajectory**", "**power-like rise with energy**"
- We believe that QCD is the fundamental theory for hadronic collisions
- What does QCD have to say about the rise of the scattering amplitudes at high energies
- Does the Pomeron fit within pQCD or is it of non-perturbative nature?
- The answers (or attempts to answers) to the above started in the 70's

In the early days of QCD

- $A(s,t) \sim \beta(t) s^{\alpha(t)}$ Regge trajectory
- The Pomeron couples to quarks as a photon (?)
- Donnachie and Landshoff (1979, 1984) fitted

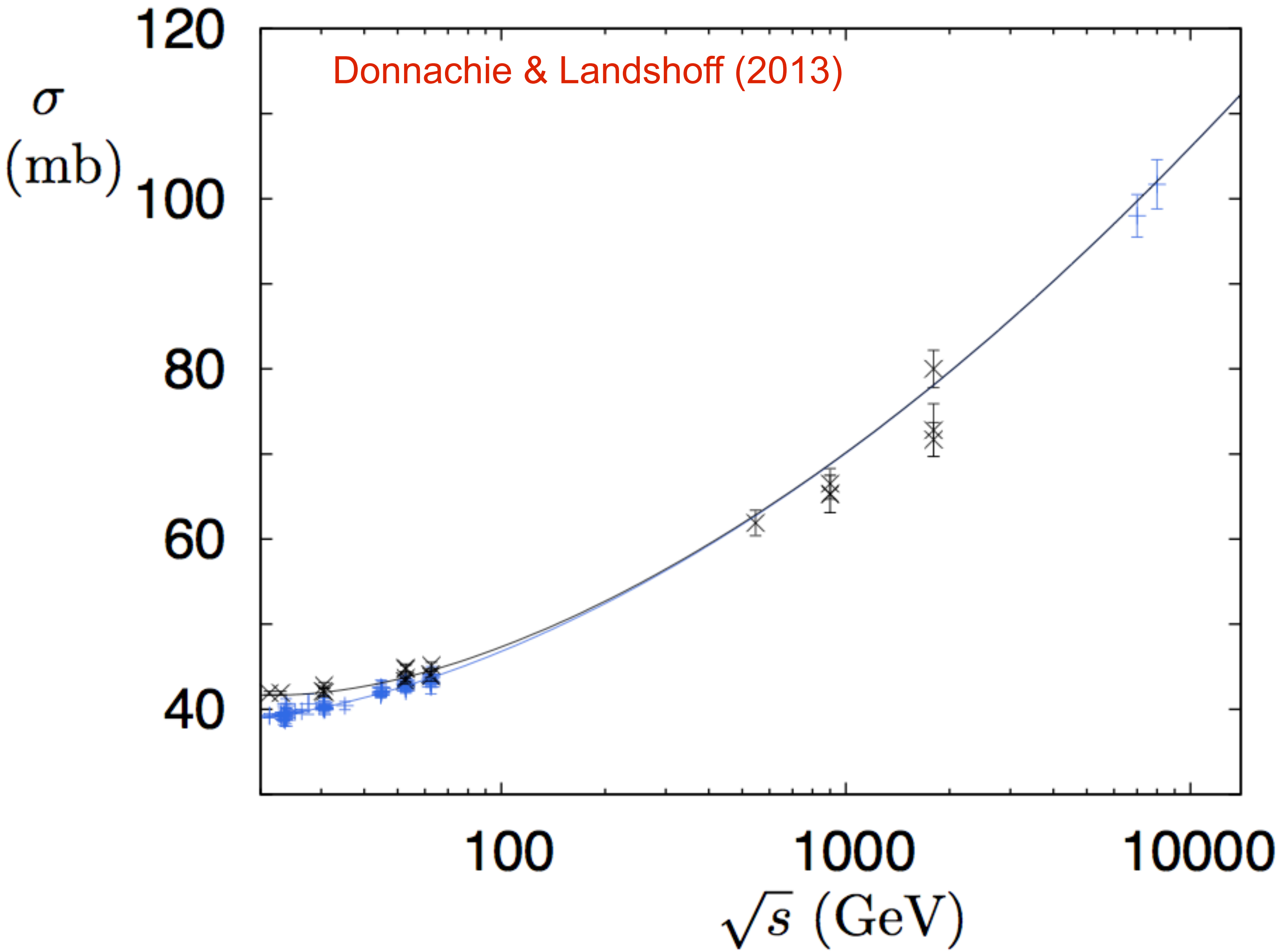
$$\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}t} = \frac{g^4 \left[3F_1(t)\right]^4}{4\pi \sin^2\left(\frac{\pi\alpha_{\mathbb{P}}(t)}{2}\right)} \left(\frac{s}{s_0}\right)^{2\alpha_{\mathbb{P}}(t)-2}$$

to the available pp data and

obtained $\alpha_{\mathbb{P}}(0) = 1.08, \quad \alpha'_{\mathbb{P}} = 0.25\mathrm{GeV}^{-2}, \quad g^4 = 3.21\mathrm{GeV}^{-2}$

- Soft Pomeron
- Low and Nussinov (1975, 1976) proposed to picture Pomeron

as a two-gluon exchange


Donnachie & Landshoff (2013)
σ (mb)
120
100
80
60
40
100
1000
10000
√s (GeV)

The QCD perturbative Pomeron

The Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

This big adventure started almost 50 years ago

L. N. Lipatov, Sov. J. Nucl. Phys. **23** (1976) 338; **1464 citations**

E. A. Kuraev, L. N. Lipatov, V. S. Fadin, Phys. Lett. B **60** (1975) 50, Sov. Phys. JETP **44** (1976) 443, Sov. Phys. JETP **45** (1977) 199. **3670 citations**

Ia. Ia. Balitsky, L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822. **4056 citations**

What we will cover and what not

- Reggeization of the gluon
- LO BFKL equation
- Intro to methodology
- NLO BFKL
- Impact factors
- Issues/problems
- Phenomenology

- CCFM
- BK
- CGC
- Saturation
- … and many more

The aim is to show you that this is “good physics”. If still things are not so clear, please do not despair, many great people found it hard in the beginning.

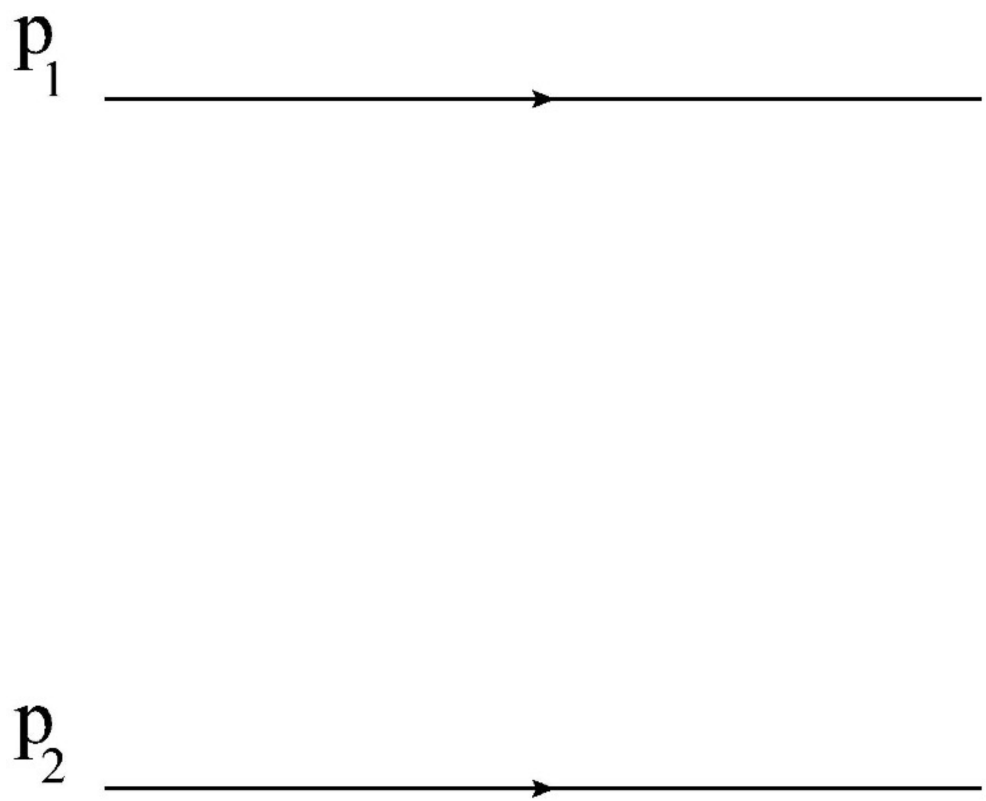
BFKL how to, step 0

Start from the simplest q q scattering, with momenta p_1 and p_2

Remember that you will have to see an power-like rise to cross sections

You will be hunting logarithms, in particular, $\ln s$

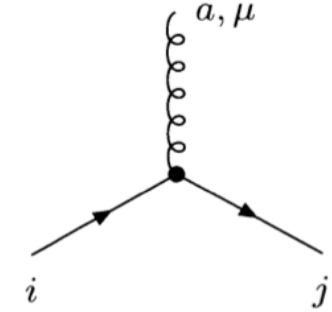
The last two points are interconnected as we will soon see



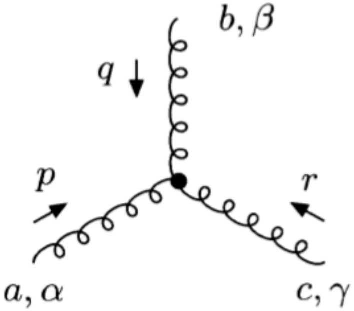
Some considerations before resummation

- Q: What is the most relevant scale in high energy scattering?
- A: The center-of-mass energy squared s
- Q: In which functional form does s appear in the Feynman diagrams?
- A: $\alpha_s^m (\ln s)^n$ If m=n, $\alpha_s^n (\ln s)^n$, leading logarithmic approximation LLA
- Q: Can one isolate those Feynman diagrams that come with a numerically important $[\alpha_s^m (\ln s)^n \sim 1]$ contribution?
- A: It depends (for this lecture the answer is yes)
- Q: Can one resum all these diagrams with important $\alpha_s^m \ln(s)^n$ contributions to all orders in α_s?
- A: It depends (for this lecture the answer is yes)

Feynman rules for QCD

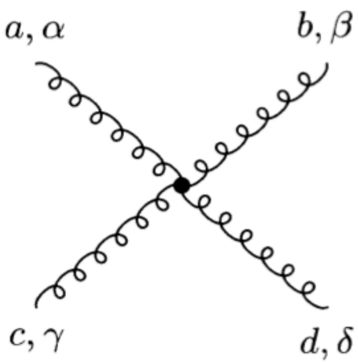


$$-\mathrm{i}g_s(t^a)_{ji}\gamma^\mu$$

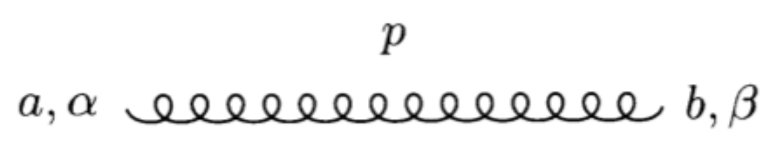


$$-g_s f_{abc}[(p-q)^\gamma g^{\alpha\beta} + (q-r)^\alpha g^{\beta\gamma} + (r-p)^\beta g^{\gamma\alpha}]$$

$$(p+q+r=0)$$



$$-\mathrm{i}g_s^2\{f_{eac}f_{ebd}[g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}]$$
$$+f_{ead}f_{ebc}[g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}]$$
$$+f_{eab}f_{ecd}[g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}]\}$$



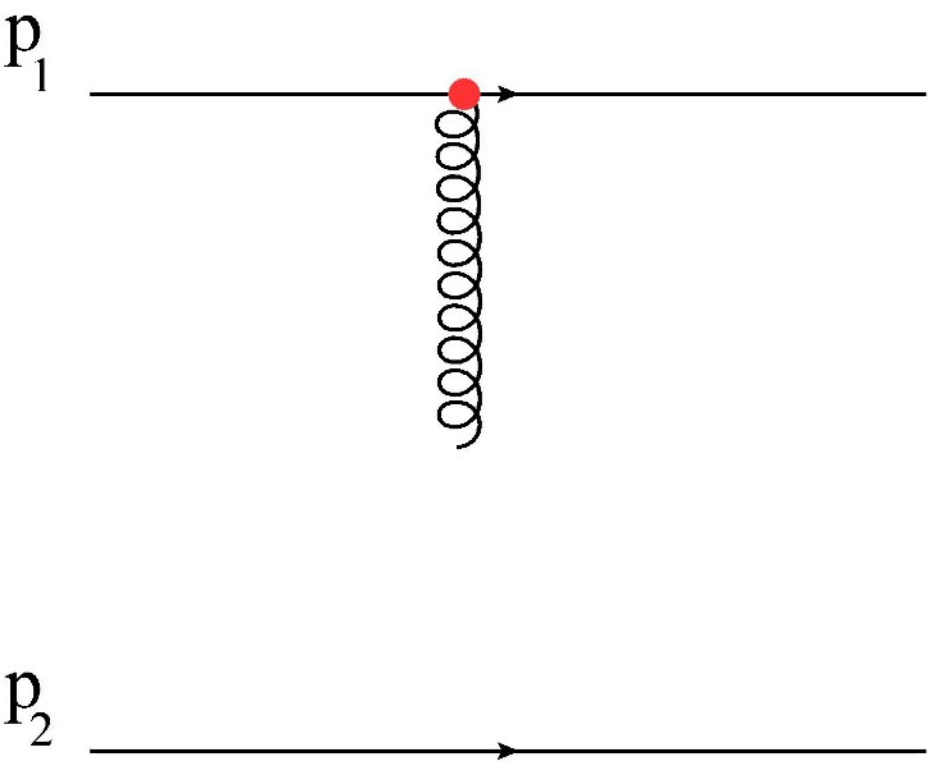
$$\Gamma^{ab}_{\alpha\beta}(p) = \delta^{ab}\left[-g_{\alpha\beta} + (1-\eta)\frac{p_\alpha p_\beta}{p^2+\mathrm{i}\epsilon}\right]\frac{\mathrm{i}}{p^2+\mathrm{i}\epsilon}$$

BFKL how to, still step 0

Assume that whatever is exchanged to the t-channel has mainly transverse components and also that it is much smaller than s.
Actually the kinematical limit we are working in is s >> |t|, u~ -s

Then the quark-gluon vertex can be written as

$$-ig_s\bar{u}(p_1)\gamma_\mu u(p_1) = -2ig_s p_1^\mu$$



Sudakov parametrization

$$A^\mu = (A^0, A^1, A^2, A^3) = (A^0, \boldsymbol{A}_\perp, A^3) = (A^0, \boldsymbol{A})$$

$$A^\pm = \frac{1}{\sqrt{2}}(A^0 \pm A^3)$$

$$A^\mu = (A^+, A^-, \boldsymbol{A}_\perp)$$

Light-cone components

$$p^\mu = \frac{1}{\sqrt{2}}(\Lambda, 0, 0, \Lambda),$$

$$n^\mu = \frac{1}{\sqrt{2}}(\Lambda^{-1}, 0, 0, -\Lambda^{-1})$$

$$p^2 = n^2 = 0, \quad p \cdot n = 1, \quad n^+ = p^- = 0$$

$$A^\mu = \alpha\, p^\mu + \beta\, n^\mu + A_\perp^\mu$$
$$= (A \cdot n)\, p^\mu + (A \cdot p)\, n^\mu + A_\perp^\mu$$

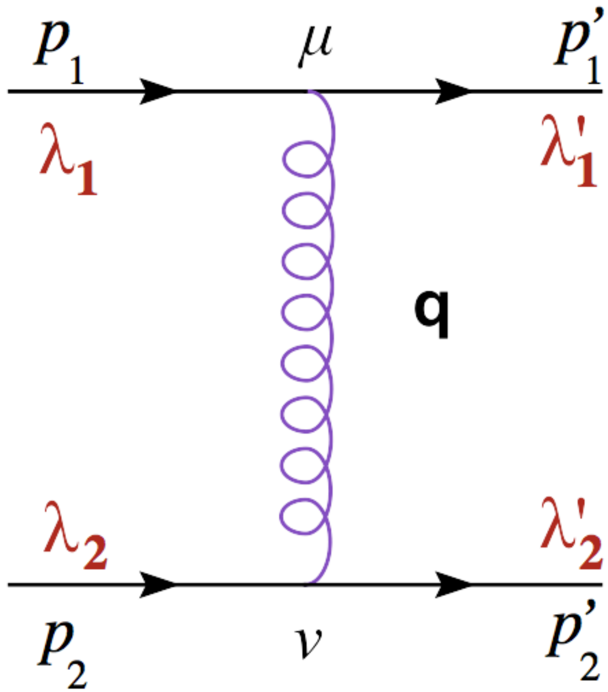
$$A^2 = 2\,\alpha\beta - \boldsymbol{A}_\perp^2$$

BFKL how to, step 1

$$q = \alpha\, p_1 + \beta\, p_2 + q_\perp$$

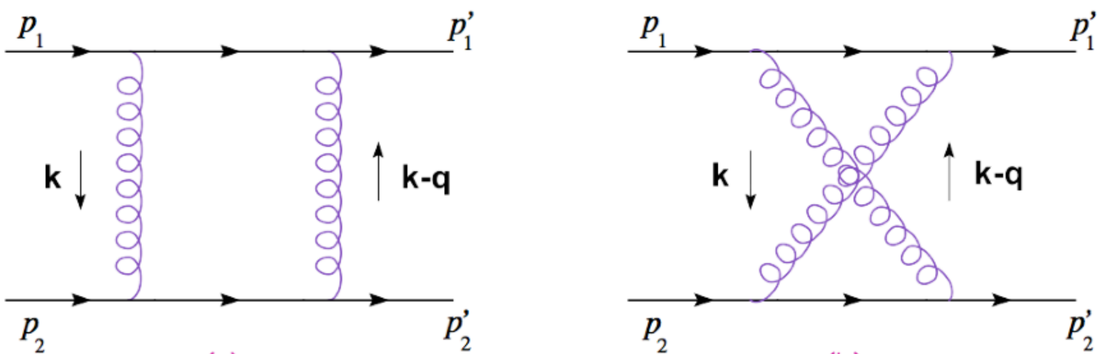
$$s = 2p_1p_2 \text{ and } t = q^2 = \alpha\beta s - \mathbf{q}^2$$

$$g_{\mu\nu} = \frac{2}{s}(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}) + g_{\mu\nu\perp}$$

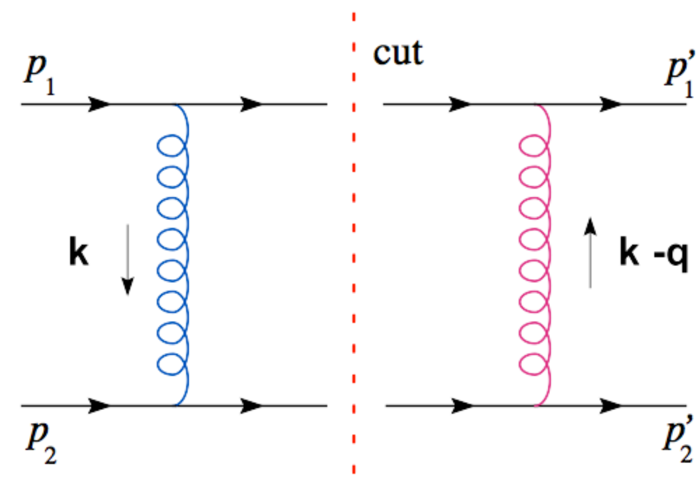


$$A^{(0)}(s,t) = 8\pi a_s t^{\alpha}_{ij} t^{\alpha}_{kl} \frac{s}{q^2} = 8\pi a_s t^{\alpha}_{ij} t^{\alpha}_{kl} \frac{s}{t}$$

BFKL how to, step 2



(a) (b)



$$ImA^{(1)}(s,t) = \frac{1}{2}\int d\Pi_2 A^{(0)}(s,k^2)A^{(0)\dagger}(s,(k-q)^2)$$

$$\int d\Pi = \int \frac{d^4k}{(2\pi)^2}\delta((p_1-k)^2)\delta((p_2+k)^2)$$

$$k = \alpha\, p_1 + \beta\, p_2 + k_\perp$$

$$d^4k = \frac{s}{2} d\alpha d\beta d^2\mathbf{k}\,.$$

$$A^{(0)}(s,k^2) \;=\; -8\pi a_s (t^\alpha_{mj} t^\alpha_{nl})\frac{s}{\mathbf{k}^2}$$

$$A^{(0)\dagger}(s,(k-q)^2) \;=\; -8\pi a_s (t^\beta_{mi} t^\beta_{nk})^* \frac{s}{(\mathbf{k}-\mathbf{q})^2}$$

$$A^{(1)}(s,t) \;=\; -4\frac{\alpha_s^2}{\pi}\,(t^\alpha t^\beta)_{ij}\,(t^\alpha t^\beta)_{kl}\ln(\frac{s}{t})s\int\frac{d^2\mathbf{k}}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}$$

Dispersion relations

BFKL how to, still step 2

$$A^{(1)}(s,t) = -\frac{16\pi\alpha_s}{N_c}(t^\alpha t^\beta)_{ij}(t^\alpha t^\beta)_{kl}\frac{s}{t}\ln(\frac{s}{t})\,\epsilon(t)$$

$$A^{(1)}_{cross}(s,t) = -\frac{16\pi\alpha_s}{N_c}(t^\alpha t^\beta)_{ij}(t^\alpha t^\beta)_{kl}\frac{u}{t}\ln(\frac{u}{t})\,\epsilon(t)$$

The two one-loop amplitudes

after setting: $$\epsilon(t) = \frac{N_c\alpha_s}{4\pi^2}\int -\mathbf{q}^2\frac{d^2\mathbf{k}}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}$$

Remember that we said we are hunting logs in s

$$\mathcal{A} = Re\mathcal{A} + iIm\mathcal{A} \sim \mathcal{B}\ln\frac{s}{t} = \mathcal{B}\ln\frac{s}{|t|} - i\pi\mathcal{B}$$

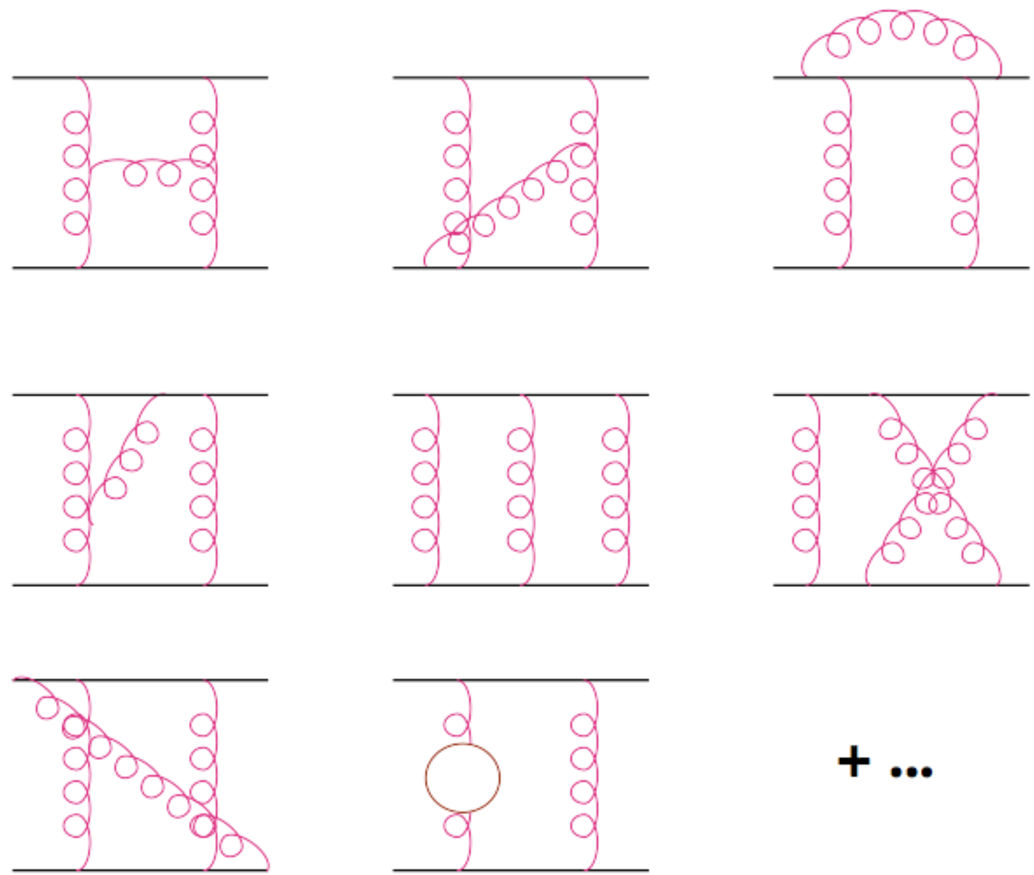
$$Re\mathcal{A} = -\tfrac{1}{\pi}\,Im\mathcal{A}\,\ln\tfrac{s}{|t|}$$

BFKL how to, step 2 final

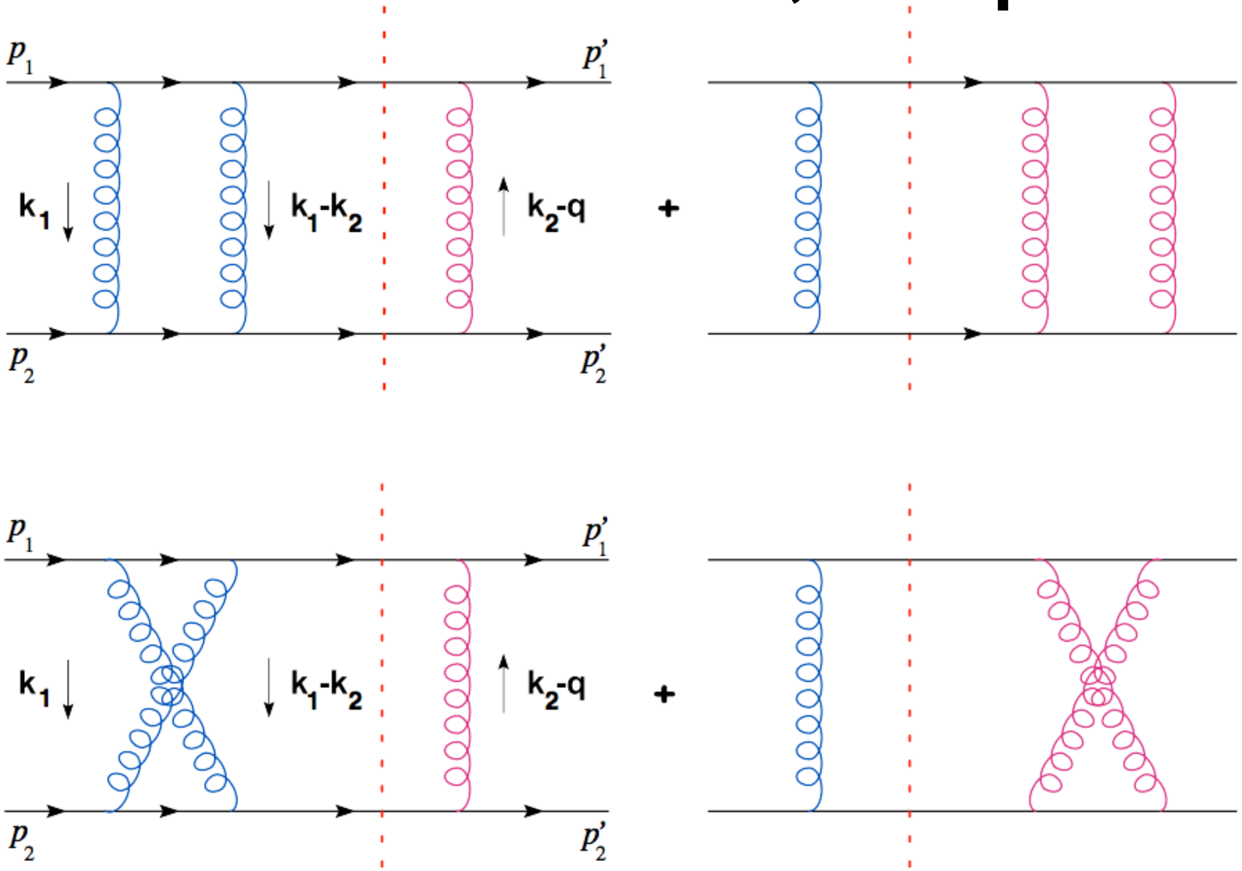
Putting together the two amplitudes for the one-loop, we obtain:

$$A_8^{(1)}(s,t) = 8\pi a_s t^{\alpha}_{ij} t^{\alpha}_{kl} \frac{s}{t} \ln\left(\frac{s}{|t|}\right) \epsilon(t) = A^{(0)} \ln\left(\frac{s}{|t|}\right) \epsilon(t)$$

These diagrams come in the next order



BFKL how to, step 3



$$A_8^{(2)}(s,t) = A^{(0)}(s,t)\,\frac{1}{2}\ln^2\left(\frac{s}{|t|}\right)\epsilon^2(t)$$

Stay on the virtual contributions, real contributions will come afterward

BFKL how to, … pause...

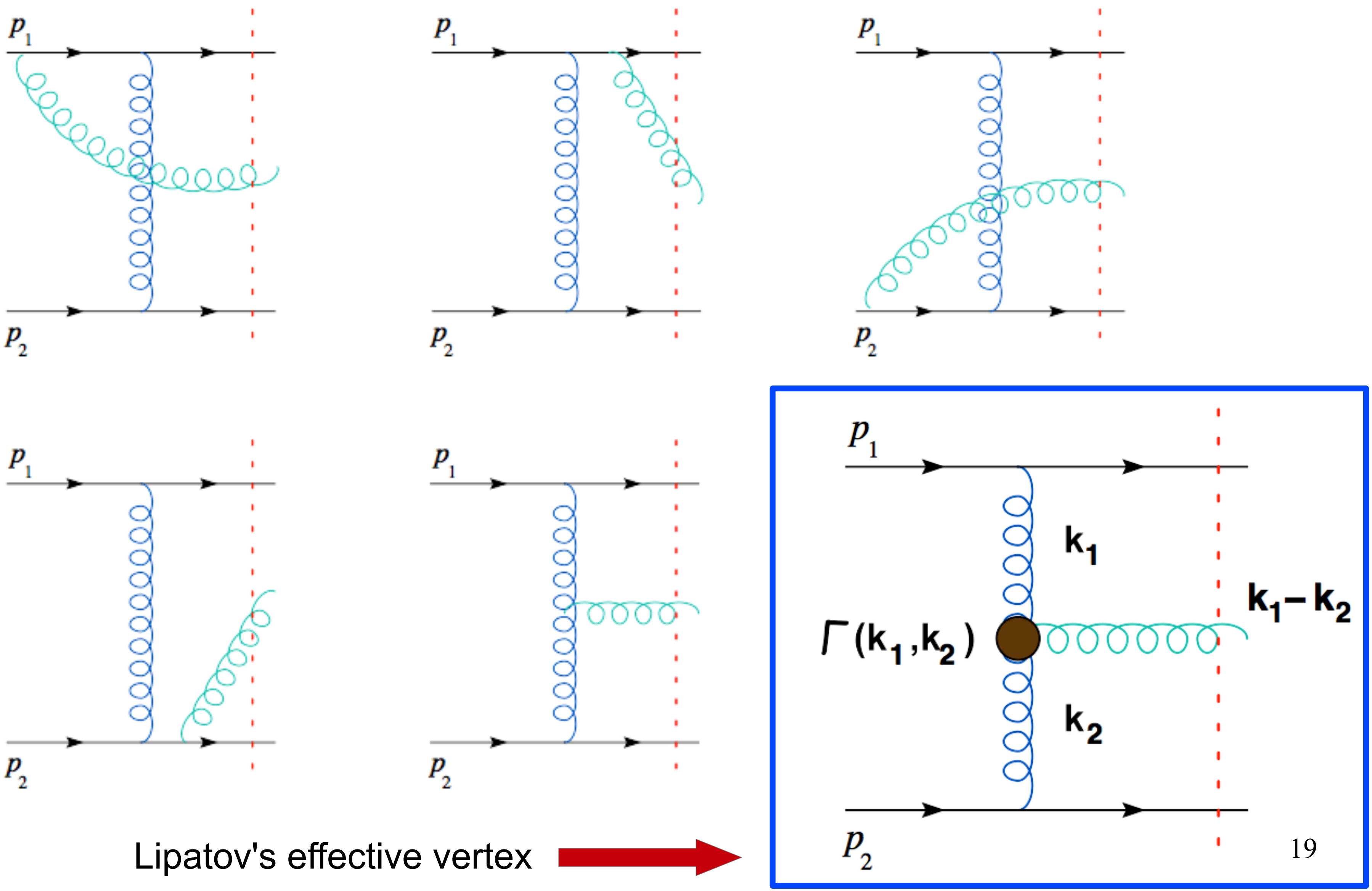
$$A_8(s,t) = A^{(0)}(s,t)\left(1 + \ln(\frac{s}{|t|})\,\epsilon(t) + \frac{1}{2}\ln^2(\frac{s}{|t|})\,\epsilon^2(t) + \ldots\right)$$

An ansatz seems natural: $$A_8(s,t) = A^{(0)}(s,t)\left(\frac{s}{|t|}\right)^{\epsilon(t)}$$

$$D_{\mu\nu}(s,q^2) = -i\frac{g_{\mu\nu}}{q^2}\left(\frac{s}{\mathbf{k}^2}\right)^{\epsilon(q^2)}$$

The reggeization of the gluon; Bootstrap equation

BFKL how to, next step, real corrections

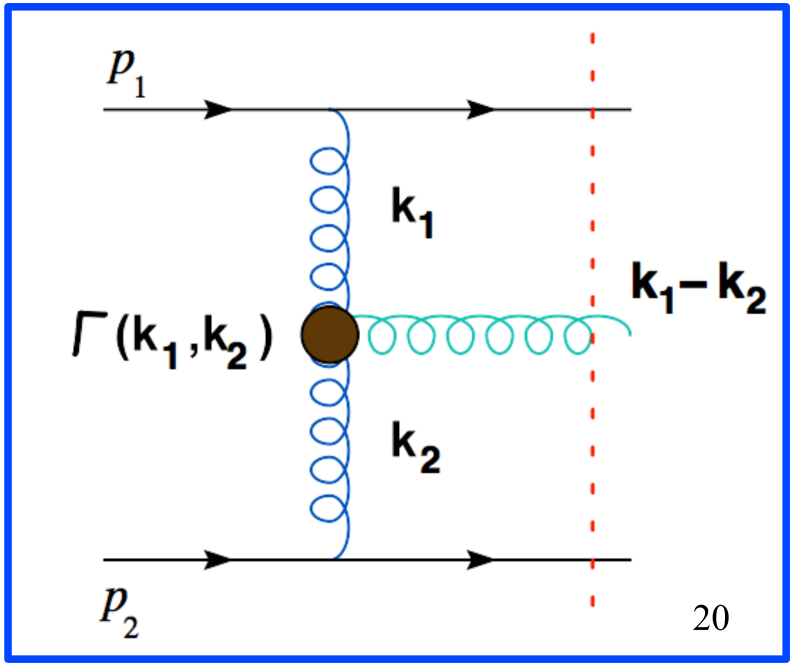


BFKL how to, next step, real corrections

$$k_1 = \alpha_1 p_1 + \beta_1 p_2 + k_{1\perp} \qquad 1 >> \alpha_1 >> \alpha_2$$
$$k_2 = \alpha_2 p_1 + \beta_2 p_2 + k_{2\perp}\,. \qquad 1 >> |\beta_2| >> |\beta_1|$$

$$\int d\Pi_3 = \frac{s^2}{4(2\pi)^5} \int d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 d^2\mathbf{k}_1 d^2\mathbf{k}_2$$
$$\delta(-\beta_1(1-\alpha_1)s - \mathbf{k}_1^2)\ \delta(\alpha_2(1+\beta_2)s - \mathbf{k}_2^2)$$
$$\delta((\alpha_1-\alpha_2)(\beta_1-\beta_2)s - (\mathbf{k}_1-\mathbf{k}_2)^2)\,.$$

$$\int d\Pi_3 = \frac{1}{4(2\pi)^5 s} \underbrace{\int_{\mathbf{k}_2^2/s}^{1} \frac{d\alpha_1}{\alpha_1}}_{\ln\left(\frac{s}{\mathbf{k}_2^2}\right)} \int d^2\mathbf{k}_1 d^2\mathbf{k}_2$$



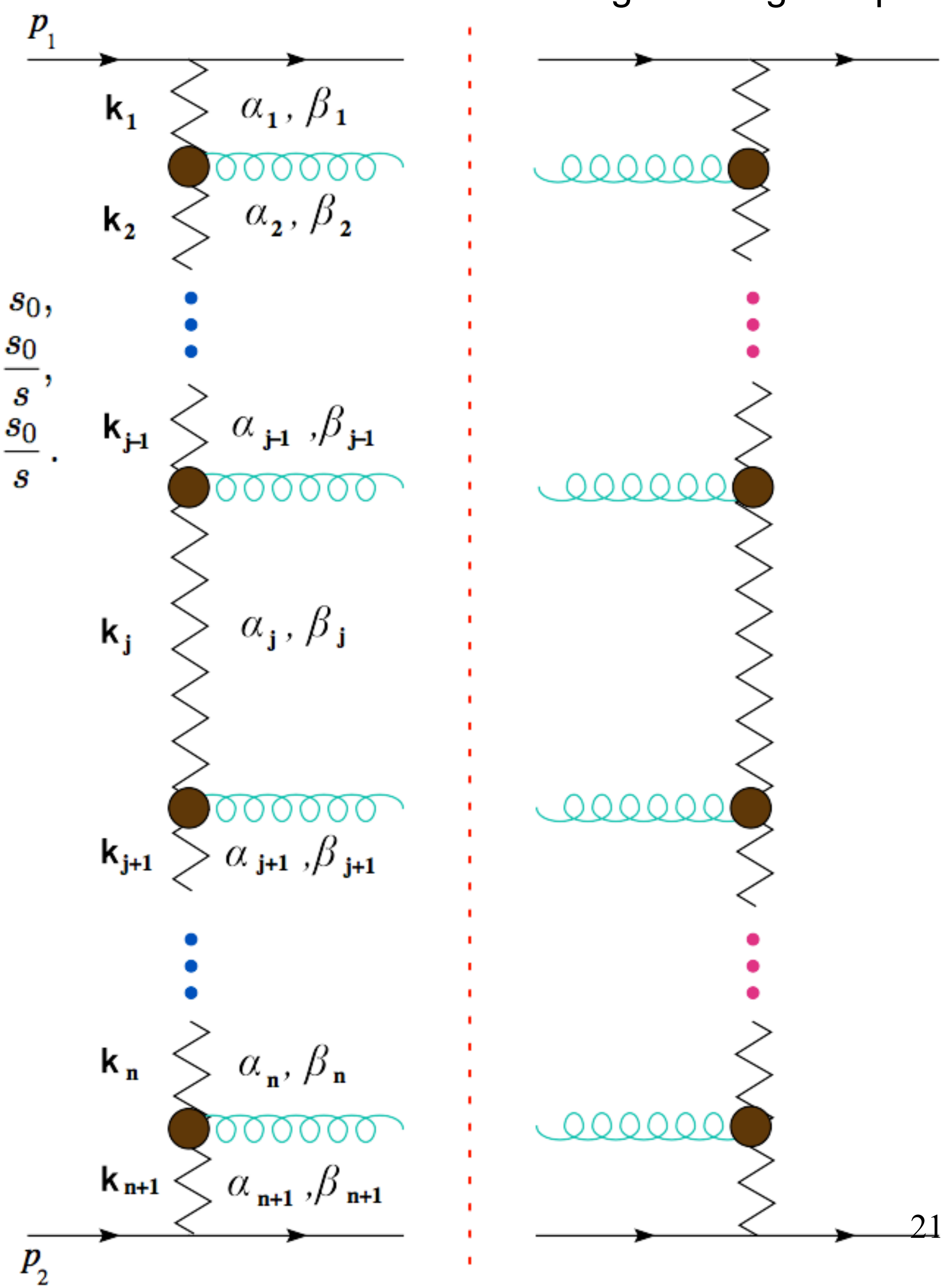
Almost there...

Now, time to iterate, assume the t-channel gluons to be reggeized gluons, use the conditions:

$$\mathbf{k}_1^2 \simeq \mathbf{k}_2^2 \simeq \ldots \mathbf{k}_i^2 \simeq \mathbf{k}_{i+1}^2 \ldots \simeq \mathbf{k}_n^2 \simeq \mathbf{k}_{n+1}^2 \gg \mathbf{q}^2 \simeq s_0,$$
$$1 \gg \alpha_1 \gg \alpha_2 \gg \ldots \alpha_i \gg \alpha_{i+1} \gg \alpha_{n+1} \gg \frac{s_0}{s},$$
$$1 \gg |\beta_{n+1}| \gg |\beta_n| \gg \ldots \gg |\beta_2| \gg |\beta_1| \gg \frac{s_0}{s}.$$

The (n+2)-body phase will now be much more complicated

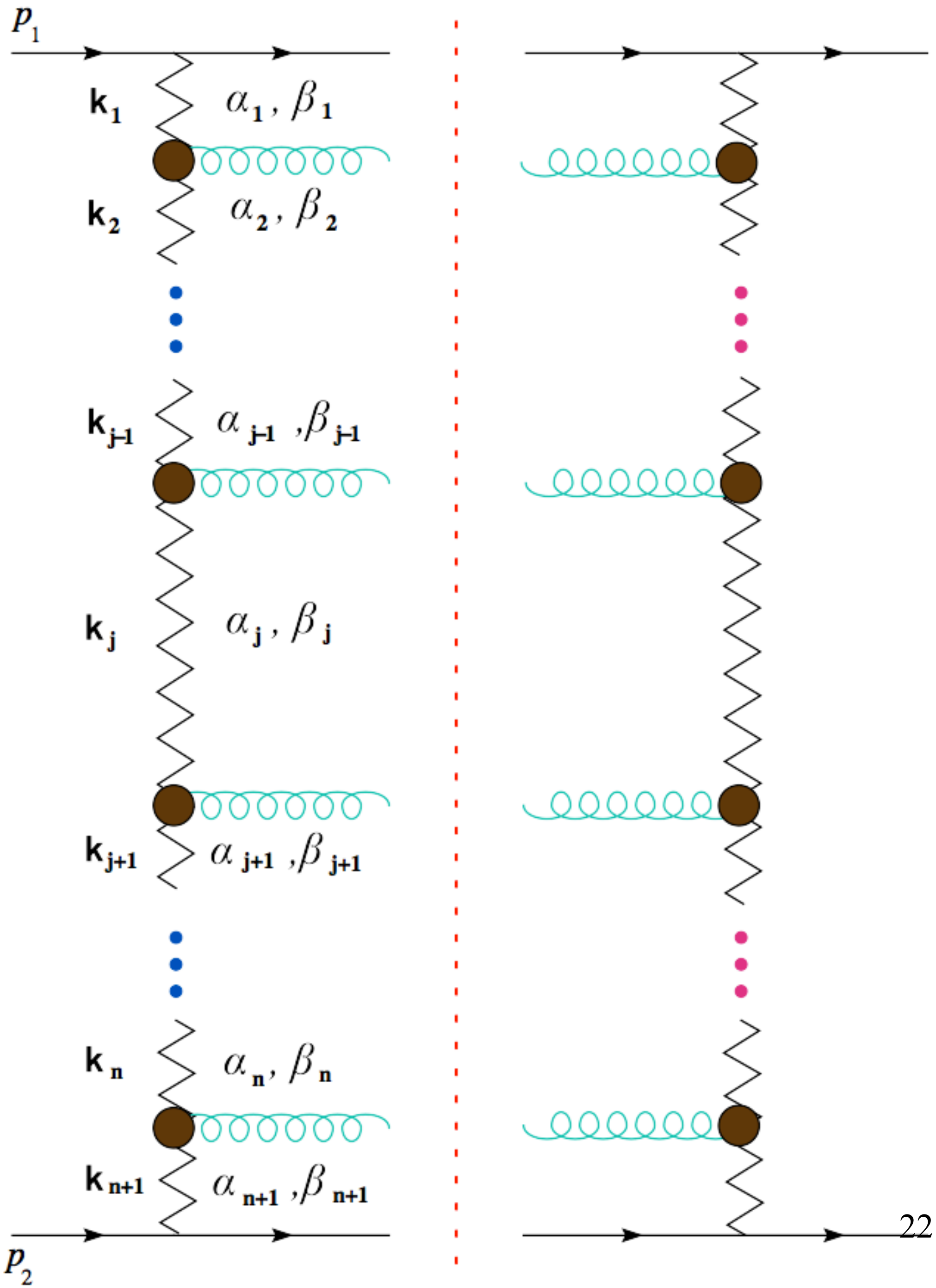
Strong ordering in rapidity



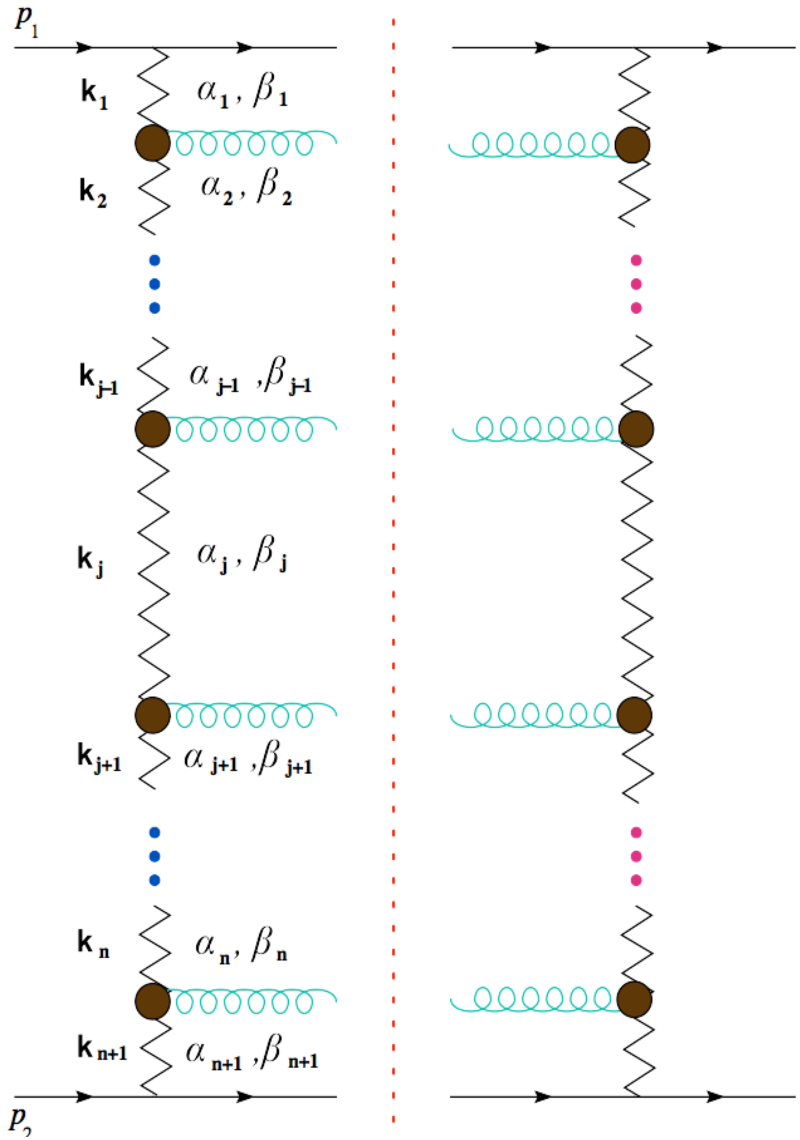
Ladder diagrams

Remember that we were calculating the imaginary part of the amplitude to the right.

This type of diagrams are called ladder diagrams



Ladder diagrams



$$\operatorname{Im} A(s,t) = \frac{1}{2}(-1)^n g_{\rho_1\sigma_1}\cdots g_{\rho_n\sigma_n}$$

$$\times \int \mathrm{d}\Pi_{n+2}\, A^{\rho_1\ldots\rho_n}_{2\to n+2}(k_1,\ldots,k_n)\, A^{\sigma_1\ldots\sigma_n\dagger}_{2\to n+2}(k_1-q,\ldots,k_n-q)$$

Almost there... but before we have to do something with the phase space ...

$$\mathrm{d}\Pi_{n+2} = \frac{s^{n+1}}{2^{n+1}\,(2\pi)^{3n+2}} \int \prod_{i=1}^{n+1} \mathrm{d}\alpha_i\,\mathrm{d}\beta_i\,\mathrm{d}^2\boldsymbol{k}_i$$
$$\times\,\delta\left(-\beta_1(1-\alpha_1)s - \boldsymbol{k}_1^2\right)\,\delta\left(\alpha_{n+1}(1+\beta_{n+1})s - \boldsymbol{k}_{n+1}^2\right)$$
$$\times \prod_{j=1}^{n} \delta\left((\alpha_j - \alpha_{j+1})(\beta_j - \beta_{j+1})s - (\boldsymbol{k}_j - \boldsymbol{k}_{j+1})^2\right)\ .$$

The (n+2)-body phase space

After integrating over β_i we obtain:

$$\mathrm{d}\Pi_{n+2} = \frac{1}{2^{n+1}\,(2\pi)^{3n+2}} \prod_{i=1}^{n} \int_{\alpha_{i+1}}^{1} \frac{\mathrm{d}\alpha_i}{\alpha_i} \int_0^1 \mathrm{d}\alpha_{n+1}$$
$$\times \prod_{j=1}^{n+1} \int \mathrm{d}^2\boldsymbol{k}_j\,\delta\left(\alpha_{n+1}s - \boldsymbol{k}^2\right)\ .$$

Mellin transform

Mellin transform
$$\widetilde{f}(\omega) = \int_1^\infty \mathrm{d}\left(\frac{s}{s_0}\right)\left(\frac{s}{s_0}\right)^{-\omega-1} f(s) \tag{1}$$

Inverse Mellin transform
$$f(s) = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}\omega \left(\frac{s}{s_0}\right)^{\omega} \widetilde{f}(\omega) \tag{2}$$

$f(s)$ is the convolution of n functions g_i

$$f(s) = \prod_{i=1}^{n} \int_{\alpha_{i+1}}^{1} \frac{\mathrm{d}\alpha_i}{\alpha_i}\, g_i\left(\frac{\alpha_{i-1}}{\alpha_i}\right) s_0\, \delta(\alpha_n s - s_0) \tag{3}$$

Mellin transform of $f(s)$:

$$\widetilde{f}(\omega) = \prod_{i=1}^{n} \int_0^1 \mathrm{d}\rho_i\, \rho_i^{\omega-1}\, g_i\left(\frac{1}{\rho_i}\right) = \prod_{i=1}^{n} \widetilde{g}_i(\omega) \tag{4}$$

Mellin and inverse Mellin transform

Remember also that to unfold the nested integration we took a Mellin transform

$$f_R(\omega, t) = \int_1^{\infty} \mathrm{d}\left(\frac{s}{|t|}\right) \left(\frac{s}{|t|}\right)^{-\omega-1} \frac{\mathrm{Im}\,\mathcal{A}_R(s,t)}{s}$$

$$\Updownarrow$$

$$\frac{\mathrm{Im}\,\mathcal{A}_R(s,t)}{s} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}\omega \left(\frac{s}{|t|}\right)^{\omega} f_R(\omega, t)$$

Mellin transformed amplitude

$$f_R(\omega, \boldsymbol{q}^2) = (4\pi\alpha_s)^2 \, \mathcal{G}_R \sum_{n=0}^{\infty} \prod_{i=1}^{n+1} \frac{\mathrm{d}^2 \boldsymbol{k}_i}{(2\pi)^2}$$

$$\times \frac{1}{\boldsymbol{k}_1^2 (\boldsymbol{k}_1 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_1^2) - \epsilon((k_1 - q)^2)}$$

$$\times (-2\alpha_s \eta_R) \, K(\boldsymbol{k}_1, \boldsymbol{k}_2)$$

$$\times \frac{1}{\boldsymbol{k}_2^2 (\boldsymbol{k}_2 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_2^2) - \epsilon((k_2 - q)^2)}$$

$$\vdots$$

$$\times (-2\alpha_s \eta_R) \, K(\boldsymbol{k}_n, \boldsymbol{k}_{n+1})$$

$$\times \frac{1}{\boldsymbol{k}_{n+1}^2 (\boldsymbol{k}_{n+1} - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_{n+1}^2) - \epsilon((k_{n+1} - q)^2)}$$

Mellin transformed amplitude

$$
\begin{aligned}
f_R(\omega, \boldsymbol{q}^2) = (4\pi\alpha_s)^2\, \mathcal{G}_R \sum_{n=0}^{\infty} \prod_{i=1}^{n+1} \frac{\mathrm{d}^2\boldsymbol{k}_i}{(2\pi)^2} \\
&\times \frac{1}{\boldsymbol{k}_1^2(\boldsymbol{k}_1-\boldsymbol{q})^2}\, \frac{1}{\omega - \epsilon(k_1^2) - \epsilon((k_1-q)^2)} \\
&\times (-2\alpha_s\eta_R)\, K(\boldsymbol{k}_1, \boldsymbol{k}_2) \\
&\times \frac{1}{\boldsymbol{k}_2^2(\boldsymbol{k}_2-\boldsymbol{q})^2}\, \frac{1}{\omega - \epsilon(k_2^2) - \epsilon((k_2-q)^2)} \\
&\vdots \\
&\times (-2\alpha_s\eta_R)\, K(\boldsymbol{k}_n, \boldsymbol{k}_{n+1}) \\
&\times \frac{1}{\boldsymbol{k}_{n+1}^2(\boldsymbol{k}_{n+1}-\boldsymbol{q})^2}\, \frac{1}{\omega - \epsilon(k_{n+1}^2) - \epsilon((k_{n+1}-q)^2)}
\end{aligned}
$$

The equation is an infinite series where each term involves multiple integrals
over the transverse momenta $\mathbf{k}_i$.
The kernel $K(k_i, k_{i+1})$ and the
propagators $\frac{1}{k_i^2(k_i-q)^2}$ appear in each term.

Mellin transformed amplitude

$$
\begin{aligned}
f_R(\omega, \boldsymbol{q}^2) = (4\pi\alpha_s)^2\, \mathcal{G}_R \sum_{n=0}^{\infty} \prod_{i=1}^{n+1} \frac{\mathrm{d}^2 \boldsymbol{k}_i}{(2\pi)^2} \\
\times \frac{1}{\boldsymbol{k}_1^2 (\boldsymbol{k}_1 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_1^2) - \epsilon((k_1 - q)^2)} \\
\times (-2\alpha_s \eta_R)\, K(\boldsymbol{k}_1, \boldsymbol{k}_2) \\
\times \frac{1}{\boldsymbol{k}_2^2 (\boldsymbol{k}_2 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_2^2) - \epsilon((k_2 - q)^2)} \\
\vdots \\
\times (-2\alpha_s \eta_R)\, K(\boldsymbol{k}_n, \boldsymbol{k}_{n+1}) \\
\times \frac{1}{\boldsymbol{k}_{n+1}^2 (\boldsymbol{k}_{n+1} - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_{n+1}^2) - \epsilon((k_{n+1} - q)^2)}
\end{aligned}
$$

Introduce $F_R(\omega, \mathbf{k}, \mathbf{q})$:
Define the function $F_R(\omega, \mathbf{k}, \mathbf{q})$ such that it encapsulates the infinite series of integrals and summations. This function will then satisfy an integral equation, which can be recursively solved.

Mellin transformed amplitude

$$
\begin{aligned}
f_R(\omega, \boldsymbol{q}^2) = (4\pi\alpha_s)^2\, \mathcal{G}_R \sum_{n=0}^{\infty} \prod_{i=1}^{n+1} \frac{\mathrm{d}^2 \boldsymbol{k}_i}{(2\pi)^2} \\
&\times \frac{1}{\boldsymbol{k}_1^2 (\boldsymbol{k}_1 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_1^2) - \epsilon((k_1 - q)^2)} \\
&\times (-2\alpha_s \eta_R)\, K(\boldsymbol{k}_1, \boldsymbol{k}_2) \\
&\times \frac{1}{\boldsymbol{k}_2^2 (\boldsymbol{k}_2 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_2^2) - \epsilon((k_2 - q)^2)} \\
&\vdots \\
&\times (-2\alpha_s \eta_R)\, K(\boldsymbol{k}_n, \boldsymbol{k}_{n+1}) \\
&\times \frac{1}{\boldsymbol{k}_{n+1}^2 (\boldsymbol{k}_{n+1} - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_{n+1}^2) - \epsilon((k_{n+1} - q)^2)}
\end{aligned}
$$

Recursive Structure:
Notice that $F_R(\omega, \mathbf{k}, \mathbf{q})$ can be defined recursively.
The initial term in the series (for $n = 0$) gives:

$$F_R(\omega, \mathbf{k}, \mathbf{q}) \to \frac{1}{\omega - \epsilon\big(k^2\big) - \epsilon\big((\mathbf{k} - \mathbf{q})^2\big)}$$

Each subsequent term in the series can be written in terms of the previous term by introducing an additional integration over $\mathbf{k}$ and multiplying by the kernel $K(\mathbf{k_i}, \mathbf{k_{i+1}})$

Mellin transformed amplitude

$$f_R(\omega, \boldsymbol{q}^2) = (4\pi\alpha_s)^2\, \mathcal{G}_R \sum_{n=0}^{\infty} \prod_{i=1}^{n+1} \frac{\mathrm{d}^2\boldsymbol{k}_i}{(2\pi)^2}$$
$$\times \frac{1}{\boldsymbol{k}_1^2(\boldsymbol{k}_1 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_1^2) - \epsilon((k_1 - q)^2)}$$
$$\times (-2\alpha_s\eta_R)\, K(\boldsymbol{k}_1, \boldsymbol{k}_2)$$
$$\times \frac{1}{\boldsymbol{k}_2^2(\boldsymbol{k}_2 - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_2^2) - \epsilon((k_2 - q)^2)}$$
$$\vdots$$
$$\times (-2\alpha_s\eta_R)\, K(\boldsymbol{k}_n, \boldsymbol{k}_{n+1})$$
$$\times \frac{1}{\boldsymbol{k}_{n+1}^2(\boldsymbol{k}_{n+1} - \boldsymbol{q})^2} \frac{1}{\omega - \epsilon(k_{n+1}^2) - \epsilon((k_{n+1} - q)^2)}$$

$$F_R(\omega, \mathbf{k}, \mathbf{q}) = \frac{1}{\omega - \epsilon\big(k^2\big) - \epsilon\big((\mathbf{k} - \mathbf{q})^2\big)} \left[1 - \frac{2\alpha_s\eta_R}{4\pi^2} \int \frac{d^2\kappa}{\kappa^2(\kappa - \mathbf{q})^2} K(\mathbf{k}, \kappa) F_R(\omega, \kappa, \mathbf{q})\right]$$

BFKL equation

Let us try to understand the BFKL equation

Let us define the following:

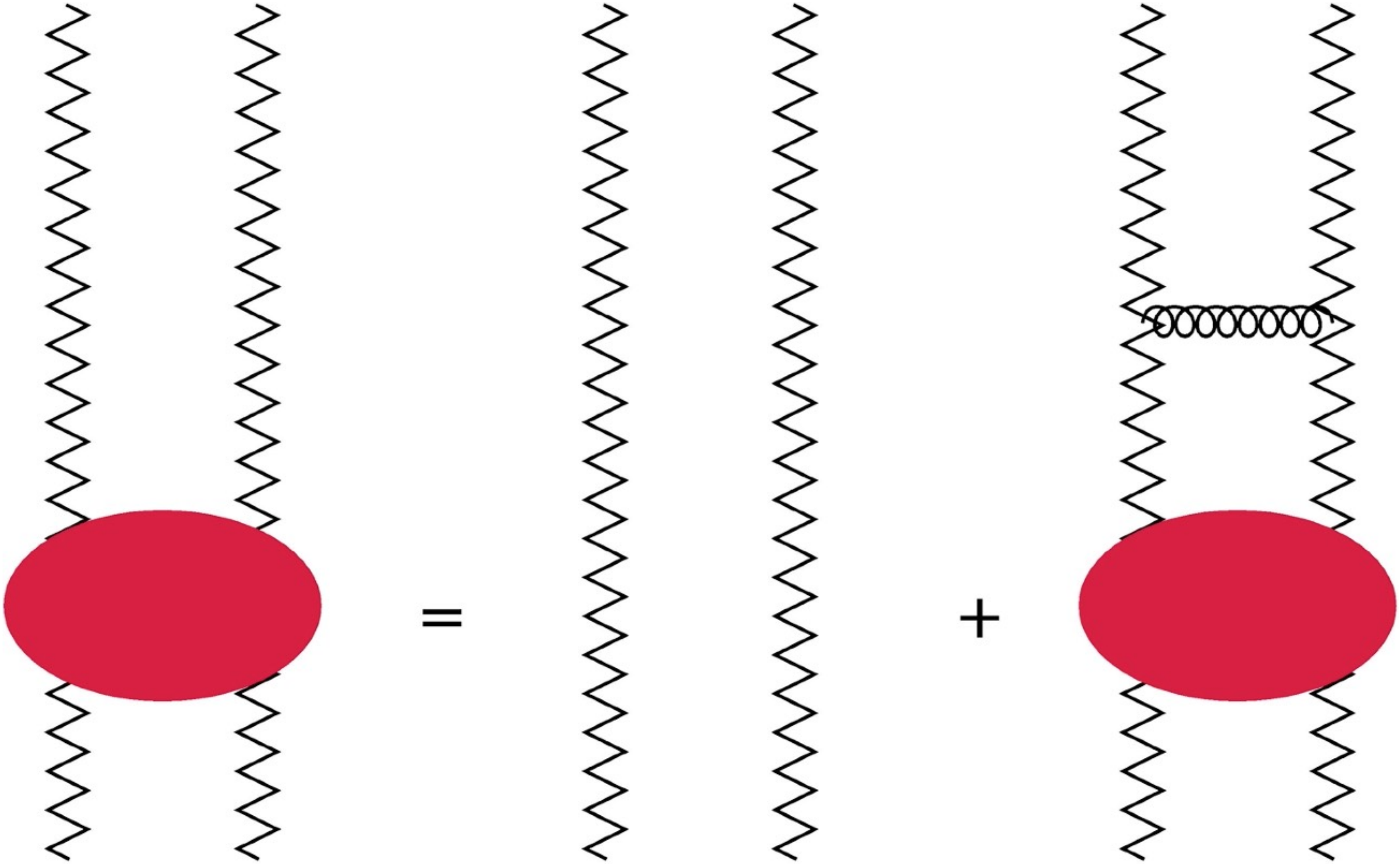
$$f_{\underline{1}}(\omega, \boldsymbol{q}^2) = (8\pi^2\alpha_s)^2 \frac{N_c^2 - 1}{4N_c} \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \int \frac{\mathrm{d}^2\boldsymbol{k}'}{(2\pi)^2} \frac{F(\omega, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q})}{\boldsymbol{k}'^2(\boldsymbol{k}-\boldsymbol{q})^2}$$

Then we will have the following integral equation in which we encode the behaviour of $f_1(\omega, \mathbf{q}^2)$:

$$\left[\omega - \epsilon(-\boldsymbol{k}^2) - \epsilon(-(\boldsymbol{k}-\boldsymbol{q})^2)\right] F(\omega, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q})$$
$$= \delta^2(\boldsymbol{k}-\boldsymbol{k}') - \frac{N_c\alpha_s}{2\pi^2} \int \mathrm{d}^2\boldsymbol{\kappa} \frac{K(\boldsymbol{k}, \boldsymbol{\kappa})}{\boldsymbol{k}^2(\boldsymbol{\kappa}-\boldsymbol{q})^2} F(\omega, \boldsymbol{\kappa}, \boldsymbol{k}', \boldsymbol{q})$$

The subscript R in f_R will be from now on 1

Let us try to understand the BFKL equation



Let us try to understand the BFKL equation

$$\mathrm{Im}\, A(s,t) = \frac{1}{2}\,(-1)^n\, g_{\rho_1\sigma_1}\cdots g_{\rho_n\sigma_n} \times \int \mathrm{d}\Pi_{n+2}\, A^{\rho_1\ldots\rho_n}_{2\to n+2}(k_1,\ldots,k_n)\, A^{\sigma_1\ldots\sigma_n\dagger}_{2\to n+2}(k_1-q,\ldots,k_n-q)$$

$$\mathrm{Im}\,\mathcal{A}_R(s,t) = \frac{1}{2}\sum_{n=0}^{\infty} 4s^2 g_s^4\, \mathcal{G}_R \int \mathrm{d}\Pi_{n+2}\, \frac{1}{\boldsymbol{k}_1^2(\boldsymbol{k}_1-\boldsymbol{q})^2}\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)+\epsilon((k_1-q)^2)} \times \prod_{i=1}^{n}\left\{\frac{g_s^2}{\boldsymbol{k}_{i+1}^2(\boldsymbol{k}_{i+1}-\boldsymbol{q})^2}\,(-2\eta_R)\,K(\boldsymbol{k}_i,\boldsymbol{k}_{i+1}) \times \left(\frac{\alpha_i}{\alpha_{i+1}}\right)^{\epsilon(k_{i+1}^2)+\epsilon((k_{i+1}-q)^2)}\right\}$$

Contraction of Lipatov's effective vertices

$$C^{\rho_i}(k_i,k_{i+1})\,C_{\rho_i}(-k_i+q,-k_{i+1}+q) = -2\left[\boldsymbol{q}^2 - \frac{\boldsymbol{k}_i^2\,(\boldsymbol{k}_{i+1}-\boldsymbol{q})^2}{(\boldsymbol{k}_i-\boldsymbol{k}_{i+1})^2} - \frac{\boldsymbol{k}_{i+1}^2\,(\boldsymbol{k}_i-\boldsymbol{q})^2}{(\boldsymbol{k}_i-\boldsymbol{k}_{i+1})^2}\right] \equiv -2\,K(\boldsymbol{k}_i,\boldsymbol{k}_{i+1})$$

Finally the BFKL equation

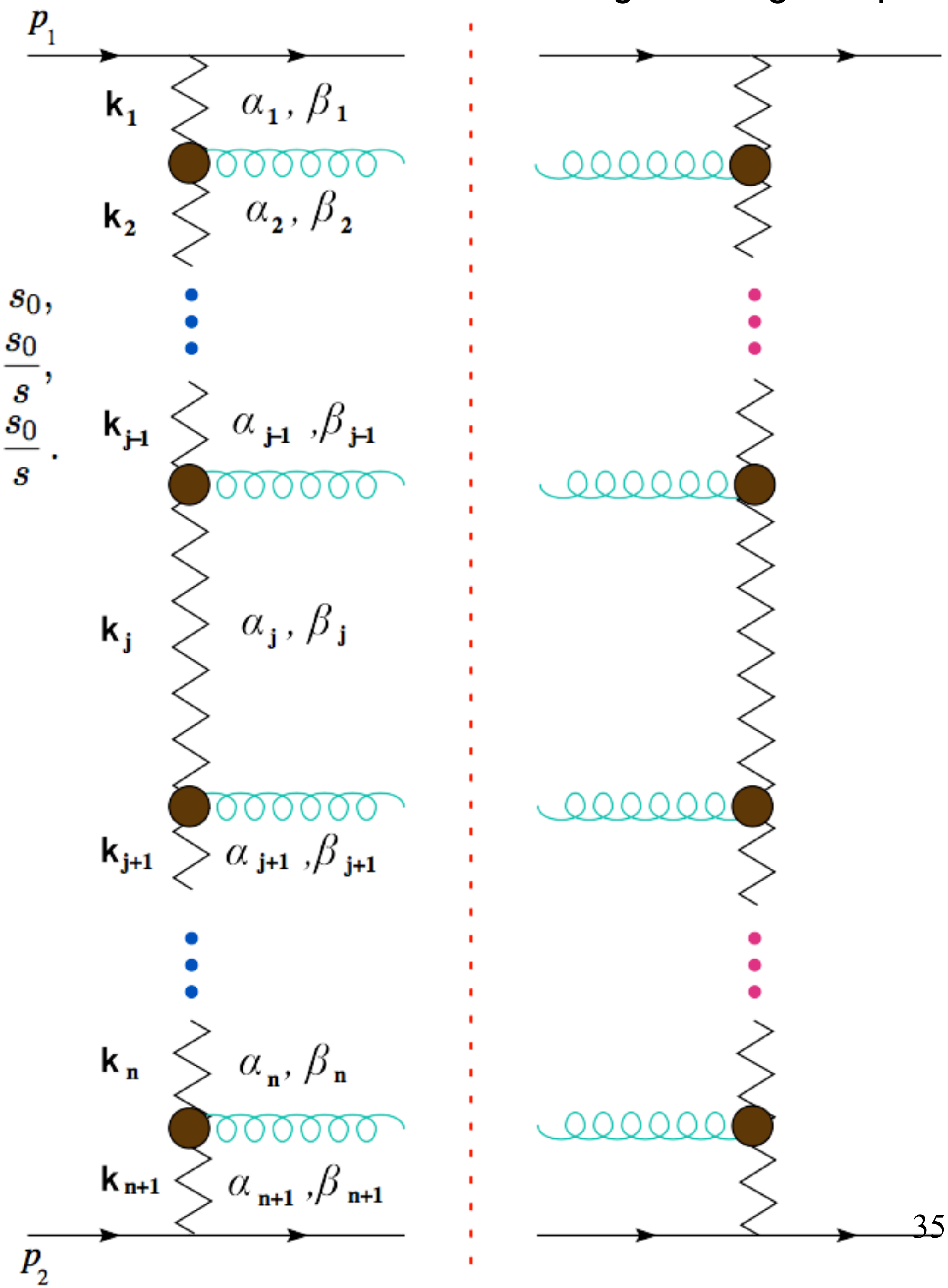
Again, time to iterate, set the t-channel gluons to reggeized gluons, use the conditions:

$$\mathbf{k}_1^2 \simeq \mathbf{k}_2^2 \simeq \ldots \mathbf{k}_i^2 \simeq \mathbf{k}_{i+1}^2 \ldots \simeq \mathbf{k}_n^2 \simeq \mathbf{k}_{n+1}^2 \gg \mathbf{q}^2 \simeq s_0,$$
$$1 \gg \alpha_1 \gg \alpha_2 \gg \ldots \alpha_i \gg \alpha_{i+1} \gg \alpha_{n+1} \gg \frac{s_0}{s},$$
$$1 \gg |\beta_{n+1}| \gg |\beta_n| \gg \ldots \gg |\beta_2| \gg |\beta_1| \gg \frac{s_0}{s}.$$

and after the Mellin transform to unfold the nested integrations over phase space, we finally get:

$$\begin{aligned}\omega f_\omega(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}) &= \delta^2(\mathbf{k}_1-\mathbf{k}_2)\\ &+\frac{\bar{\alpha}_s}{2\pi}\int d^2\mathbf{l}\Bigg\{\frac{-\mathbf{q}^2}{(\mathbf{l}-\mathbf{q})^2\mathbf{k}_1^2}f_\omega(\mathbf{l},\mathbf{k}_2,\mathbf{q})\\ &+\frac{1}{(\mathbf{l}-\mathbf{k}_1)^2}\left(f_\omega(\mathbf{l},\mathbf{k}_2,\mathbf{q}^2)-\frac{\mathbf{k}_1^2 f_\omega(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q})}{\mathbf{l}^2+(\mathbf{k}_1-\mathbf{l})^2}\right)\\ &+\frac{1}{(\mathbf{l}-\mathbf{k}_1)^2}\left(\frac{(\mathbf{k}_1-\mathbf{q})^2\mathbf{l}^2 f_\omega(\mathbf{l},\mathbf{k}_2,\mathbf{q}^2)}{(\mathbf{l}-\mathbf{q})^2\mathbf{k}_1^2}\right.\\ &\left.-\frac{(\mathbf{k}_1-\mathbf{q})^2 f_\omega(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}^2)}{(\mathbf{l}-\mathbf{q})^2(\mathbf{k}_1-\mathbf{l})^2}\right)\Bigg\},\end{aligned}$$

Strong ordering in rapidity



The BFKL equation for zero momentum transfer

$$\omega f_\omega(\mathbf{k}_1,\mathbf{k}_2) = \delta^2(\mathbf{k}_1-\mathbf{k}_2)$$
$$+\frac{\bar{\alpha_s}}{2\pi}\int\frac{d^2\mathbf{l}}{(\mathbf{l}-\mathbf{k}_1)^2}\left(f_\omega(\mathbf{l},\mathbf{k}_2)-\frac{\mathbf{k}_1^2 f_\omega(\mathbf{k}_1,\mathbf{k}_2)}{\mathbf{l}^2+(\mathbf{k}_1-\mathbf{l})^2}\right),.$$

We can rewrite the equation above very nicely as

$$\omega f_\omega(\mathbf{k}_1,\mathbf{k}_2) = \delta^2(\mathbf{k}_1-\mathbf{k}_2)+\int d^2\mathbf{l}\,\mathcal{K}(\mathbf{k}_1,\mathbf{l})\,f_\omega(\mathbf{l},\mathbf{k}_2)$$

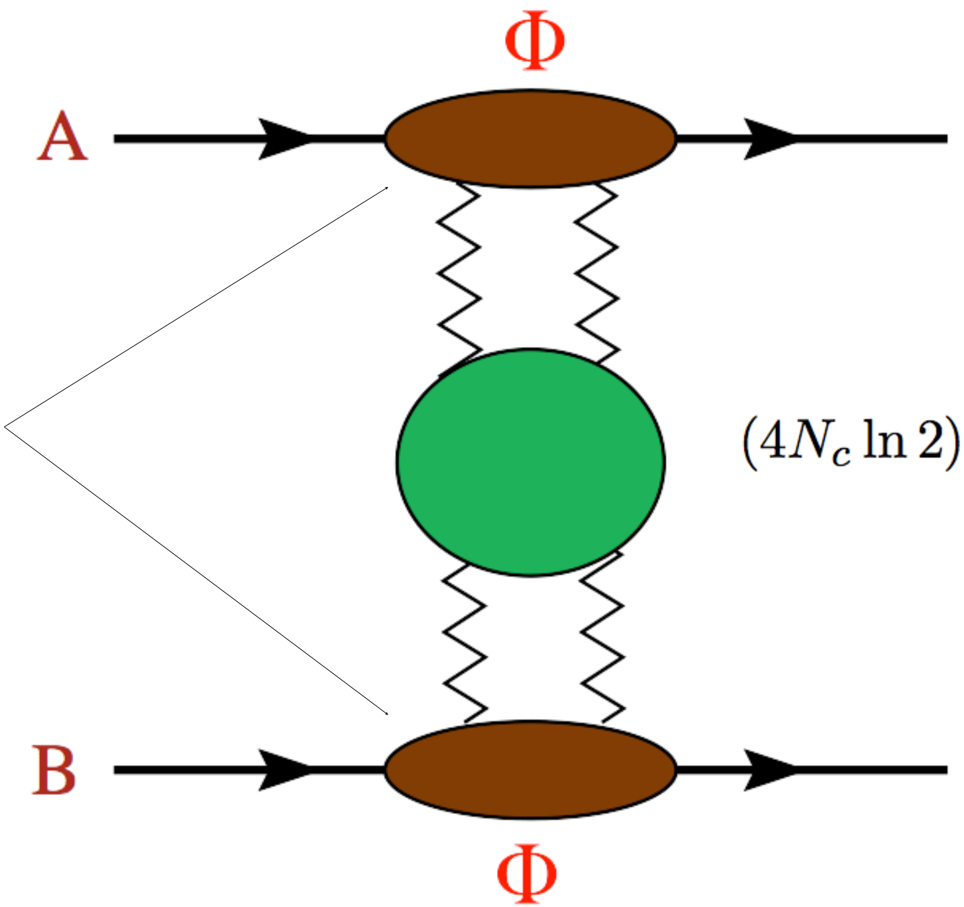
where $\mathcal{K}(\mathbf{k}_1,\mathbf{l})$ is the BFKL kernel:

$$\mathcal{K}(\mathbf{k}_1,\mathbf{l}) = \underbrace{2\epsilon(-\mathbf{k}^2)\,\delta^2(\mathbf{k}_1-\mathbf{l})}_{\mathcal{K}_{virt}} + \underbrace{\frac{N_c\alpha_s}{\pi^2}\frac{1}{(\mathbf{k}_1-\mathbf{k}_2)^2}}_{\mathcal{K}_{real}}.$$

We can go back to s-space:

$$f(s,\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\omega\left(\frac{s}{s_0}\right)^\omega f_\omega(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q})$$

A hadronic elastic amplitude



Impact factors... ?

$(4N_c \ln 2)/\pi)\alpha_s \approx 0.5.\ (\alpha_s \sim 0.2)$

$$\mathcal{A}(s,t) = i\,s\,\mathcal{C}\int \frac{d^2\mathbf{k}_1}{(2\pi)^2}\,\frac{d^2\mathbf{k}_2}{(2\pi)^2}\,\Phi_A(\mathbf{k}_1,\mathbf{q})\,\frac{f(s,\mathbf{k}_1,\mathbf{k}_2,\mathbf{q})}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2}\Phi_B(\mathbf{k}_2,\mathbf{q})$$

Let us make a summary here

- We have shown how to derive the BFKL equation at LO
- We find that the exchange of ladder-diagrams leads to an exponential rise for the total cross section
- The resummation of all these ladders gives an intercept ~ 0.5, perturbative Pomeron
- What is the connection between 'soft' and 'hard' Pomeron?
- Old ideas from Regge theory find accommodation -not in an always clear way- in QCD

The BFKL equation, again

$$\begin{aligned}
\omega\, F(\omega, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q}) &= \delta^2(\boldsymbol{k} - \boldsymbol{k}') \\
&+ \frac{N_c \alpha_s}{2\pi^2} \int \mathrm{d}^2\boldsymbol{\kappa} \left\{ \frac{-\boldsymbol{q}^2}{(\boldsymbol{\kappa} - \boldsymbol{q})^2 \boldsymbol{k}^2}\, F(\omega, \boldsymbol{\kappa}, \boldsymbol{k}', \boldsymbol{q}) \right. \\
&+ \frac{1}{(\boldsymbol{\kappa} - \boldsymbol{k})^2} \left[F(\omega, \boldsymbol{\kappa}, \boldsymbol{k}', \boldsymbol{q}) - \frac{\boldsymbol{k}^2\, F(\omega, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q})}{\boldsymbol{\kappa}^2 + (\boldsymbol{k} - \boldsymbol{\kappa})^2} \right] \\
&+ \frac{1}{(\boldsymbol{\kappa} - \boldsymbol{k})^2} \left[\frac{(\boldsymbol{k} - \boldsymbol{q})^2\, \boldsymbol{\kappa}^2\, F(\omega, \boldsymbol{\kappa}, \boldsymbol{k}', \boldsymbol{q})}{(\boldsymbol{\kappa} - \boldsymbol{q})^2\, \boldsymbol{k}^2} \right. \\
&- \left. \left. \frac{(\boldsymbol{k} - \boldsymbol{q})^2\, F(\omega, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q})}{(\boldsymbol{\kappa} - \boldsymbol{q})^2 + (\boldsymbol{k} - \boldsymbol{\kappa})^2} \right] \right\}
\end{aligned}$$

To complete the story...

Suppose now that we know $F(\omega, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q})$

The we take an inverse Mellin transform to go back to s-space

$$F(s, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q}) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}\omega \left(\frac{s}{|t|} \right)^{\omega} F(\omega, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q})$$

To recover the imaginary part of the ladder diagrams all we need to do is:

$$\frac{\mathrm{Im}\, \mathcal{A}_{\underline{1}}(s,t)}{s} = (8\pi^2 \alpha_s)^2 \, \frac{N_c^2 - 1}{4N_c} \int \frac{\mathrm{d}^2 \boldsymbol{k}}{(2\pi)^2} \int \frac{\mathrm{d}^2 \boldsymbol{k}'}{(2\pi)^2} \, \frac{F(s, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q})}{\boldsymbol{k}'^2 (\boldsymbol{k} - \boldsymbol{q})^2}$$

The BFKL equation for zero momentum transfer, **q**=0

$$\omega\, F(\omega, \boldsymbol{k}, \boldsymbol{k}') = \delta^2(\boldsymbol{k}-\boldsymbol{k}') + \frac{N_c\alpha_s}{\pi^2}\int \frac{\mathrm{d}^2\boldsymbol{\kappa}}{(\boldsymbol{k}-\boldsymbol{\kappa})^2} \times \left[F(\omega, \boldsymbol{\kappa}, \boldsymbol{k}') - \frac{\boldsymbol{k}^2}{\boldsymbol{\kappa}^2 + (\boldsymbol{k}-\boldsymbol{\kappa})^2} F(\omega, \boldsymbol{k}, \boldsymbol{k}')\right]$$

Or symbolically:

$$\omega\, F(\omega, \boldsymbol{k}, \boldsymbol{k}') = \delta^2(\boldsymbol{k}-\boldsymbol{k}') + \int \mathrm{d}^2\boldsymbol{\kappa}\, \mathcal{K}(\boldsymbol{k}, \boldsymbol{\kappa})\, F(\omega, \boldsymbol{\kappa}, \boldsymbol{k}')$$

where $$\mathcal{K}(\boldsymbol{k}, \boldsymbol{\kappa}) = 2\,\epsilon(-\boldsymbol{k}^2)\,\delta^2(\boldsymbol{k}-\boldsymbol{\kappa}) + \frac{N_c\alpha_s}{\pi^2}\frac{1}{(\boldsymbol{k}-\boldsymbol{\kappa})^2}$$

$$\mathcal{K}_{\mathrm{virt}}(\boldsymbol{k}, \boldsymbol{\kappa}) = 2\,\epsilon(-\boldsymbol{k}^2)\,\delta^2(\boldsymbol{k}-\boldsymbol{\kappa})$$

$$\mathcal{K}_{\mathrm{real}}(\boldsymbol{k}, \boldsymbol{\kappa}) = \frac{N_c\alpha_s}{\pi^2}\frac{1}{(\boldsymbol{k}-\boldsymbol{\kappa})^2}$$

SOLVING THE BFKL EQUATION

Solution for zero momentum transfer

Let us write symbolically:

$$\omega F = \mathbb{1} + \mathcal{K} \otimes F$$

By solving the equation we mean finding eigenfunctions such that:

$$\mathcal{K} \otimes \phi_\alpha = \omega_\alpha \phi_\alpha$$

The eigenfunction obey the completeness relation:

$$\sum_\alpha \phi_\alpha(\boldsymbol{k})\, \phi^*_\alpha(\boldsymbol{k}') = \delta^2(\boldsymbol{k} - \boldsymbol{k}')$$

Then the solution to the first equation will be:

$$F(\omega, \boldsymbol{k}, \boldsymbol{k}') = \sum_\alpha \frac{\phi_\alpha(\boldsymbol{k})\, \phi^*_\alpha(\boldsymbol{k}')}{\omega - \omega_\alpha}$$

α denotes a set of indices that can be discrete or continuous and the summation symbol can hide an integration

Solution for zero momentum transfer

Let us write symbolically:

$$\omega F = 1\!\!1 + \mathcal{K} \otimes F$$

By solving the equation we mean finding eigenfunctions such that:

$$\mathcal{K} \otimes \phi_\alpha = \omega_\alpha \phi_\alpha$$

Actually, if we use polar coordinates

$$\boldsymbol{k} \equiv (|\boldsymbol{k}|, \vartheta)$$

the eigenfunctions are:

$$\phi_{n\nu}(|\boldsymbol{k}|, \vartheta) = \frac{1}{\pi\sqrt{2}} \left(\boldsymbol{k}^2\right)^{-\frac{1}{2}+\mathrm{i}\nu} \mathrm{e}^{in\vartheta}$$

obeying:

$$\int \mathrm{d}^2\boldsymbol{k}\, \phi_{n\nu}(\boldsymbol{k})\, \phi_{n'\nu'}(\boldsymbol{k}) = \delta_{nn'}\, \delta(\nu - \nu')$$

whereas the eigenvalues are:

$$\omega_n(\nu) = -\frac{2\alpha_s N_c}{\pi} \mathrm{Re} \left[\psi\left(\frac{|n|+1}{2} + \mathrm{i}\nu\right) - \psi(1)\right]$$

Solution for zero momentum transfer

The solution will then be:

$$F(\omega, \boldsymbol{k}, \boldsymbol{k}') = \frac{1}{2\pi^2 \left(\boldsymbol{k}^2 \boldsymbol{k}'^2\right)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n(\vartheta - \vartheta')} \int_{\infty}^{+\infty} \mathrm{d}\nu \, \frac{\mathrm{e}^{\mathrm{i}\nu \, \ln\left(\frac{\boldsymbol{k}^2}{\boldsymbol{k}'^2}\right)}}{\omega - \omega_n(\nu)}$$

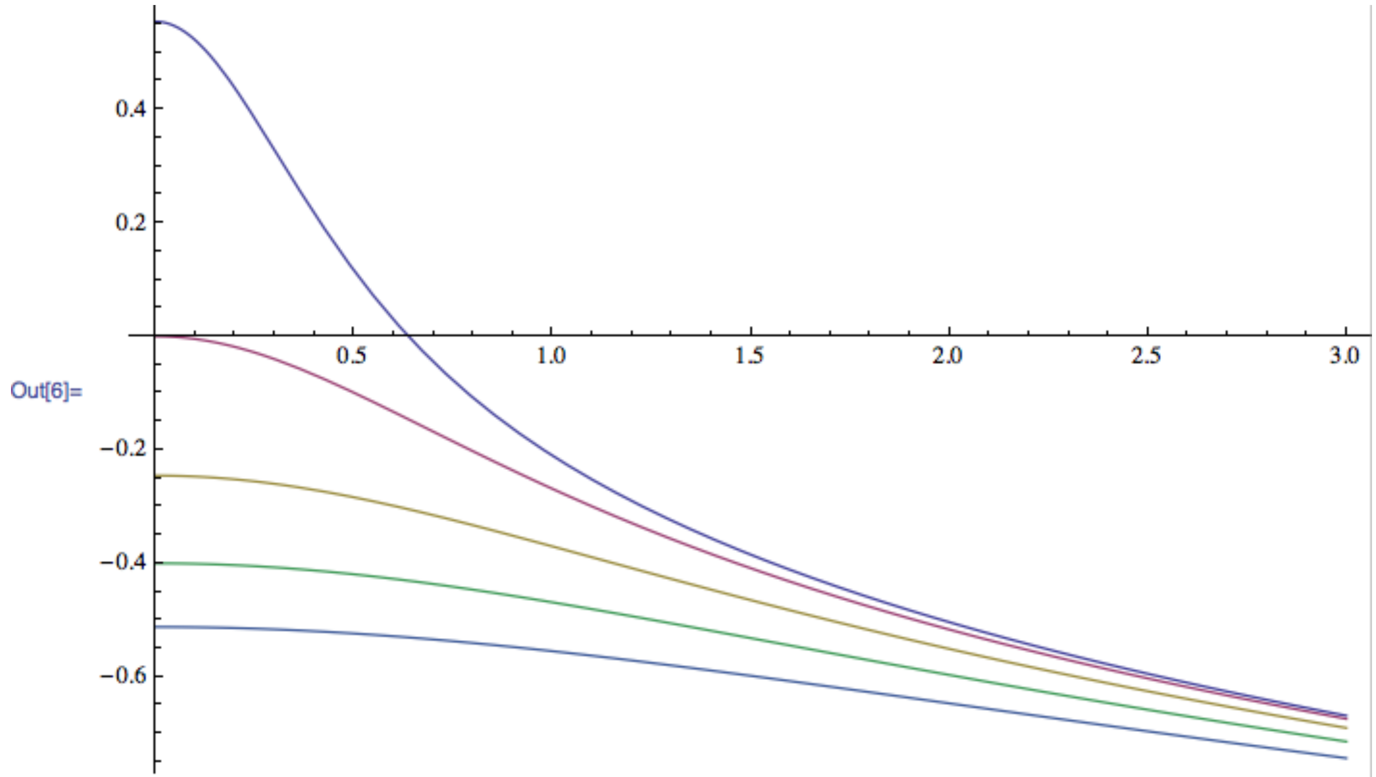
Here, n is also called conformal spin, it is connected to the angular information encoded in the gluon Green's function $F(\omega, \mathbf{k}, \mathbf{k}')$.

Solution for zero momentum transfer

Hands on... Let us use Mathematica to plot things and draw conclusions

```
omega[n_, v_] := Module[{asBar = 1/5},
   Return[2 asBar (PolyGamma[0, 1] -
     Re[PolyGamma[(Abs[n] + 1)/2 + I v]])]];

Plot[{omega[0, ν], omega[1, ν], omega[2, ν],
  omega[3, ν], omega[4, ν]}, {ν, 0, 3}]
```



Solution for zero momentum transfer

$$F(\omega, \boldsymbol{k}, \boldsymbol{k}') = \frac{1}{2\pi^2 \left(\boldsymbol{k}^2 \boldsymbol{k}'^2\right)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n(\vartheta - \vartheta')} \int_{\infty}^{+\infty} \mathrm{d}\nu \, \frac{\mathrm{e}^{\mathrm{i}\nu \, \ln\left(\frac{\boldsymbol{k}^2}{\boldsymbol{k}'^2}\right)}}{\omega - \omega_n(\nu)}$$

Retain only the n=0 term, this from the analysis before

$$F(\omega, \boldsymbol{k}, \boldsymbol{k}') = \frac{1}{2\pi^2 \left(\boldsymbol{k}^2 \boldsymbol{k}'^2\right)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \mathrm{d}\nu \, \frac{\mathrm{e}^{\mathrm{i}\,\nu \, \ln\left(\frac{\boldsymbol{k}^2}{\boldsymbol{k}'^2}\right)}}{\omega - \omega_0(\nu)}$$

Expanding around zero where we have the maximum gives:

$$\omega_0(\nu) = \frac{N_c \alpha_s}{\pi} \left(4 \, \ln 2 - 14 \, \zeta(3) \, \nu^2 + \ldots\right)$$

Solution for zero momentum transfer

$$\omega_0(\nu) = \frac{N_c \alpha_s}{\pi} \left(4\, \ln 2 - 14\, \zeta(3)\, \nu^2 + \ldots\right)$$

Set: $$\lambda = \frac{N_c \alpha_s}{\pi}\, 4\, \ln 2\,, \quad \lambda' = \frac{N_c \alpha_s}{\pi}\, 28\, \zeta(3)$$

Take the inverse Mellin transform

$$F(s, \boldsymbol{k}, \boldsymbol{k}') = \frac{1}{2\pi^2 \left(\boldsymbol{k}^2 \boldsymbol{k}'^2\right)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \mathrm{d}\nu \, \left(\frac{s}{\boldsymbol{k}^2}\right)^{\omega_0(\nu)} \mathrm{e}^{\mathrm{i}\, \nu\, \ln\left(\frac{\boldsymbol{k}^2}{\boldsymbol{k}'^2}\right)}$$

$$F(s, \boldsymbol{k}, \boldsymbol{k}') = \frac{1}{\sqrt{2\pi^3 \lambda' \, \boldsymbol{k}^2 \boldsymbol{k}'^2}} \, \frac{1}{\sqrt{\ln(s/\boldsymbol{k}^2)}} \times \left(\frac{s}{\boldsymbol{k}^2}\right)^{\lambda} \exp\left[-\frac{\ln^2(\boldsymbol{k}^2/\boldsymbol{k}'^2)}{2\lambda' \, \ln(s/\boldsymbol{k}^2)}\right]$$

Pomeron solution of the BFKL equation

Solution for zero momentum transfer

$$F(s,\boldsymbol{k},\boldsymbol{k}') = \frac{1}{\sqrt{2\pi^3\lambda'\,\boldsymbol{k}^2\boldsymbol{k}'^2}}\,\frac{1}{\sqrt{\ln(s/\boldsymbol{k}^2)}} \times \left(\frac{s}{\boldsymbol{k}^2}\right)^{\lambda} \exp\left[-\frac{\ln^2(\boldsymbol{k}^2/\boldsymbol{k}'^2)}{2\lambda'\,\ln(s/\boldsymbol{k}^2)}\right]$$

$$\alpha_{I\!P}(0) = 1 + \lambda = 1 + \frac{N_c\alpha_s}{\pi}\,4\,\ln 2$$

QCD Pomeron intercept way too large in comparison to the soft Pomeron intercept

Solution for zero momentum transfer

Again in Mathematica:

```
omega[n_, v_] := Module[{asBar = 1/5},
   Return[2 asBar (PolyGamma[0, 1] -
     Re[PolyGamma[(Abs[n] + 1)/2 + I v]])]];
```

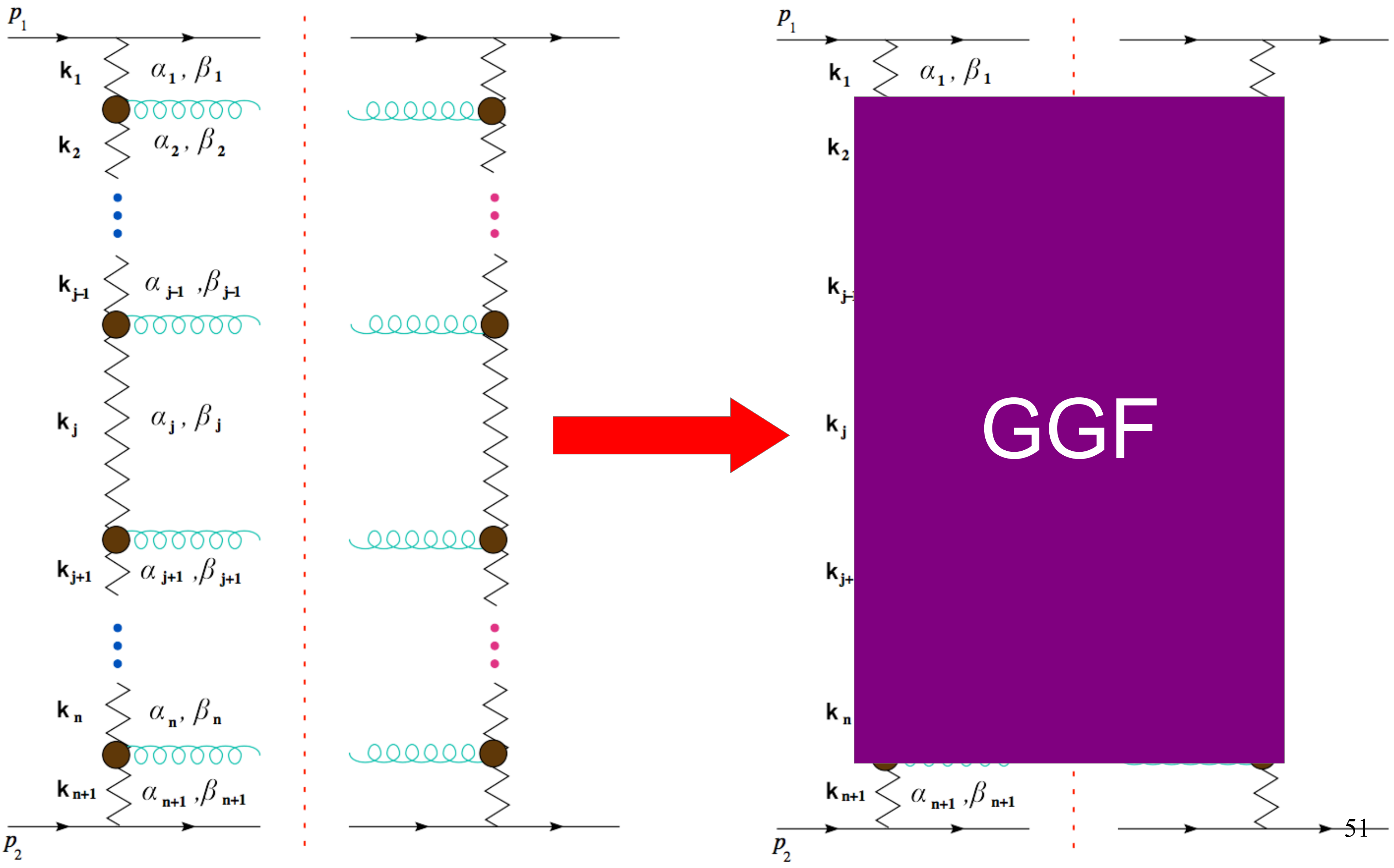
```
analytic[n_, Y_, ka_, kb_, angle_] :=
NIntegrate[Exp[I*n*angle]/(2Pi^2)/ka/kb*2*Exp[omega[n,v]Y]*
Cos[2 Log[(ka/kb)] v], {v, 0, Infinity}, WorkingPrecision -> 20];
```

Now you can calculate the LO gluon Green's function for a given rapidity Y, conformal spin n, and certain momenta of the reggeized gluons.

Note: Many times, in the literature, the leading eigenvalue is denoted as χ_0. It is also called sometimes as the LO BFKL kernel!

$$\chi_0(\nu) = -2\,\mathrm{Re}\left\{\psi\left(\frac{1}{2}+i\nu\right) - \psi(1)\right\}$$

The gluon Green's function



Unintegrated gluon distributions

$$\frac{\partial F(s,\boldsymbol{k},\boldsymbol{k}')}{\partial \ln(s/\boldsymbol{k}^2)} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}\omega \left(\frac{s}{\boldsymbol{k}^2}\right)^{\omega} \omega\, F(\omega,\boldsymbol{k},\boldsymbol{k}')$$

$$\frac{\partial F(s,\boldsymbol{k},\boldsymbol{k}')}{\partial \ln(s/\boldsymbol{k}^2)} = \frac{N_c \alpha_s}{\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{\kappa}}{(\boldsymbol{k}-\boldsymbol{\kappa})^2} \times \left[F(s,\boldsymbol{\kappa},\boldsymbol{k}') - \frac{\boldsymbol{k}^2}{\boldsymbol{\kappa}^2 + (\boldsymbol{k}-\boldsymbol{\kappa})^2} F(s,\boldsymbol{k},\boldsymbol{k}')\right]$$

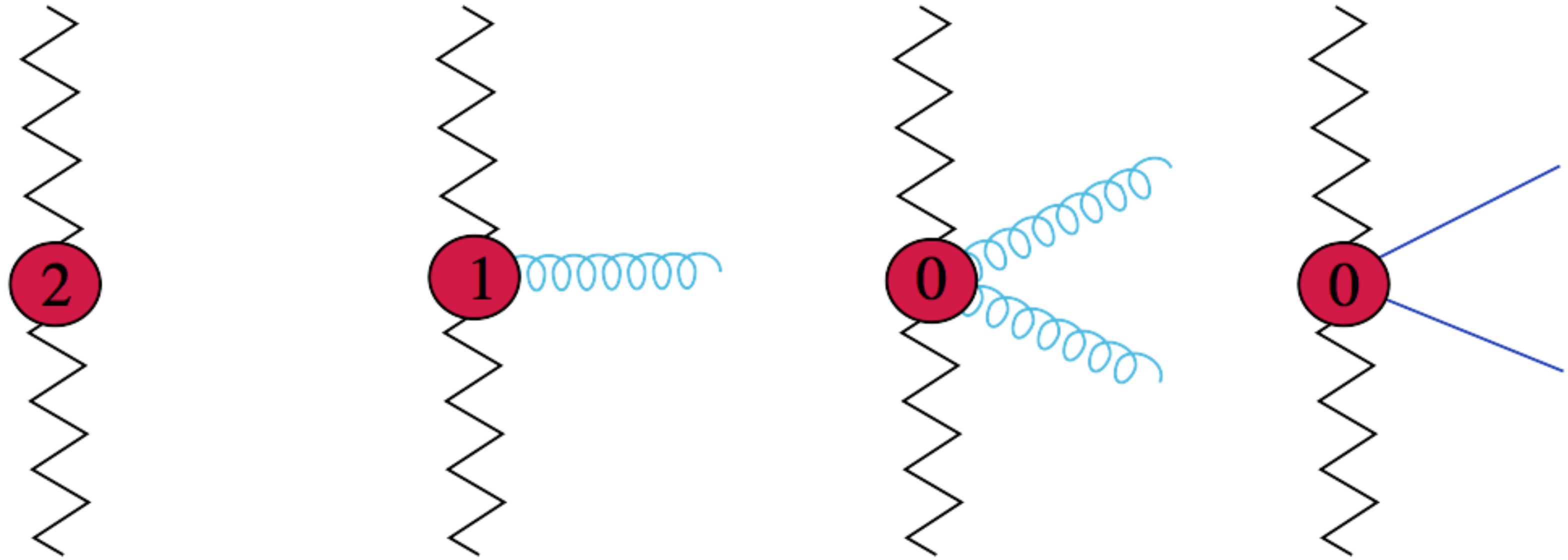
Evolution eq. in rapidity

$$f(x,\boldsymbol{k}_\perp^2) \equiv \frac{\partial\left[xg(x,\boldsymbol{k}_\perp^2)\right]}{\partial \ln \boldsymbol{k}_\perp^2}$$

Unintegrated gluon distribution: the probability to find a gluon with longitudinal momentum fraction x and transverse momentum $\boldsymbol{k}$

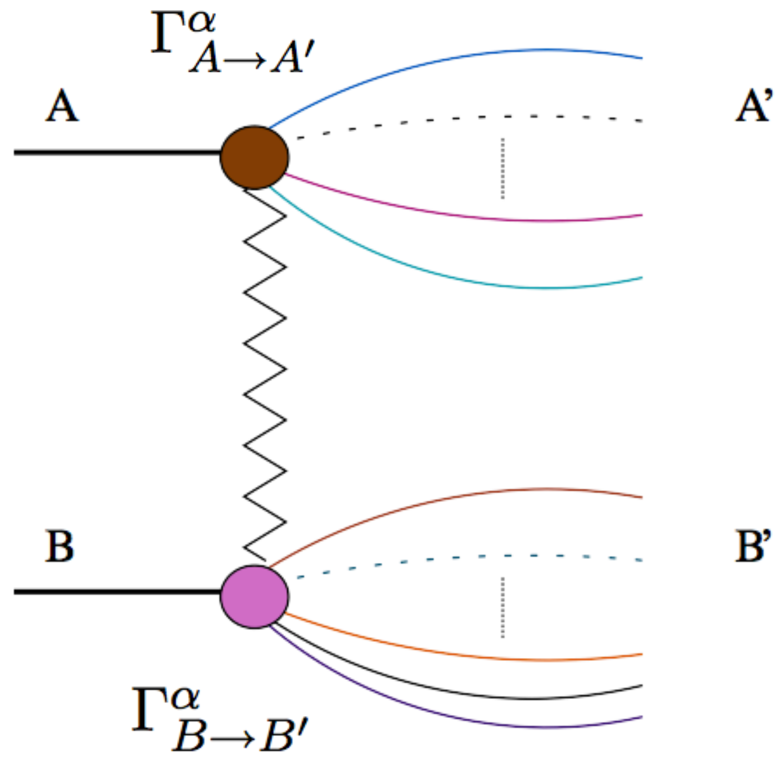
DIS

NLO BFKL



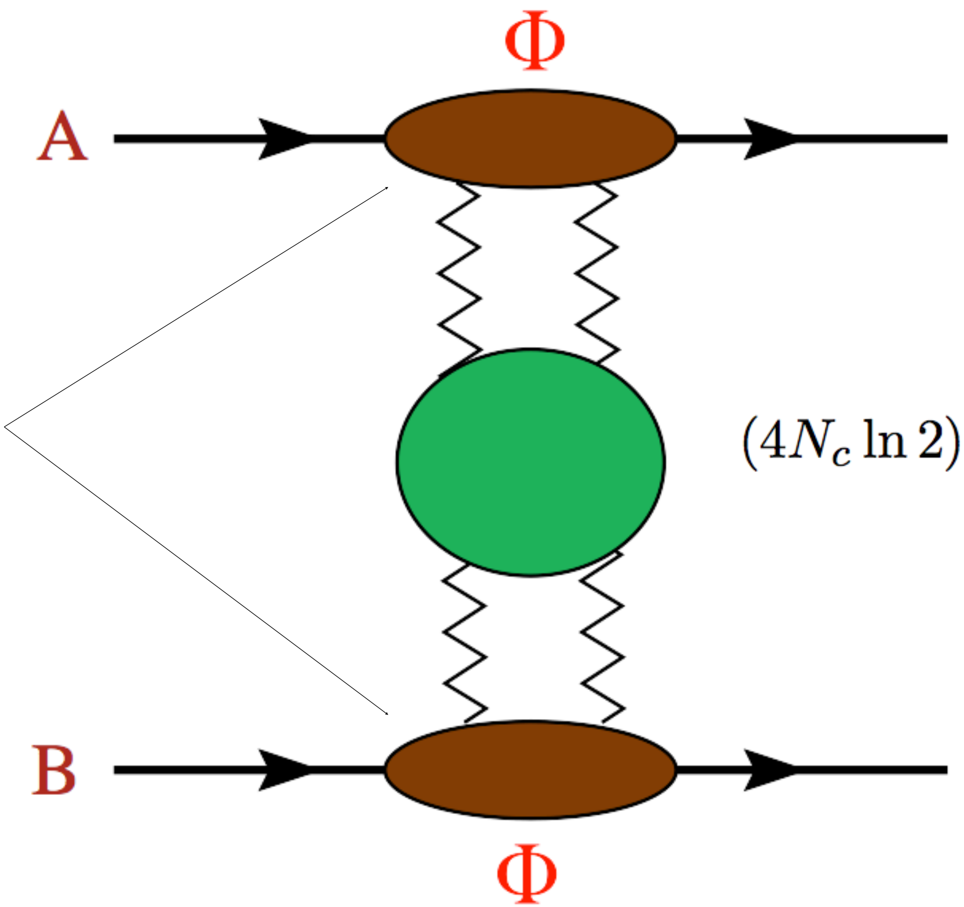
2–loop trajectory   1–loop g emission   pair production

Regge ansatz



$$\mathcal{M}_{AB} = \frac{s}{t}\Gamma^{\alpha}_{A\to A'}\left[\left(\frac{s}{-t}\right)^{\omega(t)} + \left(\frac{-s}{-t}\right)^{\omega(t)}\right]\Gamma^{\alpha}_{B\to B'}$$

A hadronic elastic amplitude



Impact factors... ?

$(4N_c \ln 2)/\pi)\alpha_s \approx 0.5.\ (\alpha_s \sim 0.2)$

$$\mathcal{A}(s,t) = i\, s\, \mathcal{C} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{d^2\mathbf{k}_2}{(2\pi)^2} \Phi_A(\mathbf{k}_1, \mathbf{q}) \frac{f(s, \mathbf{k}_1, \mathbf{k}_2, \mathbf{q})}{\mathbf{k}_2^2(\mathbf{k}_1 - \mathbf{q})^2} \Phi_B(\mathbf{k}_2, \mathbf{q})$$

Impact factors

Impact factors are effective couplings of the BFKL gluon Green's function to the colliding projectiles

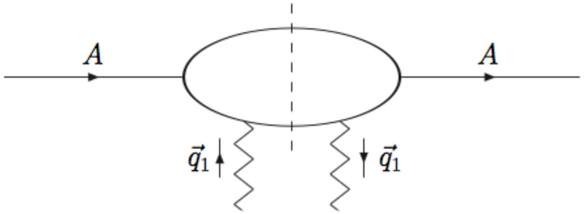
They are process dependent objects

One needs to calculate them at a certain order of the perturbative expansion, preferably the same one as that of the BFKL gluon Green's function.

It is not an easy task to calculate impact factors to NLO.

Impact factors

- **Impact factors** are **process-dependent**;



only very few have been calculated in the NLA:

- colliding partons
  [V.S. Fadin, R. Fiore, M.I. Kotsky, A. Papa (2000)]
  [M. Ciafaloni and G. Rodrigo (2000)]

- $\gamma^* \longrightarrow V$, with $V = \rho^0, \omega, \phi$, forward case
  [D.Yu. Ivanov, M.I. Kotsky, A. Papa (2004)]

- forward jet production
  [J. Bartels, D. Colferai, G.P. Vacca (2003)]
  [F. Caporale, D.Yu. Ivanov, B. M., A. Papa, A. Perri (2012)]
  (small-cone approximation) [D.Yu. Ivanov, A. Papa (2012)]

- forward identified hadron production [D.Yu. Ivanov, A. Papa (2012)]

- $\gamma^* \longrightarrow \gamma^*$
  [J. Bartels *et al.* (2001) →]
  [I. Balitsky, G.A. Chirilli (2011)-(2014)]

Taken from a talk by B. Murdaca

Bibliography II (indicative)

- Forshaw & Ross, “Quantum Chromodynamics and the Pomeron”
- Barone & Predazzi, High Energy Particle Diffraction
- Ioffe, Fadin & Lipatov, “Quantum Chromodynamics: Perturbative and Nonperturbative Aspects”
- Kovchegov & Levin, “Quantum Chromodynamics at High Energy”
- Many many review articles...