# *The Colour Glass Condensate 2*

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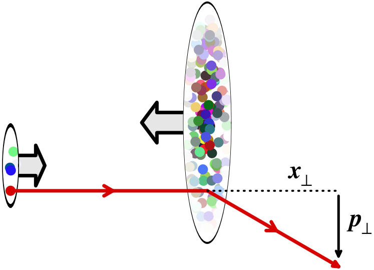
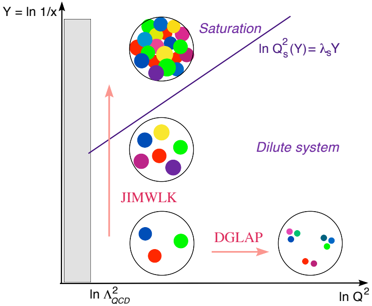


# Outline

## Lecture 1

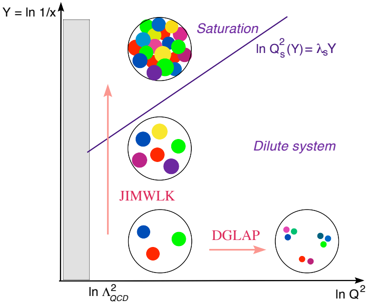
- Experimental motivations for small-$x$ gluons
- Introduction to the parton picture and to gluon saturation
- Main output: the saturation momentum $Q_s^2(x, A)$ at $x \leq 0.01$ and $A \gg 1$

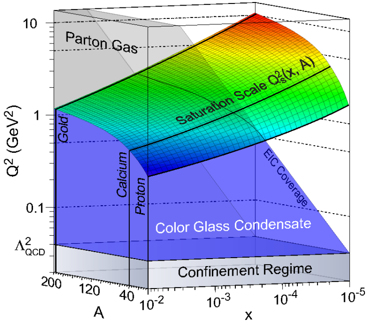
## Lecture 2

- The CGC effective theory in a nutshell
- Forward particle production in proton-nucleus $(pA)$ collisions
  - duality gluon saturation $\longleftrightarrow$ multiple scattering
  - eikonal approximation & Wilson lines
  - hybride factorisation
  - unintegrated gluon distribution
  - Cronin peak

- Characteristic transverse momentum for the onset of gluon saturation

$$Q_s^2(x,A) \simeq \alpha_s \frac{xG_A(x,Q_s^2)}{\pi R_A^2} \sim \frac{A^{1/3}}{x^{\lambda_s}} \quad \text{with} \quad \lambda_s \simeq 0.25$$

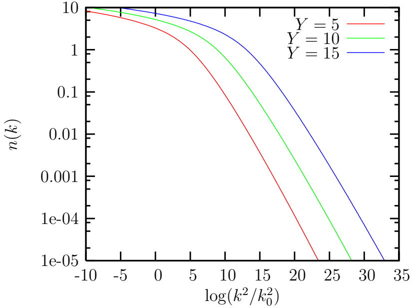



- $x \sim 10^{-3}$ (EIC): $Q_s^2 \sim 2$ GeV$^2$ for Pb or Au
- $x \sim 10^{-5}$ (LHC): $Q_s^2 \sim 10$ GeV$^2$ for Pb and $\sim 1$ GeV$^2$ for a proton

# The saturation front

- $Q_s(x)$ is also the typical transverse momentum for gluons with $x \ll 1$




- Gluon occupation number:

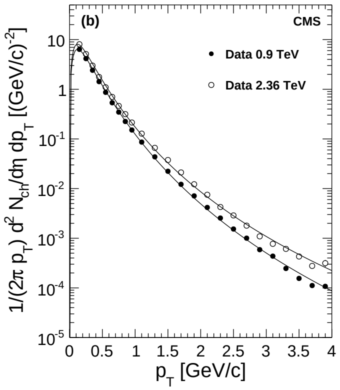
$$n(x, k_\perp) \simeq \frac{1}{\alpha_s} \begin{cases} 1 & \text{for } k_\perp \lesssim Q_s(x)\,, \\ \dfrac{Q_s^2(x)}{k_\perp^2} & \text{for } k_\perp \gg Q_s(x)\,. \end{cases}$$

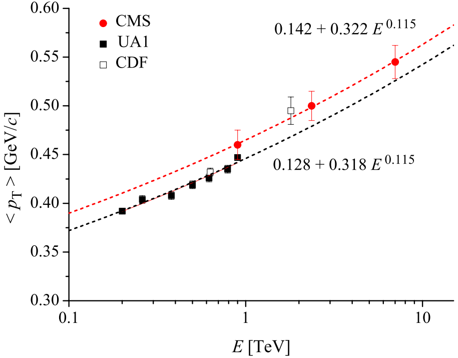
- Occupation numbers are rapidly decreasing when increasing $k_\perp$ above $Q_s$
- When increasing $Y = \ln(1/x)$: a front which progresses towards larger $k_\perp$
- For sufficiently large $Y$: $Q_s^2(Y) \gg \Lambda_{\rm QCD}^2 \implies \alpha_s\big(Q_s(Y)\big) \ll 1 \implies$ pQCD

# Average $p_\perp$ in $pp$ (LHC) and $p\bar{p}$ (Tevatron)

- The saturated gluons are released by the collision and fragment into hadrons in the final state

- Typical transverse momentum: $\langle p_T \rangle \propto Q_s(x) \sim E^{\lambda_s/2}$ $\quad (E \equiv \sqrt{s})$

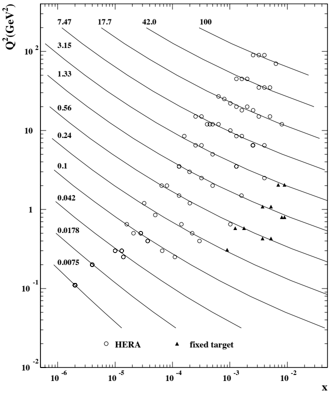


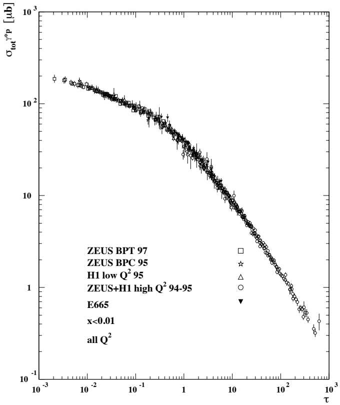




*(McLerran and Praszalowicz, 2010)*

- DIS cross-section $\sigma(x, Q^2)$ is *a priori* a function of 2 variables
- At small $x$, proton structure involves one intrinsic scale $Q_s(x)$

  $\Longrightarrow$ physics should depend upon the ratio $Q^2/Q_s^2(x)$ : geometric scaling

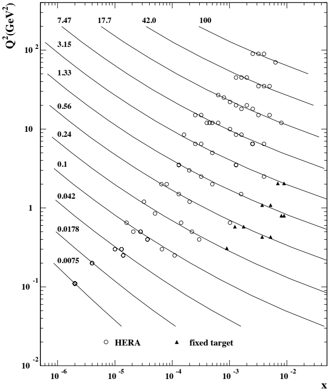


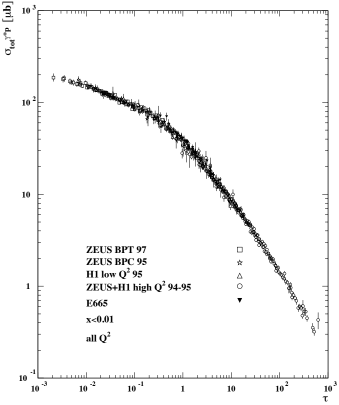



# Geometric scaling at HERA

- DIS cross-section $\sigma(x, Q^2)$ is *a priori* a function of 2 variables
- DIS cross–section at HERA *(Staśto, Golec-Biernat, Kwieciński, 2000)*

  $\sigma(x, Q^2)$ vs. $\tau \equiv Q^2/Q_s^2(x) \propto Q^2 x^{0.3}$ : $x \leq 0.01, \quad Q^2 \leq 450$ GeV$^2$

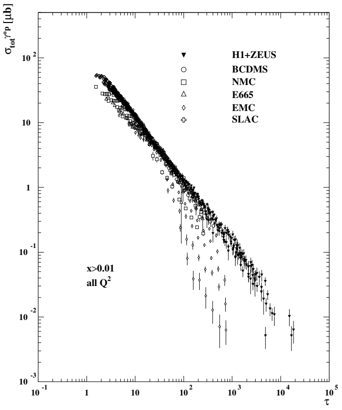


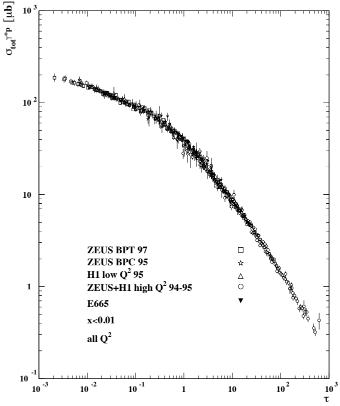




- Left: data in $(x, Q^2)$ plane. Right: cross-section as a function of $\tau$

# Geometric scaling at HERA

- DIS cross-section $\sigma(x, Q^2)$ is *a priori* a function of 2 variables




- No scaling for the HERA data corresponding to larger values $x > 0.01$
- Theory (CGC): *E.I., Itakura, McLerran, '02; Mueller, Triantafyllopoulos, '02*

- Particle multiplicity $\mathrm{d}N/\mathrm{d}\eta$: number of hadrons per unit rapidity near $\eta = 0$
- Saturated gluons which are released by the collision and hadronise

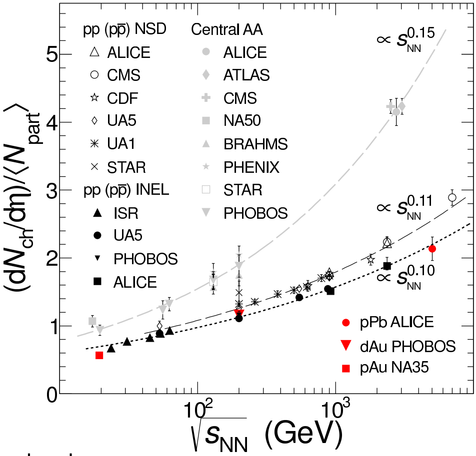
$$\frac{\mathrm{d}N}{\mathrm{d}\eta} \propto xG(x, Q_s^2) \propto Q_s^2(x)$$

▷ which value for $x$ ?

$$x \simeq \frac{k_\perp}{\sqrt{s}} \quad \& \quad k_\perp \sim Q_s$$

$$Q_s^2(x) \propto \frac{1}{x^{\lambda_s}} \sim s^{\frac{\lambda_s}{2+\lambda_s}}$$

$$\lambda_s \simeq 0.2 \div 0.3$$




- Qualitatively consistent with the data

# Color Glass Condensate

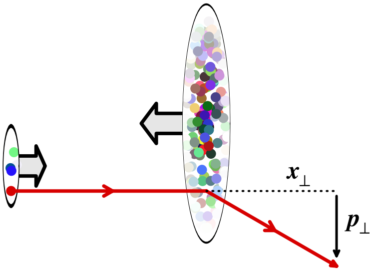
- "Colour": gluons carry the non-Abelian, SU$(3)$, charge of QCD
- "Glass": in the infinite momentum frame of the hadron, gluons are frozen (by Lorentz time dilation) in random configurations: the BFKL cascades
- "Condensate": gluons at saturation have large occupation numbers $n \sim 1/\alpha_s$
- They can faithfully be described as strong & random classical colour fields with $x$-dependent gauge-invariant correlations

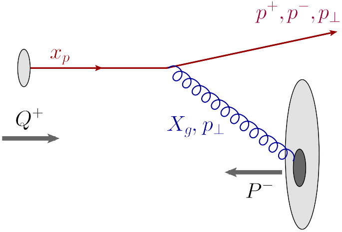
$$n(x, \boldsymbol{k}_\perp) \sim \langle A_a^i(\boldsymbol{k}_\perp) A_a^i(-\boldsymbol{k}_\perp) \rangle_x \sim \frac{1}{\alpha_s} \implies A_a^i \sim \frac{1}{g}$$

- An effective field theory :
  - classical equations or motion: Yang-Mills equations, Wilson lines
  - quantum evolution with decreasing $x$: BK/JIMWLK equations
  - higher order quantum corrections computable in pQCD
  - some non-perturbative ingredients: initial conditions at low energy, transverse geometry, effective gluon mass...

# Particle production in $pA$ collisions

- A quark initially collinear with the proton acquires a transverse momentum $p_\perp \sim Q_s$ via (generally, multiple) scattering off the gluons in the nucleus
- In practice: p+Pb collisions at the LHC and d+Au collisions at RHIC

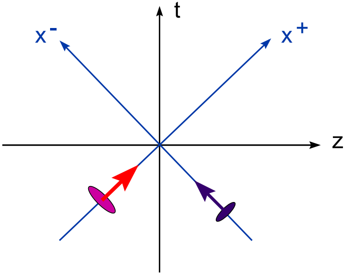



- Single scattering to start with: a $2 \to 1$ process: $q(0_\perp)g(p_\perp) \to q(p_\perp)$
  - contrast to collinear factorisation, which starts with a $2 \to 2$ process
  - semi-hard intrinsic $p_T$ (typically, $p_\perp \sim Q_s$) makes the difference

# Light-cone variables

- Four-momentum: $p^\mu = (p^0, p^1, p^2, p^3) \equiv (p_0, \boldsymbol{p}_\perp, p_z) = (p^+, p^-, \boldsymbol{p}_\perp)$




$$p^\pm = \frac{p_0 \pm p_z}{\sqrt{2}}\,, \quad x^\pm = \frac{t \pm z}{\sqrt{2}}$$

$$p \cdot x = p^+ x^- + p^- x^+ - \boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp$$

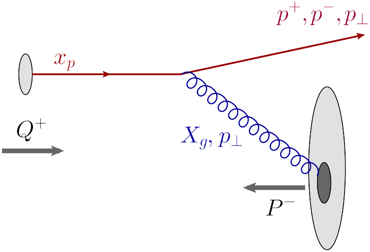
$$\mathrm{d}t\mathrm{d}z = \mathrm{d}x^+\mathrm{d}x^-, \quad p^2 = 2p^+p^- - p_\perp^2$$

- Ultrarelativistic right mover: $p_z \simeq p_0 \gg p_\perp$ & $z \simeq t$
  - $p^+ \simeq \sqrt{2} p_z$ (LC longitudinal momentum) & $p^- \simeq 0$ (LC energy)
  - $x^+ = \sqrt{2} t$ (LC time) & $x^- \simeq 0$ (LC longitudinal coordinate)
- Left mover: the roles of $x^+$ and $x^-$ (or $p^+$ and $p^-$) get interchanged
- Similar LC components for all 4-vectors/tensors: $A_a^\pm(x) = (A_a^0 \pm A_a^3)/\sqrt{2}$

# Particle production in $pA$ collisions

- Express kinematics in terms of LC variables : $p^\mu = (p^+, p^-, \boldsymbol{p}_\perp)$
  - initial quark (a right mover): $q^\mu = (x_p Q^+, 0, \boldsymbol{0}_\perp)$
  - initial gluon (a left mover): $k^\mu = (0, X_g P_N^-, \boldsymbol{p}_\perp)$
  - final quark: $p^\mu = q^\mu + k^\mu = (x_p Q^+, X_g P_N^-, \boldsymbol{p}_\perp)$
- The (pseudo)rapidity of the final quark ($2p^+p^- = p_\perp^2$):

$$\eta = y = \frac{1}{2}\ln\frac{p^+}{p^-} \implies p^\pm = \frac{p_\perp}{\sqrt{2}}\,\mathrm{e}^{\pm\eta}$$




- COM frame: $Q^+ = P^- = \sqrt{s/2}$

$$x_p = \frac{p_\perp}{\sqrt{s}}\,\mathrm{e}^{\eta}\,, \quad X_g = \frac{p_\perp}{\sqrt{s}}\,\mathrm{e}^{-\eta}$$

$$X_g \ll x_p \quad \text{when} \quad \eta > 1$$

- Forward production $\leftrightarrow$ small $X_g$

# Particle production in $pA$ collisions

- Express kinematics in terms of LC variables : $p^\mu = (p^+, p^-, \boldsymbol{p}_\perp)$
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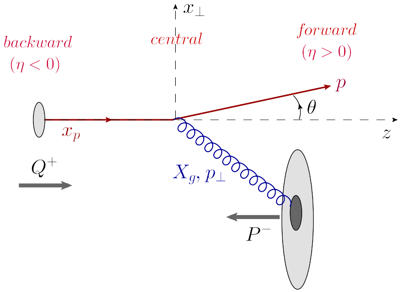


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$X_g \ll x_p$ when $\eta > 1$

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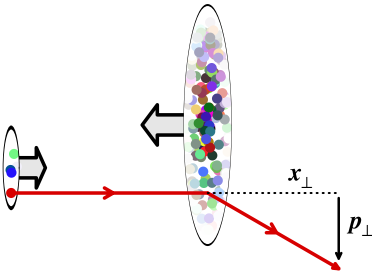
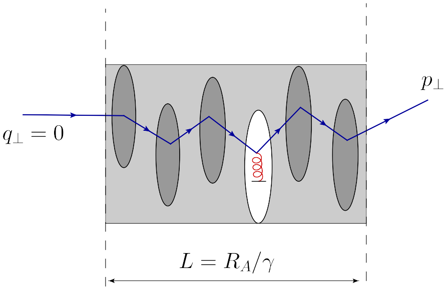
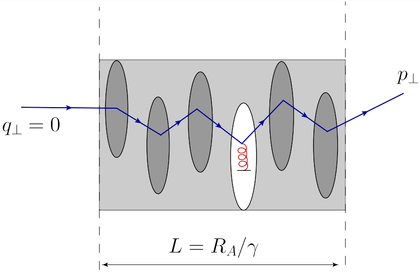
# Multiple scattering

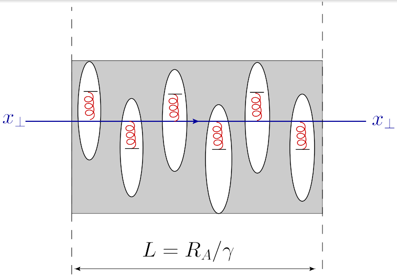



- The quark scatters off the color sources within the nucleons that it crosses
  - a longitudinal tube with length $L \sim R_A/\gamma$ with $R_A = R_0 A^{1/3}$
- Collisions off different nucleons are independent from each other
  - a random walk in transverse momentum: $p_\perp$–broadening
- The transverse coordinate of the quark is not modified by the scattering

# Eikonal approximation




- By the uncertainty principle, the quark is delocalised in the transverse plane over a distance $\lambda_\perp \sim 1/p_\perp$: smaller deviations $\Delta x_\perp \ll \lambda_\perp$ don't matter

$$\Delta x_\perp \simeq \frac{p_\perp}{p^+} L \ll \lambda_\perp \sim \frac{1}{p_\perp} \iff p_\perp^2 L \ll p^+ : \quad \text{true at high energy}$$

- Using $L = \frac{R}{\gamma} \sim \frac{1}{P_N^-}$ and $p^+ P_N^- \sim s$, this is the same as $s \gg p_\perp^2$

- The corrections to the eikonal approx are suppressed by $p_\perp/\sqrt{s}$

# Wilson lines

- In the eikonal approx, the quark $S$-matrix reduces to a Wilson line

$$\hat{S}_q = \mathrm{T}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}(x)} \quad \text{with} \quad \mathcal{L}_{\rm int}(x) = j_a^\mu(x) A_\mu^a(x)$$

  - in general, $j_a^\mu$ and $A_a^\mu$ are quantum operators; e.g. $j_a^\mu = g\bar{\psi}\gamma^\mu t^a \psi$
  - at high energy they can be replaced by their classical expectation values: their fluctuations are frozen

- The (classical) color current density of the right-moving quark: $v^\mu = \delta^{\mu +}$

$$j_a^\mu(x) = \delta^{\mu +} g t^a \delta(x^-)\delta^{(2)}(\boldsymbol{x}_\perp - \boldsymbol{x}_\perp^0) \; : \quad \text{indep. of LC time } x^+$$

$$\hat{S}_q \simeq V(\boldsymbol{x}_\perp^0) \;\; \text{with} \;\; V(\boldsymbol{x}_\perp) \equiv \mathrm{T} \exp\left\{ig \int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}_\perp) t^a\right\}$$

- An exponential : multiple scattering is resummed to all orders

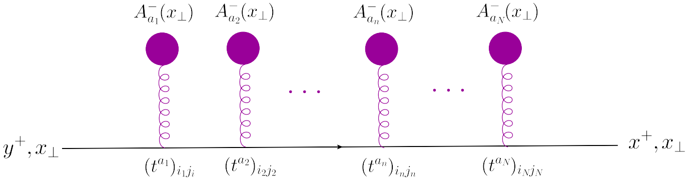
- A unitary matrix: $VV^\dagger = 1 \Longrightarrow$ a rotation of the quark color state

# Wilson lines

- A Wilson line at transverse coordinate $\boldsymbol{x}_\perp$ extending from $y^+$ to $x^+ > y^+$:

$$V_{x^+y^+}(\boldsymbol{x}_\perp) = \mathrm{T}\exp\left\{ig\int_{y^+}^{x^+} \mathrm{d}z^+ A_a^-(z^+, \boldsymbol{x}_\perp)t^a\right\}, \quad \mathrm{T}: \text{ordering in } z^+$$

- Best understood with a discretisation of time: $z_n^+ = n\epsilon,\ n = 1, \cdots, N$




$$V_N(\boldsymbol{x}_\perp) = \mathrm{e}^{\mathrm{i}g\epsilon A_N^-}\, \mathrm{e}^{\mathrm{i}g\epsilon A_{N-1}^-} \ldots \mathrm{e}^{\mathrm{i}g\epsilon A_1^-} \qquad \left(A_n^- \equiv A_a^-(z_n^+, \boldsymbol{x}_\perp)t^a\right)$$

$$V_N^\dagger(\boldsymbol{x}_\perp) = \mathrm{e}^{-\mathrm{i}g\epsilon A_1^-} \ldots \mathrm{e}^{-\mathrm{i}g\epsilon A_{N-1}^-}\mathrm{e}^{-\mathrm{i}g\epsilon A_N^-} \qquad \text{(reverse ordering)}$$

- Easy to check unitarity: $V_N(\boldsymbol{x}_\perp)\, V_N^\dagger(\boldsymbol{x}_\perp) = 1$

- A Wilson line at transverse coordinate $\boldsymbol{x}_\perp$ extending from $y^+$ to $x^+ > y^+$:

$$V_{x^+y^+}(\boldsymbol{x}_\perp) = \mathrm{T}\exp\left\{ig\int_{y^+}^{x^+} \mathrm{d}z^+ A^-_a(z^+, \boldsymbol{x}_\perp)t^a\right\}, \quad \mathrm{T}: \text{ ordering in } z^+$$

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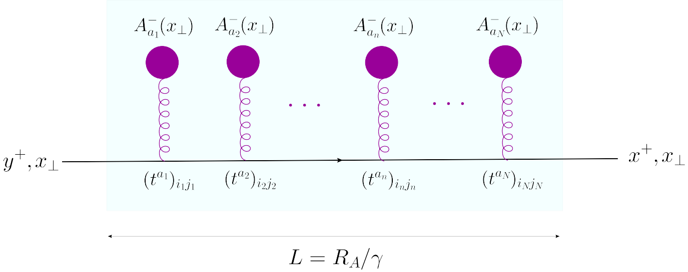


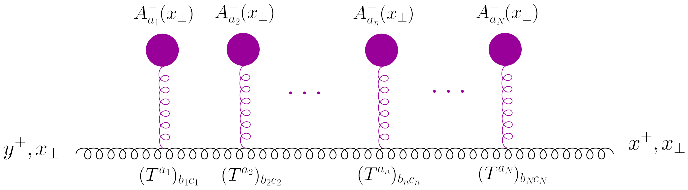

- The support of the field restricted to the longitudinal extent of the target

- $L \to 0$ when $\gamma \to \infty$: a shockwave ("pancake")

# Wilson lines

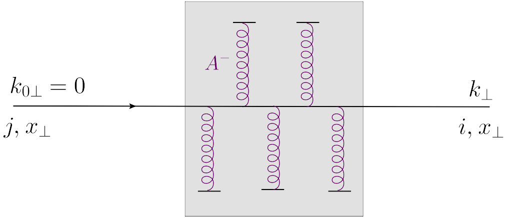
- Similar definition in any other representation, e.g. the adjoint one:

$$U_{x^+y^+}(\boldsymbol{x}_\perp) = \mathrm{T}\exp\left\{ig\int_{y^+}^{x^+} \mathrm{d}z^+ A_a^-(z^+,\boldsymbol{x}_\perp)T^a\right\}, \quad \mathrm{T}: \text{ ordering in } z^+$$




- A right-moving probe gluon propagating in the colour field of the target
- Wilson lines: fundamental degrees of freedom for scattering at high energies
- Frozen ($x^-$-independent) background field $A_a^-(x^+,\boldsymbol{x}_\perp)$, to be averaged over at the level of the cross-section

- The amplitude for the quark to emerge with a transverse momentum $\boldsymbol{k}_\perp$ after crossing the nuclear target

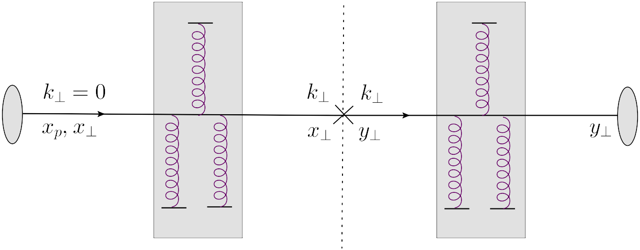



$$\mathcal{M}_{ij}(\boldsymbol{k}_\perp) = \int \mathrm{d}^2\boldsymbol{x}_\perp\, \mathrm{e}^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{k}_\perp}\, V_{ij}(\boldsymbol{x}_\perp)$$

- the Fourier transform of the Wilson line
- color rotation: $i$, $j$ color indices for the fundamental repres.

# The cross-section

- Take the modulus squared of the amplitude $|\mathcal{M}_{ij}(\boldsymbol{k}_\perp)|^2$
- Sum over the final colour indices, average over the initial ones
- Average over the target color field $A^-$ weight function)
- Multiply with the quark PDF of the proton $(k^+ = x_p Q^+)$



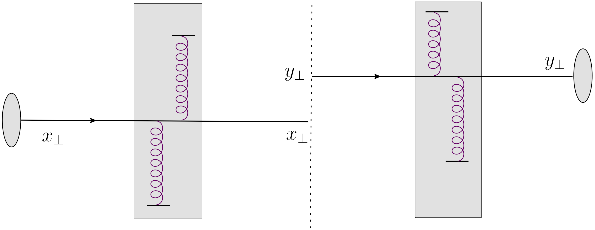

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}_\perp} \simeq x_p q(x_p, Q^2) \frac{1}{N_c} \left\langle \sum_{ij} |\mathcal{M}_{ij}(\boldsymbol{k}_\perp)|^2 \right\rangle_{X_g}$$

# Dipole picture for $pA$ collisions

- The sum over color indices ⇒ the trace of a product of 2 Willson lines:

$$S(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; X_g) \equiv \frac{1}{N_c} \left\langle \mathrm{tr}\left[V(\boldsymbol{x}_\perp) V^\dagger(\boldsymbol{y}_\perp)\right] \right\rangle_{X_g}$$

  - $V(\boldsymbol{x}_\perp)$ for the quark in the direct amplitude
  - $V^\dagger(\boldsymbol{y}_\perp)$ for the quark in the complex conjugate amplitude



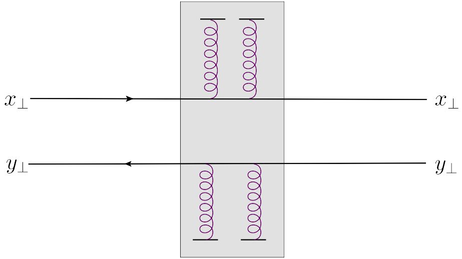

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p) \int \frac{\mathrm{d}^2\boldsymbol{x}\mathrm{d}^2\boldsymbol{y}}{(2\pi)^2} \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}} \, S(\boldsymbol{x}, \boldsymbol{y}; X_g)$$

# Dipole picture for $pA$ collisions

- The sum over color indices $\Rightarrow$ the trace of a product of 2 Willson lines:

$$S(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; X_g) \equiv \frac{1}{N_c} \left\langle \mathrm{tr}\left[V(\boldsymbol{x}_\perp) V^\dagger(\boldsymbol{y}_\perp)\right] \right\rangle_{X_g}$$

- Formally, the same as the elastic $S$-matrix for a $q\bar{q}$ color dipole




$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p) \int \frac{\mathrm{d}^2\boldsymbol{x} \mathrm{d}^2\boldsymbol{y}}{(2\pi)^2} \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}} \, S(\boldsymbol{x}, \boldsymbol{y}; X_g)$$

- Emerging dipole picture for transverse momentum broadening in $pA$

# Hybride factorisation

- The cross-section for quark production in $pA$ takes a factorised form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p)\, \frac{\alpha_s}{k_\perp^2}\, \mathcal{F}_g(X_g, k_\perp)$$

  - quark PDF for the incoming proton
  - cross-section $\frac{\alpha_s}{k_\perp^2}$ for gluon absorption by the quark
  - unintegrated gluon distribution for the nuclear target

$$\mathcal{F}_g(X_g, k_\perp) = \frac{k_\perp^2}{\alpha_s} \int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, S(\boldsymbol{x}, \boldsymbol{y}; X_g)$$

- Rewrite the $k_\perp^2$ factor as the effect of 2 transverse spatial derivatives

# Hybride factorisation

- The cross-section for quark production in $pA$ takes a factorised form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p)\, \frac{\alpha_s}{k_\perp^2}\, \mathcal{F}_g(X_g, k_\perp)$$

  - quark PDF for the incoming proton
  - cross-section $\frac{\alpha_s}{k_\perp^2}$ for gluon absorption by the quark
  - unintegrated gluon distribution for the nuclear target

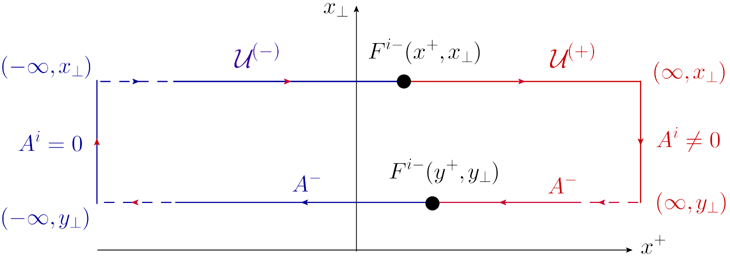
$$\mathcal{F}_g(X_g, k_\perp) = \frac{1}{\alpha_s}\int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, \frac{1}{N_c}\left\langle \mathrm{tr}\left[\partial^i V(\boldsymbol{x})\, \partial^i V^\dagger(\boldsymbol{y})\right]\right\rangle_{X_g}$$

- Rewrite the $k_\perp^2$ factor as the effect of 2 transverse spatial derivatives
- Use the following mathematical identity (with $F_a^{i-} = \partial^i A_a^-$): [Exercice !]

$$\partial^i V(\boldsymbol{x}_\perp) = ig \int \mathrm{d}x^+\, V_{\infty, x^+}(\boldsymbol{x}_\perp)\, F_a^{i-}(x^+, \boldsymbol{x}_\perp) t^a\, V_{x^+, -\infty}(\boldsymbol{x}_\perp)$$

# The dipole gluon TMD

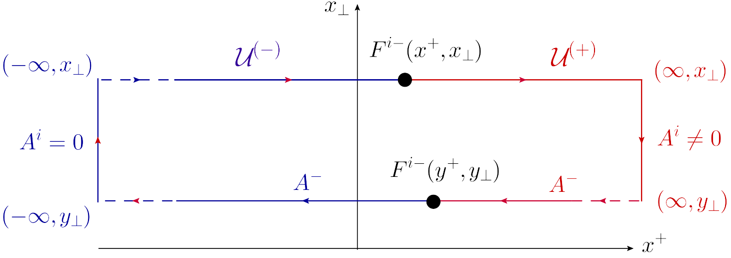
$$\mathcal{F}_g(X_g, k_\perp) = \frac{1}{N_c}\int_{x,y} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}} \left\langle \mathrm{tr}\left[F^{i-}(x)\,\mathcal{U}^{(-)}(x,y)\,F^{i-}(y)\,\mathcal{U}^{(+)\dagger}(x,y)\right]\right\rangle$$




- A gauge-invariant 2-point correlation of the chromo-electric fields $F^{i-}$
- A process-dependent gluon transverse momentum dependent distribution
  - different processes ⇒ different contours, different gluon TMDs
- Non-linear effects encoded in $\mathcal{U}^{(\pm)}$: gluon saturation

# The dipole gluon TMD

$$\mathcal{F}_g(X_g, k_\perp) = \frac{1}{N_c}\int_{x,y} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}} \left\langle \mathrm{tr}\left[F^{i-}(x)\,\mathcal{U}^{(-)}(x,y)\,F^{i-}(y)\,\mathcal{U}^{(+)\dagger}(x,y)\right]\right\rangle$$




- In the dilute regime at large $k_\perp \gg Q_s(X_g)$, fields are weak: $\mathcal{U}^{(\pm)} \simeq 1$
  - the universal "unintegrated gluon distribution" of $k_\perp$-factorisation
- The gluon PDF is contour independent as well (since $\boldsymbol{x}_\perp \to \boldsymbol{y}_\perp$)
- Collinear/TMD/$k_\perp$–factorisations emerge from CGC at large $k_\perp \gg Q_s$

# The dipole scattering amplitude

$$\mathcal{F}_g(k_\perp) = \frac{k_\perp^2}{\alpha_s} \int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, S(\boldsymbol{x},\boldsymbol{y}), \quad S(\boldsymbol{x},\boldsymbol{y}) = \frac{\mathrm{tr}}{N_c}\left\langle V(\boldsymbol{x})V^\dagger(\boldsymbol{y})\right\rangle$$

- The Fourier transform selects $r \sim 1/k_\perp$, where $r \equiv |\boldsymbol{x}-\boldsymbol{y}|$ is the dipole size
- When $k_\perp \to \infty$, $r \to 0$ and $S(\boldsymbol{x},\boldsymbol{y}) \to 1$ : no scattering
  - a zero-size dipole = a colorless, point-like object, like a photon
- The relevant quantity is the dipole scattering amplitude: $T = 1 - S$
  - weak scattering $\Longleftrightarrow S \approx 1 \Longleftrightarrow T \ll 1$
- The target looks dense for $k_\perp \lesssim Q_s$, but dilute for $k_\perp \gg Q_s$
- The dipole scattering is strong for $r \gtrsim 1/Q_s$, but weak for $r \ll 1/Q_s$
- Duality between gluon saturation and multiple scattering

# The single scattering approximation

- Dilute target $k_\perp \gg Q_s \Leftrightarrow$ weak scattering $r \ll 1/Q_s$: double expansion
  - expand the Wilson lines to quadratic order in $A^-$: single scattering
  - expand to quadratic order in $r$: leading twist
- Weak field expansion:

$$V(\boldsymbol{x}) \simeq 1 + ig \int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}) t^a$$
$$- \frac{g^2}{2} \int_{x^+, y^+} \left[\theta(x^+ - y^+) t^a t^b + \theta(y^+ - x^+) t^b t^a\right] A_a^-(x^+, \boldsymbol{x}) A_a^-(y^+, \boldsymbol{x})$$

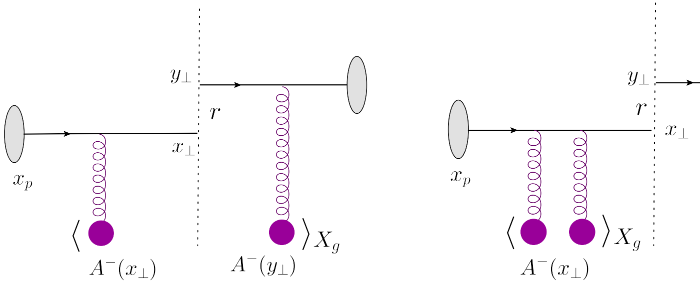
- Dipole amplitude in the single scattering approximation: 2-gluon exchange

$$T_0(\boldsymbol{x}, \boldsymbol{y}) = \frac{g^2}{4N_c} \left\langle \left[A_a^-(\boldsymbol{x}) - A_a^-(\boldsymbol{y})\right]^2 \right\rangle$$

$$A_a^-(\boldsymbol{x}) \equiv \int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}), \quad \langle A_a^- \rangle = 0, \quad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$

# The single scattering approximation (2)

$$T_0(\boldsymbol{x}, \boldsymbol{y}; X_g) = \frac{g^2}{4N_c} \left\langle \left[ A_a^-(\boldsymbol{x}) - A_a^-(\boldsymbol{y}) \right]^2 \right\rangle_{X_g}$$



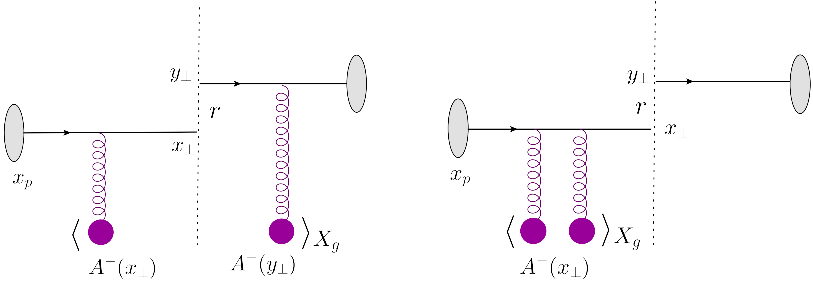

- Small dipole size expansion: $A_a^-(\boldsymbol{x}) - A_a^-(\boldsymbol{y}) \simeq r^i F_a^{i-}(\boldsymbol{b})$, $\boldsymbol{b} = \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y})$
- Recall: the gluon distribution

$$xG(x, Q^2) \propto \int \mathrm{d}^2\boldsymbol{b} \, \left\langle F_a^{i-}(\boldsymbol{b}) \, F_a^{i-}(\boldsymbol{b}) \right\rangle_x \quad \text{with} \quad Q^2 \sim 1/r^2$$

- The scattering of a small dipole: a direct measurement of $xG(x, Q^2)$

# The single scattering approximation (2)

$$T_0(\boldsymbol{x},\boldsymbol{y};X_g) \simeq \frac{\pi^2\alpha_s r^2}{2N_c}\,\frac{\mathrm{d}\,xG(x,4/r^2)}{\mathrm{d}^2\boldsymbol{b}} \quad \text{with} \quad x = X_g$$

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# Multiple scattering and unitarity

- Assume a homogeneous nucleus with radius $R_A$

$$T_0(r;x) \simeq \pi^2 \, \frac{\alpha_s r^2}{2N_c} \, \frac{xG(x, 4/r^2)}{\pi R_A^2}$$

- $T_0(r) \propto r^2$: colour transparency (a small dipole cannot scatter)

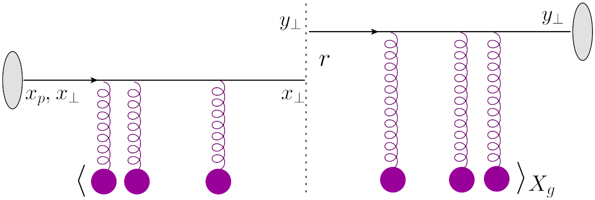
- $T_0$ increases with $r$ but cannot exceed the unitarity limit: $T = 1 - S \leq 1$

# Multiple scattering and unitarity

- Assume a homogeneous nucleus with radius $R_A$

$$T_0(r;x) \simeq \pi^2 \frac{\alpha_s r^2}{2N_c} \frac{xG(x,4/r^2)}{\pi R_A^2} \;\sim 1 \quad \text{when} \quad 2/r \sim Q_s(x)$$

- $T_0(r) \propto r^2$: colour transparency (a small dipole cannot scatter)
- $T_0$ increases with $r$ but cannot exceed the unitarity limit: $T = 1 - S \leq 1$
- When $r \sim 1/Q_s$, $T_0 \sim 1$ and multiple scattering becomes important




- Complicated in general, but easy to compute for independent colour sources

# Multiple scattering in the MV model

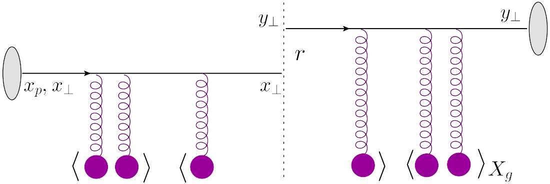
- McLerran-Venugopalan model: a large nucleus $\approx AN_c$ valence quarks

$$xG_A^{(0)}(x,Q^2)=\frac{\alpha_s AN_c C_F}{\pi}\ln\frac{Q^2}{\Lambda^2}$$

- The dipole scattering amplitude in the single-scattering approximation:

$$T_0(r;x)=\frac{r^2Q_A^2}{4}\ln\frac{4}{r^2\Lambda^2}\quad \text{with}\quad Q_A^2\equiv\frac{2\alpha_s^2 C_F A}{R_A^2}\propto A^{1/3}$$

- Independent sources $\Rightarrow$ Gaussian approximation




- The multiple scattering series exponentiates: $S(r)=\mathrm{e}^{-T_0(r)}$

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- $S(r)$ : Probability for the $q\bar{q}$ pair to survive in a color singlet ("dipole") state

$$S(r) = \mathrm{e}^{-T_0(r)} \equiv 1 - T(r)$$

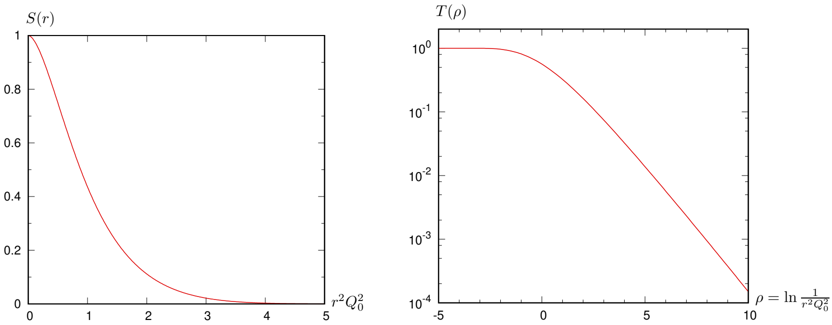
- Saturation momentum $Q_s$: conventionally defined as $T_0(r) = 1$ for $r = \frac{2}{Q_s}$

$$Q_s^2(A) = Q_A^2 \ln \frac{Q_s^2(A)}{4\Lambda^2} \propto A^{1/3} \ln A^{1/3}$$

- Duality: saturation in the target $\Leftrightarrow$ multiple scattering for the probe

# Dipole scattering (MV model)

- Left: the dipole $S$-matrix $S$ as a function of $r^2Q_s^2$




- Right: the dipole amplitude $T \equiv 1 - S$ as a function of $\rho \equiv \ln(1/r^2Q_s^2)$
  - small dipole $r \ll 1/Q_s \Longrightarrow$ large values for $\rho$ : $T \simeq T_0 \sim r^2Q_s^2 = \mathrm{e}^{-\rho}$
  - large dipole $r \gtrsim 1/Q_s \Longrightarrow$ negative $\rho$ : $T = 1$

# Momentum broadening (MV model)

- Quark production in $pA$ collisions: the transverse momentum distribution

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\;\mathrm{e}^{-\frac{1}{4}r^2Q_A^2\,\ln\frac{1}{r^2\Lambda^2}} = \tilde{S}(\boldsymbol{k})$$

- Would-be a Gaussian integration ... if there were not for the logarithm
- Competition between $r \sim 1/k_\perp$ (Fourier phase) and $r \sim 1/Q_s$ ($S$-matrix)
- The bulk of the distribution lies around $k_\perp \sim Q_s$
  - replace $1/r^2 \to Q_s^2$ within the argument of the log

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{1}{\pi Q_s^2}\,\mathrm{e}^{-k_\perp^2/Q_s^2}$$

  - a Gaussian distribution: a random walk in $\boldsymbol{k}$

$$\langle k_\perp^2\rangle \equiv \int \mathrm{d}^2\boldsymbol{k}\,k_\perp^2\,\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = Q_s^2(A)$$

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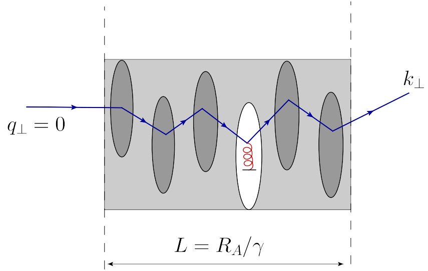


$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{1}{\pi Q_s^2}\,\mathrm{e}^{-k_\perp^2/Q_s^2}$$

$$\langle k_\perp^2 \rangle = Q_s^2(A) \sim L \sim A^{1/3}$$

- Transverse momentum broadening via multiple soft scattering: diffusion

# Momentum broadening (MV model)

- Quark production in $pA$ collisions: the transverse momentum distribution

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\ \mathrm{e}^{-\frac{1}{4}r^2Q_A^2\ln\frac{1}{r^2\Lambda^2}} = \tilde{S}(\boldsymbol{k})$$

- Would-be a Gaussian integration ... if there were not for the logarithm
- Competition between $r\sim 1/k_\perp$ (Fourier phase) and $r\sim 1/Q_s$ ($S$-matrix)
- A larger value $k_\perp \gg Q_s$ can be acquired via a single hard scattering
  - integral cut off at $r\sim 1/k_\perp$ by the exponential
  - $rQ_s \ll 1 \implies$ one can expand $S\simeq 1-T_0$ (one scattering)

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{Q_A^2}{\pi k_\perp^4} \quad \text{for} \quad k_\perp \gg Q_s$$

- Power-law tail reflecting Coulomb scattering

# Momentum broadening (MV model)

- Quark production in $pA$ collisions: the transverse momentum distribution

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\;\mathrm{e}^{-\frac{1}{4}r^2 Q_A^2 \ln\frac{1}{r^2\Lambda^2}} = \tilde{S}(\boldsymbol{k})$$

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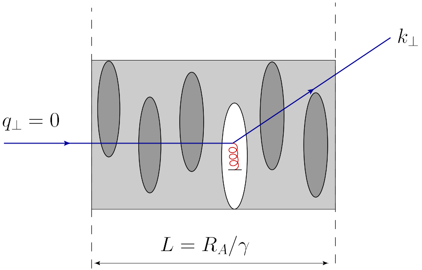


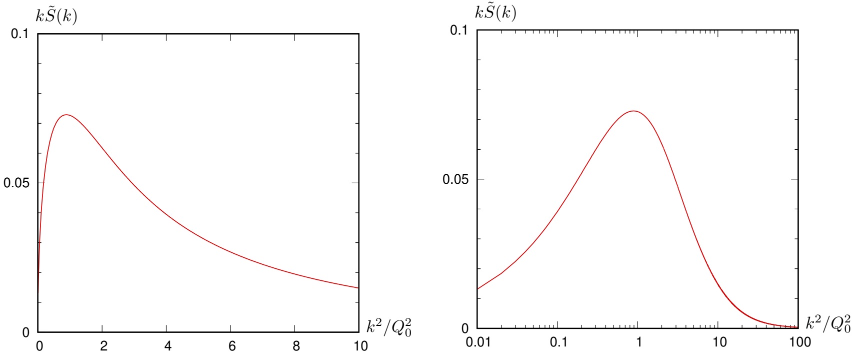

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{Q_A^2}{\pi k_\perp^4} \quad \text{for} \quad k_\perp \gg Q_s$$

- "Molière scattering" (1948)

- Not the famous French dramatist, but a German physicist !

# The dipole gluon distribution

- Left: the Fourier transform $k_\perp \tilde{S}(k_\perp)$
  - the $k_\perp$–distribution of the produced quark in $pA$ collisions
  - also that of the gluons in the nucleus ! (the "gluon dipole TMD")




- Right: the same function, but in logarithmic units
  - peaked at $k \simeq Q_s$, power-law tail at $k \gg Q_s$

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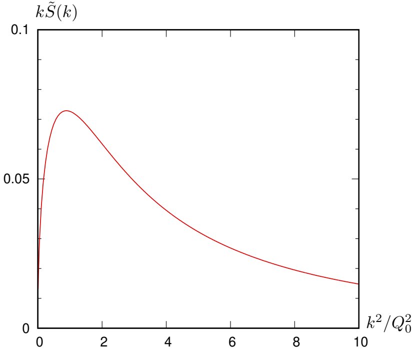
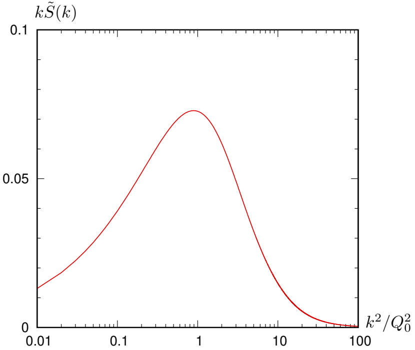
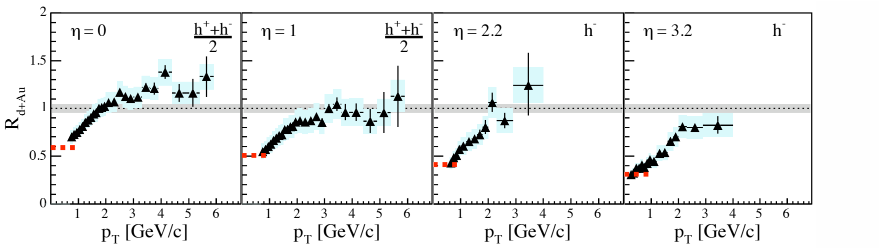



- An interesting consequence for hadron production in $pA$ collisions:
  - the Cronin peak

# The nuclear modification factor at RHIC

$$R_{pA} \equiv \frac{1}{A} \frac{\mathrm{d}\sigma_{pA}/\mathrm{d}^2p_\perp \mathrm{d}\eta}{\mathrm{d}\sigma_{pp}/\mathrm{d}^2p_\perp \mathrm{d}\eta}$$




- Would be 1 if $pA =$ incoherent superposition of $pp$ collisions
  - any deviation from unity is a signature of nuclear (high density) effects
- The RHIC data (d+Au) show two interesting nuclear effects
  - central rapidity ($\eta \simeq 0$): $R_{\mathrm{d+Au}} > 1$ for $p_\perp \gtrsim 2$ GeV ("Cronin peak")
  - forward rapidities ($\eta \gtrsim 2$): $R_{\mathrm{d+Au}} < 1$ (nuclear suppresion)

# Midrapidity: the Cronin peak

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta\mathrm{d}^2\boldsymbol{k}} = x_p q(x_p)\, \frac{\alpha_s}{k_\perp^2}\, \mathcal{F}_g(X_g, k_\perp), \qquad \mathcal{F}_g = \frac{k_\perp^2}{\alpha_s}\, \tilde{S}(X_g, k_\perp)$$

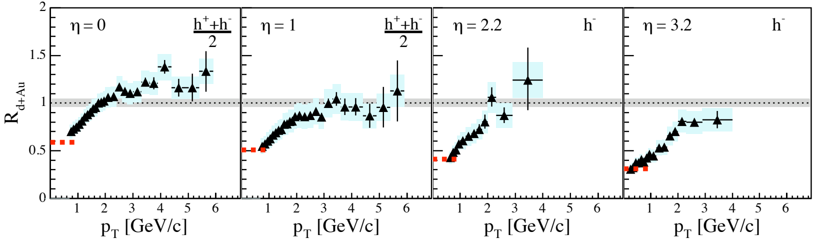
- $R_{pA}(k_\perp, \eta)$ measures the ratio of the gluon TMDs in the nucleus and respectively the (target) proton, in given bins in $k_\perp$ and $X_g$
- d+Au collisions at RHIC: $\sqrt{s} = 200$ GeV, $k_\perp \sim 2$ GeV and $\eta \approx 0$
  - $X_g \simeq 0.01 \Longrightarrow$ little evolution: the target proton is dilute
  - the nucleus is dense ($A \gg 1$), but described by the MV model
- Multiple scattering simply redistributes gluons in $k_\perp$
  - proton: a distribution $\mathcal{F}_p(k_\perp) \sim 1/k_\perp^2$ all the way down to $\Lambda$
  - nucleus: distribution depleted at $k_\perp < Q_s$ and peaked at $k_\perp \sim Q_s$

$$R_{pA}(k_\perp) = \frac{\mathcal{F}_A(k_\perp)}{A\,\mathcal{F}_p(k_\perp)} = \begin{cases} < 1 & \text{for } \; k_\perp \ll Q_s, \\ > 1 & \text{for } \; k_\perp \sim Q_s, \\ \simeq 1 & \text{for } \; k_\perp \gg Q_s \end{cases}$$

# Forward rapidities: $R_{pA}$ suppression

- Why is the Cronin peak washed out when increasing $\eta$ (decreasing $X_g$) ?
- The gluon distribution in the proton rises faster than that in the nucleus
  - radiation (DGLAP, BFKL) in the dilute tail at $p_\perp > Q_s$
  - the transverse phase-space is larger for the proton than for the nucleus,

$$\ln \frac{p_\perp^2}{\Lambda^2} > \ln \frac{p_\perp^2}{Q_s^2(A)}$$




- A quantitative understanding requires including the high energy evolution