The Colour Glass Condensate 3

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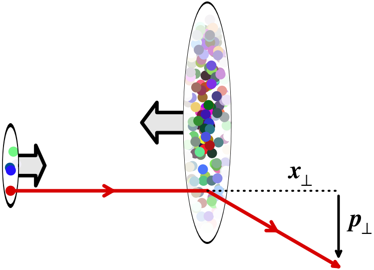
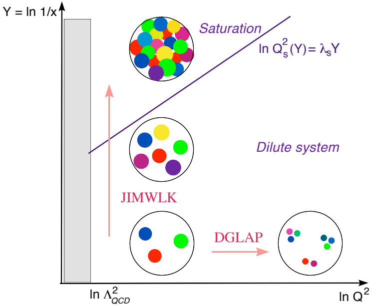
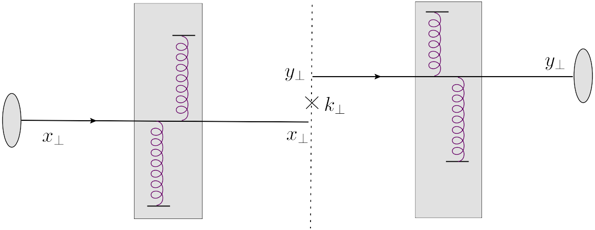




Particle production in pA collisions



- Dipole picture for the cross-section for quark production:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p) \int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, S(\boldsymbol{x},\boldsymbol{y};X_g)$$

- The elastic S-matrix for an effective quark-antiquark colour dipole

$$S(\boldsymbol{x},\boldsymbol{y};X_g) = \frac{1}{N_c}\left\langle \mathrm{tr}\left[V(\boldsymbol{x})V^\dagger(\boldsymbol{y})\right]\right\rangle_{X_g} = 1 - T(\boldsymbol{x},\boldsymbol{y};X_g)$$

Multiple scattering and unitarity

- Small dipole $r = |\boldsymbol{x} - \boldsymbol{y}| \ll 1/Q_s$: single scattering approximation

$$T_0(r;x) \simeq \pi^2 \,\frac{\alpha_s r^2}{2N_c}\,\frac{xG(x,4/r^2)}{\pi R_A^2}$$

- A direct measure of the gluon distribution on the resolution scale $Q^2 \sim 1/r^2$
- $T_0(r) \propto r^2$: colour transparency (a small dipole cannot scatter)
- T_0 increases with r but cannot exceed the unitarity limit: $T = 1 - S \leq 1$

Multiple scattering and unitarity

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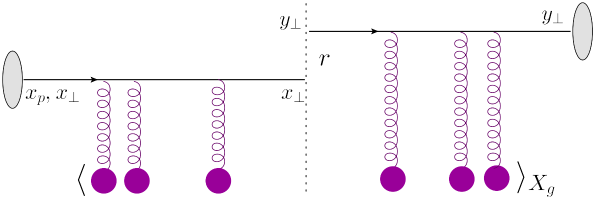
$$T_0(r;x) \simeq \pi^2 \frac{\alpha_s r^2}{2N_c} \frac{xG(x,4/r^2)}{\pi R_A^2} \sim 1 \quad \text{when} \quad 2/r \sim Q_s(x)$$

- A direct measure of the gluon distribution on the resolution scale $Q^2 \sim 1/r^2$
- $T_0(r) \propto r^2$: colour transparency (a small dipole cannot scatter)
- T_0 increases with r but cannot exceed the unitarity limit: $T = 1 - S \leq 1$
- When $r \sim 1/Q_s$, $T_0 \sim 1$ and multiple scattering becomes important



- Complicated in general, but easy to compute for independent colour sources

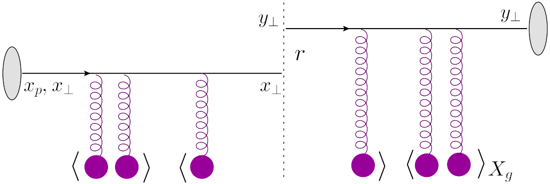
- McLerran-Venugopalan model: a large nucleus $\approx AN_c$ valence quarks

$$xG_A^{(0)}(x,Q^2) = AN_c\,\frac{\alpha_s C_F}{\pi}\ln\frac{Q^2}{\Lambda^2}$$

- The dipole amplitude in the single-scattering approximation:

$$T_0(r;x) = \frac{r^2 Q_A^2}{4}\ln\frac{4}{r^2\Lambda^2} \quad \text{with} \quad Q_A^2 \equiv \frac{2\alpha_s^2 C_F A}{R_A^2} \propto A^{1/3}$$

- Independent sources $\Rightarrow$ Gaussian approximation



- The multiple scattering series exponentiates: $S(r) = \mathrm{e}^{-T_0(r)}$

Multiple scattering in the MV model

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- $S(r)$: Probability for the $q\bar{q}$ pair to survive in a color singlet ("dipole") state

$$S(r) = \mathrm{e}^{-T_0(r)} \equiv 1 - T(r)$$

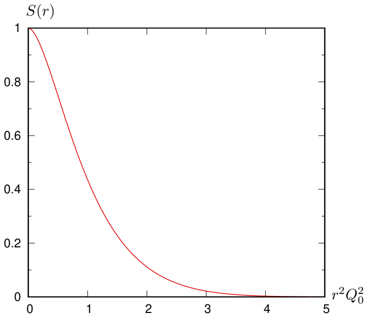
- Saturation momentum Q_s: conventionally defined as $T_0(r) = 1$ for $r = \frac{2}{Q_s}$

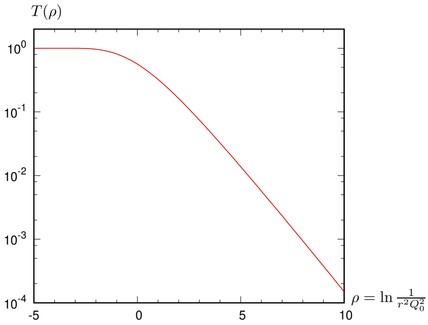
$$Q_s^2(A) \,=\, Q_A^2 \, \ln \frac{Q_s^2(A)}{\Lambda^2} \,\propto\, A^{1/3} \ln A^{1/3} \quad \text{(no } x \text{ dependence)}$$

- Duality: saturation in the target $\Leftrightarrow$ multiple scattering for the probe

Dipole scattering (MV model)

- Left: the dipole S-matrix S as a function of $r^2Q_s^2$





- Right: the dipole amplitude $T \equiv 1 - S$ as a function of $\rho \equiv \ln(1/r^2Q_s^2)$
  - small dipole $r \ll 1/Q_s \Longrightarrow$ large values for ρ : $T \simeq T_0 \sim r^2Q_s^2 = \mathrm{e}^{-\rho}$
  - large dipole $r \gtrsim 1/Q_s \Longrightarrow$ negative ρ : $T = 1$

Momentum broadening

- Quark production in pA collisions: the transverse momentum distribution

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\;\mathrm{e}^{-\frac{1}{4}r^2Q_A^2\,\ln\frac{1}{r^2\Lambda^2}} = \tilde{S}(\boldsymbol{k})$$

- Would-be a Gaussian integration ... if there were not for the logarithm
- Competition between $r \sim 1/k_\perp$ (Fourier phase) and $r \sim 1/Q_s$ (S-matrix)
- The bulk of the distribution lies around $k_\perp \sim Q_s$
  - replace $1/r^2 \to Q_s^2$ within the argument of the log

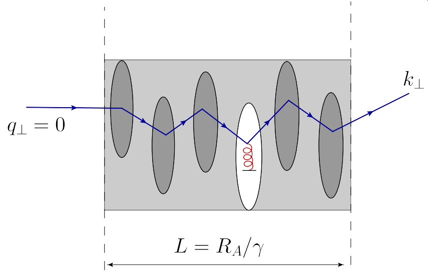
$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{1}{\pi Q_s^2}\,\mathrm{e}^{-k_\perp^2/Q_s^2}$$

  - a Gaussian distribution: a random walk in $\boldsymbol{k}$

$$\langle k_\perp^2\rangle \equiv \int \mathrm{d}^2\boldsymbol{k}\,k_\perp^2\,\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = Q_s^2(A)$$

Molière scattering

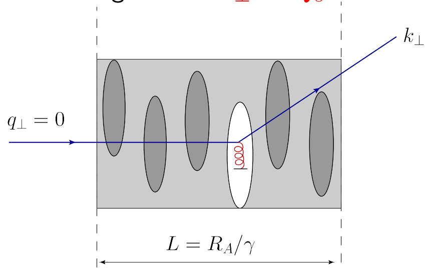
- Transverse momentum broadening via multiple soft scattering (diffusion)



$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{1}{\pi Q_s^2}\,\mathrm{e}^{-k_\perp^2/Q_s^2}$$

$$\langle k_\perp^2\rangle = Q_s^2(A) \sim L \sim A^{1/3}$$

- A larger value $k_\perp \gg Q_s$ can be acquired via a single hard scattering



- Expand $S \simeq 1 - T_0$
- Power-law tail:

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{Q_A^2}{\pi k_\perp^4} \quad \text{for} \quad k_\perp \gg Q_s$$

Molière theory (1947)

- Not the famous French dramatist, but a German physicist !

Molière Theory

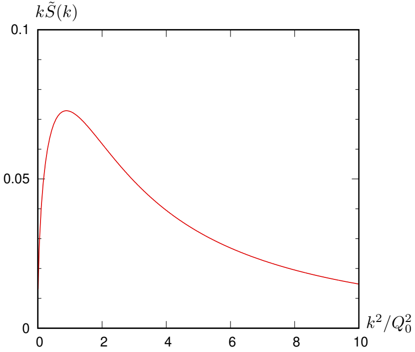
Many statistical processes obey the Central Limit Theorem: the random sum of many small displacements is a Gaussian distribution. However, the displacements must all be *small* , in a sense that can be made mathematically precise. Scattering from the screened Coulomb field of the atomic nucleus does not obey the CLT because the single scattering cross section falls off only as $\mathbf{1/\theta^4}$, too slowly. Therefore the angular distribution is approximately Gaussian for small angles but eventually tends to a 'single scattering tail' $\approx \mathbf{1/\theta^4}$.

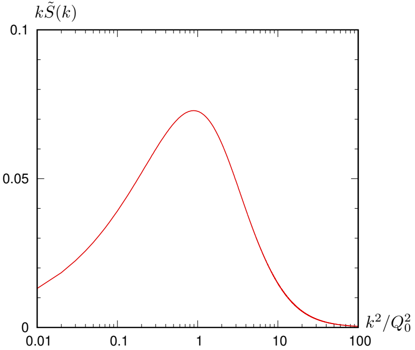
All this was well understood by many investigators who worked on multiple scattering in the 1920's and 30's but it was Molière who in 1947 published the definitive theory, uniting the Gaussian region with the large-angle region in a precise and elegant way. He computed the distributions of both the *space* and

Ref: *Multiple scattering, Harvard*

The dipole gluon distribution

- Left: the Fourier transform $k_\perp \tilde{S}(k_\perp)$
  - the $k_\perp$–distribution of the produced quark in pA collisions
  - also that of the gluons in the nucleus ! (the "gluon dipole TMD")





- Right: the same function, but in logarithmic units
  - peaked at $k \simeq Q_s$, power-law tail at $k \gg Q_s$

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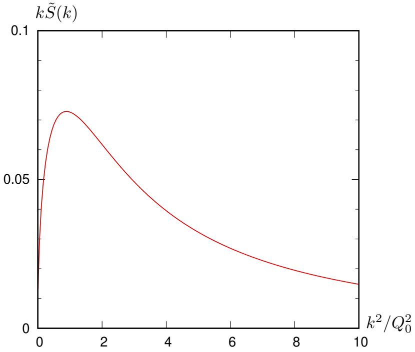
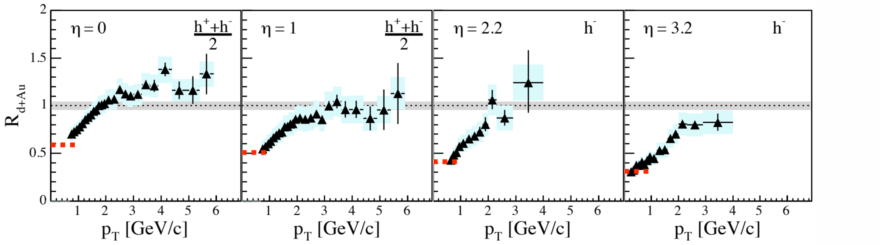


- An interesting consequence for hadron production in pA collisions:
  - the Cronin peak

The nuclear modification factor at RHIC

$$R_{pA} \equiv \frac{1}{A} \frac{\mathrm{d}\sigma_{pA}/\mathrm{d}^2p_\perp \mathrm{d}\eta}{\mathrm{d}\sigma_{pp}/\mathrm{d}^2p_\perp \mathrm{d}\eta}$$



- Would be 1 if $pA =$ incoherent superposition of pp collisions
  - any deviation from unity is a signature of nuclear (high density) effects
- The RHIC data (d+Au) show two interesting nuclear effects
  - central rapidity ($\eta \simeq 0$): $R_{\mathrm{d+Au}} > 1$ for $p_\perp \gtrsim 2$ GeV ("Cronin peak")
  - forward rapidities ($\eta \gtrsim 2$): $R_{\mathrm{d+Au}} < 1$ (nuclear suppresion)

Midrapidity: the Cronin peak

$$\frac{d\sigma}{d\eta d^2\boldsymbol{k}} \simeq x_p q(x_p)\,\frac{\alpha_s}{k_\perp^2}\,\mathcal{F}_A(X_g, k_\perp), \qquad \mathcal{F}_A(k_\perp) \sim k_\perp^2\,\tilde{S}_A(k_\perp)$$

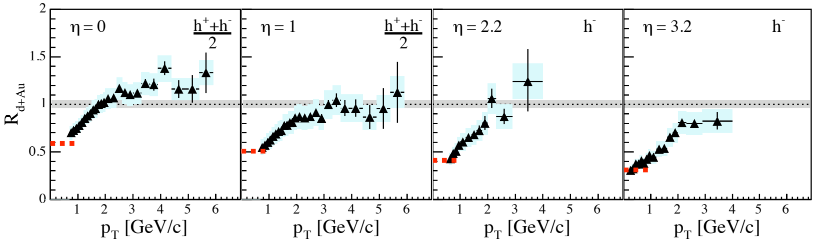
- $R_{pA}(k_\perp, \eta) =$ ratio of the gluon TMDs in the nucleus and a (target) proton
- d+Au collisions at RHIC: $\sqrt{s} = 200$ GeV, $k_\perp \sim 2$ GeV and $\eta \approx 0$
  - $X_g \simeq 0.01 \Longrightarrow$ little evolution: the target proton is dilute
  - the nucleus is dense $(A \gg 1)$, but no evolution: MV model
- Multiple scattering simply redistributes gluons in $k_\perp$
  - proton: a distribution $\mathcal{F}_p(k_\perp) \sim 1/k_\perp^2$ all the way down to Λ
  - nucleus: distribution depleted at $k_\perp < Q_s$ and peaked at $k_\perp \sim Q_s$

$$R_{pA}(k_\perp) = \frac{\mathcal{F}_A(k_\perp)}{A\,\mathcal{F}_p(k_\perp)} = \begin{cases} < 1 & \text{for} \quad k_\perp \ll Q_s, \\ > 1 & \text{for} \quad k_\perp \sim Q_s, \\ \simeq 1 & \text{for} \quad k_\perp \gg Q_s \end{cases}$$

Forward rapidities: R_{pA} suppression

- Why is the Cronin peak washed out when increasing η (decreasing X_g) ?
- The gluon distribution in the proton rises faster than that in the nucleus
  - radiation (DGLAP, BFKL) in the dilute tail at $p_\perp > Q_s$
  - the transverse phase-space is larger for the proton than for the nucleus,

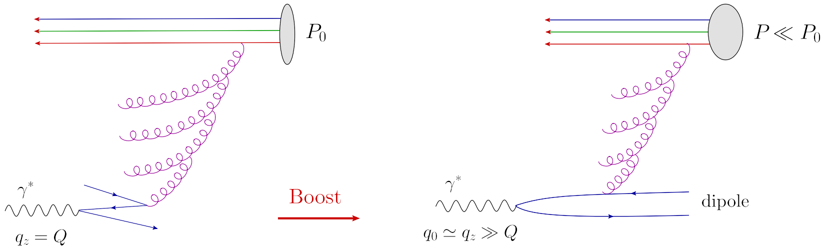
$$\ln \frac{p_\perp^2}{\Lambda^2} > \ln \frac{p_\perp^2}{Q_s^2(A)}$$



- A quantitative understanding requires including the high energy evolution

DIS in the dipole frame

- Start in the Breit frame: $P_0^\mu = P_0(1, \mathbf{0}_\perp, -1)$ & $q_0^\mu = (0, \mathbf{0}_\perp, Q)$
- Make a large boost $(\beta \simeq 1,\ \gamma \gg 1)$ in the positive z direction:
  - $P^\mu = P(1, \mathbf{0}_\perp, -1)$ with $P \ll P_0$
  - $q^\mu = (q_0, \mathbf{0}_\perp, q_z)$ with $q_0 = \gamma\beta Q \simeq q_z = \gamma Q \gg Q$
- The photon is ultrarelativistic $\Rightarrow$ the physical picture is changing
  - the $q\bar{q}$ pair can now be seen as a part of the photon wavefunction



- γ^* fluctuates into a $q\bar{q}$ color dipole, which then scatters off the proton

The photon coherence time

- It is first useful to change to light-cone (LC) notations: $k^\mu = (k^+, k^-, \boldsymbol{k}_\perp)$
  - $P^\mu = \delta^{\mu -} P_N^-$ (the momentum of a single nucleon inside the target)
  - $q^\mu = (q^+, q^-, \mathbf{0}_\perp)$ with $2q^+q^- = -Q^2$ and $q^+ \gg Q$
- Follow the evolution of the dipole: a right-mover $\Rightarrow$ the LC time is x^+
  - for the left-moving target, x^+ is a longitudinal coordinate
- The validity of the dipole picture requires a strong separation in time:
  - the lifetime Δt_γ of the $q\bar{q}$ fluctuation $\gg$ the width L of the target

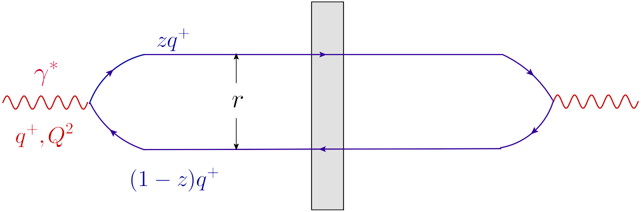
$$\Delta t_\gamma \simeq \frac{2q^+}{Q^2} \gg L = \frac{R_A}{\gamma} = \frac{R_A M_N}{P_N^-} \sim \frac{A^{1/3}}{P_N^-}$$

- This is precisely the condition of small Bjorken x:

$$x_{\rm Bj} \equiv \frac{Q^2}{2P_N^- q^+} \ll A^{-1/3}$$

Dipole factorization for DIS

- Large time separation ensures factorisation of the cross-section in time:
  - sequential process: photon decay $\gamma^* \to q\bar{q}$ followed by the collision
- It furthermore ensures the validity of the eikonal approximation
  - the dipole transverse size r is not modified by the scattering



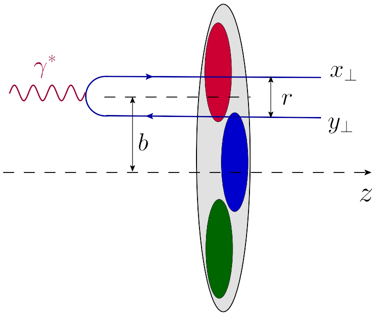
$$\sigma_{\gamma^* p}(Q^2, x) = \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \, \left|\Psi_{\gamma^* \to q\bar{q}}(r, z; Q^2)\right|^2 \sigma_{\rm dipole}(r, x)$$

- γ^* wavefunction $\Psi_{\gamma^* \to q\bar{q}}(r, z; Q^2)$: QED perturbation theory
- The dipole cross-section $\sigma_{\rm dipole}$: QCD scattering and evolution

The dipole cross-section

- Optical theorem: total cross-section = twice the elastic scattering amplitude

$$\sigma_{\rm dipole}(r, x) = 2\int {\rm d}^2\boldsymbol{b}\; T(\boldsymbol{r}, \boldsymbol{b}, x)$$



- $\boldsymbol{r} = \boldsymbol{x}_\perp - \boldsymbol{y}_\perp$: dipole size
- $\boldsymbol{b} = (\boldsymbol{x}_\perp + \boldsymbol{y}_\perp)/2$: impact parameter
- $T(\boldsymbol{r}, \boldsymbol{b}, x) = 1 - S$: dipole amplitude
- $T \le 1$: unitarity bound

- 2 Wilson lines: $V(\boldsymbol{x}_\perp)$ for the quark (q) and $V^\dagger(\boldsymbol{y}_\perp)$ for the antiquark $(\bar q)$
  - an antiquark has charge $(-g)$ and propagates backwards in time
- Elastic scattering: $q\bar q$ remains in a colour singlet state

$$S(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; x) = \frac{1}{N_c}\left\langle {\rm tr}\big(V(\boldsymbol{x}_\perp)V^\dagger(\boldsymbol{y}_\perp)\big)\right\rangle_x$$

Photon wavefunction vs. dipole scattering

$$\sigma_{\gamma^* p}(Q^2, x) = \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \left|\Psi_{\gamma^* \to q\bar{q}}(r, z; Q^2)\right|^2 \, 2\pi R_A^2 \, T(r, x)$$

- $|\Psi_{\gamma^* \to q\bar{q}}|^2$ restricts the integrations to $z(1-z) r^2 Q^2 \lesssim 1$
- Dipole amplitude interpolates between weak and strong scattering:

$$T(r, x) \simeq \begin{cases} r^2 Q_s^2(A, x) & \text{for } rQ_s \ll 1 \text{ (color transparency)} \\ 1 & \text{for } rQ_s \gtrsim 1 \text{ (saturation)} \end{cases}$$

- Sensitivity to saturation requires $r \gtrsim 1/Q_s$, hence $z(1-z)Q^2 \lesssim Q_s^2$
- $z(1-z) \leq 1/4 \;\Rightarrow$ 2 possibilities: semi-hard DIS $Q^2 \lesssim Q_s^2$
  - sensitive to saturation irrespective of the value of z
- ... or hard DIS, $Q^2 \gg Q_s^2$, but very small $z(1-z) \lesssim \frac{Q_s^2}{Q^2} \ll 1$

Inclusive DIS: $F_2(x, Q^2)$

- A $q\bar{q}$ configuration with $z(1-z) \ll 1$ is an "aligned jet" (Bjorken)
  - one quark carries the quasi-totality of photon longitudinal momentum
- Interesting since characterised by relatively large dipole sizes $r \gg 1/Q$
- When $Q^2 \gg Q_s^2$: inclusive DIS is indeed controlled by aligned jets ...

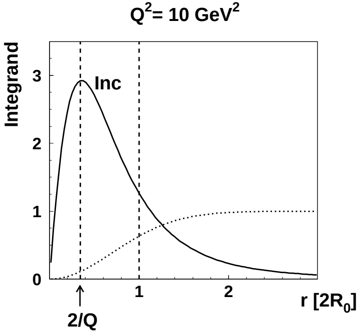
$$Q_s^2(A, x) \ll z(1-z)Q^2 \ll Q^2$$

  - but which are not large enough to suffer strong scattering: $r \ll 1/Q_s$
- Total cross-section at high Q^2: logarithm from aligned jets

$$\sigma_{\gamma^* A} \sim 2\pi R_A^2 \, \frac{Q_s^2(A, x)}{Q^2} \, \ln \frac{Q^2}{Q_s^2(A, x)}$$

- N.B. Saturation effects may still be visible at small x, e.g. geometric scaling

- The integrand of $\sigma_{\gamma^* p}(Q^2, x)$ with a Gaussian model for $T(r)$

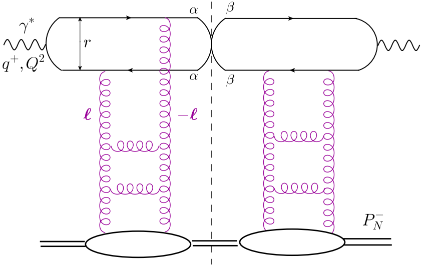


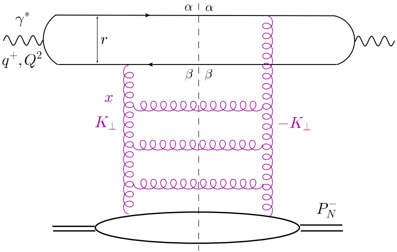
- In this calculation: $Q_s = 1$ GeV. The value $r = 2/Q_s$ is indicated as "$r = 1$"
  - $Q^2 = 10$ GeV^2: integrand peaked at $r = 2/Q \ll 2/Q_s$
  - $Q^2 = 0.8$ GeV^2: integrand peaked at $r \sim 2/Q_s$

K. Golec-Biernat, M. Wusthoff, hep-ph/9807513

Elastic scattering

- The $q\bar{q}$ dipole can also suffer an elastic scattering ("diffraction")
  - colour singlet also in the final state: colourless exchange (Pomeron)
- Compare elastic scattering (left) and inelastic (right)





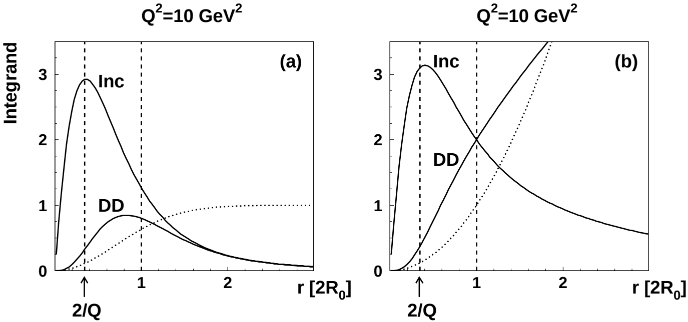
- Colour traces closed in the amplitude (left), resp. in the cross-section (right)

$$\sigma_{\rm el}(Q^2, x) = \int {\rm d}^2 r \int {\rm d}z \left|\Psi(r, z; Q^2)\right|^2 2\pi R_A^2 \, |T(r, x)|^2$$

- The weak scattering regime is now strongly suppressed

$$|T(r,x)|^2 \simeq \begin{cases} r^4 Q_s^4(A,x) & \text{for } rQ_s \ll 1 \text{ (color transparency)} \\ 1 & \text{for } rQ_s \gtrsim 1 \text{ (strong scattering)} \end{cases}$$

- $\sigma_{\rm el}(Q^2,x)$ is controlled by strong scattering even when $Q^2 \gg Q_s^2(A,x)$
  - aligned jet configurations with $z(1-z)Q^2 \sim Q_s^2$, hence $r \sim 1/Q_s$:



- Integrand would explose at large r in the absence of saturation: right plot

Diffraction = strong scattering

$$|T(r,x)|^2 \simeq \begin{cases} r^4 Q_s^4(A,x) & \text{for } rQ_s \ll 1 \text{ (color transparency)} \\ 1 & \text{for } rQ_s \gtrsim 1 \text{ (strong scattering)} \end{cases}$$

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  - aligned jet configurations with $z(1-z)Q^2 \sim Q_s^2$, hence $r \sim 1/Q_s$:

$$\text{Elastic cross-section :} \qquad \sigma_{\rm el} \sim 2\pi R_A^2 \, \frac{Q_s^2(A,x)}{Q^2}$$

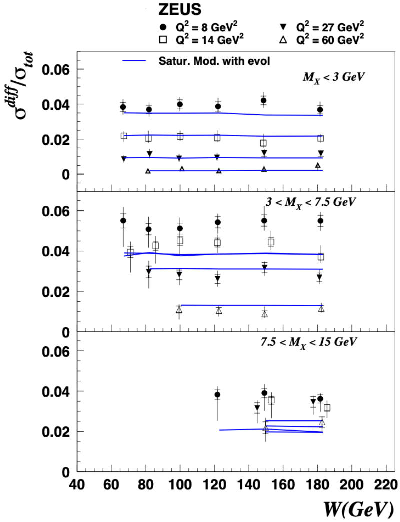
- Compare to the total cross-section

$$\sigma_{\gamma^* A} \sim 2\pi R_A^2 \, \frac{Q_s^2(A,x)}{Q^2} \, \ln \frac{Q^2}{Q_s^2(A,x)}$$

- Elastic cross-section is smaller ... but only by a log ! 14% at HERA
- Both cross-section should roughly show the same dependences with x and A

One vs two Pomerons

- This was not expected prior to saturation (Regge theory)



gure 9: *The ratio of $\sigma_{diff}/\sigma_{tot}$ versus the $\gamma^* p$ energy W. The data is from ZEUS and the solid*

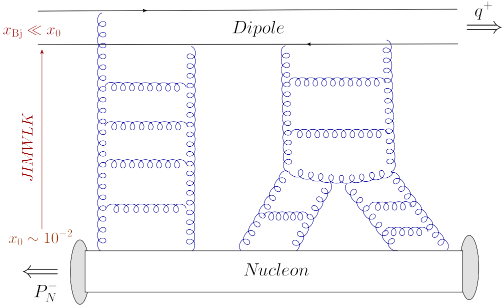
- $\sigma_{\mathrm{tot}} = 1$ Pomeron vs. $\sigma_{\mathrm{el}} = 2$ Pomerons
- First saturation argument: *Al Mueller, '97*
- x-dependence confirmed in ep at HERA *(Bartels, Golec-Biernat, Kowalski, 2002)*
- Almost the same scaling as σ_{tot}:

$$\frac{\sigma_{el}}{\sigma_{tot}} \sim \frac{1}{\ln\left(Q^2/Q_s^2\right)}$$

- Check A-dependence at the EIC

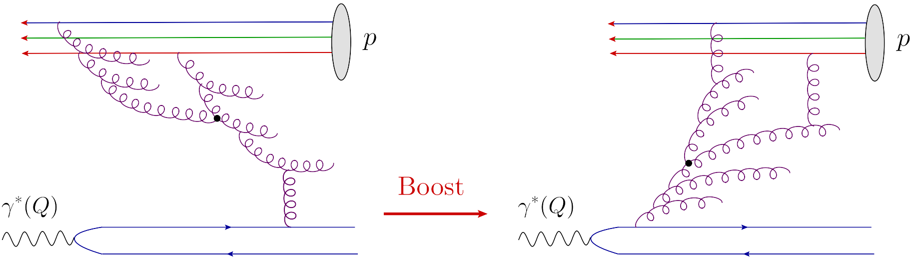
$$S(\boldsymbol{x},\boldsymbol{y};Y) = \int [DA^-]\, W_Y[A^-]\, \frac{1}{N_c}\mathrm{tr}\big(V_{\boldsymbol{x}} V^\dagger_{\boldsymbol{y}}\big), \quad Y = \ln\frac{1}{x_{\rm Bj}}$$

- JIMWLK: functional equation for the CGC weight function $W_Y[A^-]$



- gluon emissions with smaller and smaller $x = k^-/P_N^-$, down to $x_{\rm Bj}$
- initial condition at some $x_0 \sim 10^{-2}$, where e.g. MV model applies
- non-linear effects in both the evolution (gluon saturation) and the collision (multiple scattering)

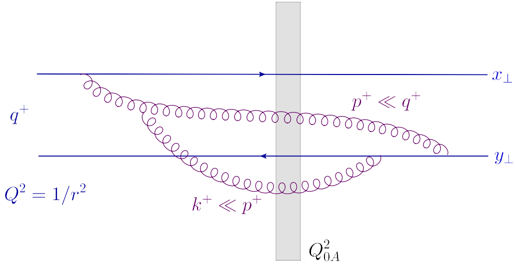
- Balitsky-Kovchegov equation for the dipole S-matrix $S(\boldsymbol{x},\boldsymbol{y};Y)$



  - gluon emissions by the projectile (the colour dipole)
  - smaller and smaller values for $z \equiv k^+/q^+$
  - non-linear effects in the collision but not also in the evolution
  - gluon number fluctuations in the evolution: dilute system

- Physical picture & calculation in the dipole LC gauge: $A_a^+ = 0$

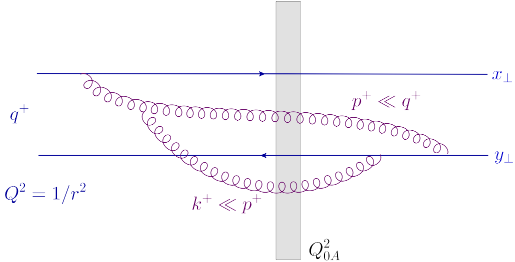
- BFKL: Probability $\sim \alpha_s \ln(1/z)$ for emitting a soft $(z \ll 1)$ gluon
- Elastic scattering: gluons must be emitted and reabsorbed within the dipole



$$z_2 = \frac{k^+}{q^+} \ll z_1 = \frac{p^+}{q^+} \ll 1\,, \qquad Q^2_{s0} \lesssim p^2_\perp,\, k^2_\perp \lesssim Q^2$$

- Strong ordering in z, no ordering in $k_\perp$ $\Rightarrow$ decreasing lifetimes: $\Delta x^+ \sim \frac{2zq^+}{k^2_\perp}$
- Leading logarithmic approx: resum $(\bar{\alpha} Y)^n$ with $Y = \ln(1/z_{\min})$

The longitudinal phase-space



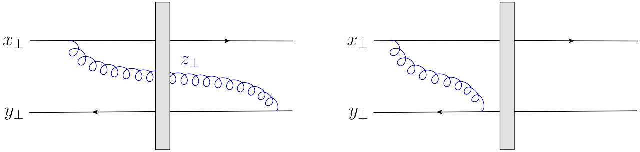
- Gluons must live longer than the longitudinal extent of the target...
  - for the high energy approximations (e.g. eikonal) to apply
- ... but cannot live longer than the coherence time of the photon

$$\tau_\gamma = \frac{2q^+}{Q^2} \gtrsim \tau_g = \frac{zq^+}{k_\perp^2} \gg \tau_A = \frac{1}{P_N^-} \implies \frac{k_\perp^2}{Q^2} \gtrsim z \gg \frac{k_\perp^2}{2P_N^- q^+}$$

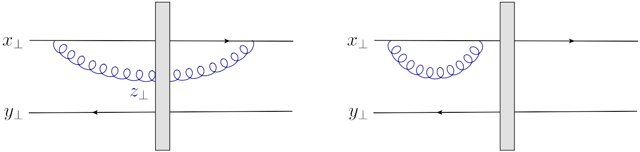
- Rapidity phase-space: $Y = \ln \frac{2P_N^- q^+}{Q^2} = \ln \frac{1}{x_{\rm Bj}}$ ... as it should

One step in the BK evolution (1)

- To construct the evolution equation, it is enough to look at the first emission
- The gluon can be exchanged between the quark and the antiquark



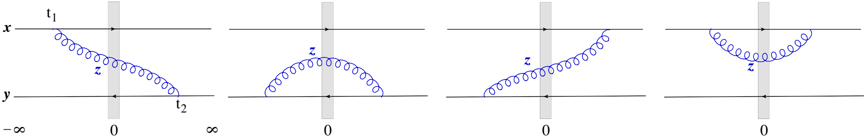
- ... or be emitted and reabsorbed by a same fermion ("self-energy graph")



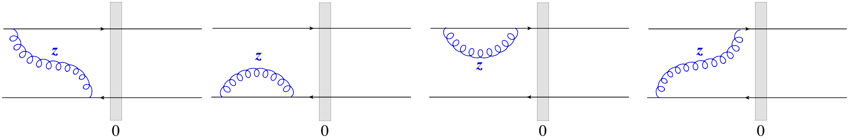
- In both cases, the gluon may cross the shockwave ... or not !

One step in the BK evolution (2)

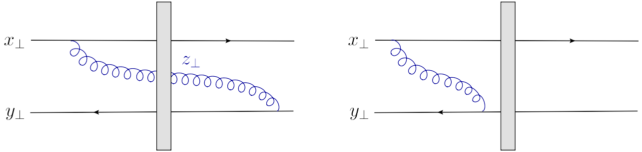
- All possible attachments and time orderings for the gluon vertices
- The soft gluon crosses the shockwave
  - the system which scatters: a 3-parton system $(q\bar{q}g)$



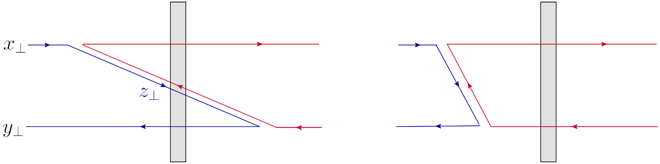
- Non-crossing: only the original $q\bar{q}$ dipole interacts



- Gluon exchange graphs for definiteness: crossing & non-crossing:



- At large N_c, a gluon can be replaced by a quark-antiquark pair
  - gluon emission by a dipole ≈ dipole splitting into 2 dipoles

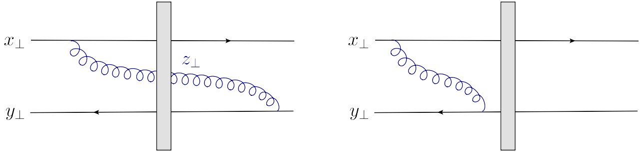


$$\frac{\partial \langle \hat{S}_{\boldsymbol{xy}} \rangle_Y}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}} \big[\langle \hat{S}_{\boldsymbol{xz}} \hat{S}_{\boldsymbol{zy}} \rangle_Y - \langle \hat{S}_{\boldsymbol{xy}} \rangle_Y \big]$$

- Dipole kernel $\mathcal{M}_{\boldsymbol{xyz}}$: BFKL kernel in the dipole picture *(Al Mueller, 1990)*

The BK equation *(Balitsky, '96; Kovchegov, '99)*

- Gluon exchange graphs for definiteness: crossing & non-crossing:



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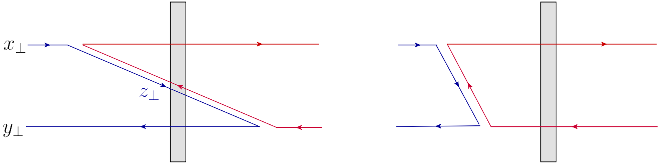

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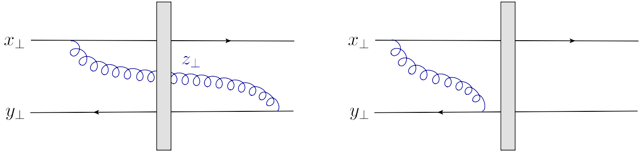
- At large N_c, expectation values of colorless operators factorize

- Gluon exchange graphs for definiteness: crossing & non-crossing:



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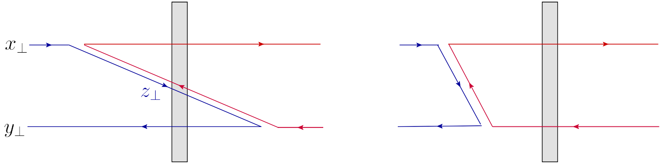


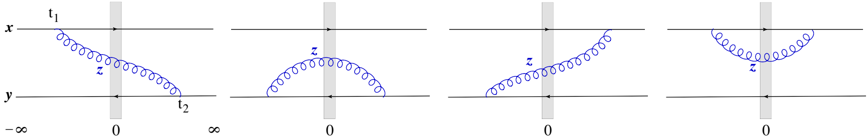
$$\frac{\partial S_Y(\boldsymbol{x},\boldsymbol{y})}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\big]$$

- Convenient notation: $\bar{\alpha} \equiv \alpha_s N_c/\pi$

The dipole kernel

$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2}\right]^2$$

- The sum of the emission probabilities for the 4 possible gluon attachements :



  - similarly for the non-crossing contributions

- Soft gluon emissions can be computed in the classical approximation
- The Weisäcker-Williams classical field by a relativistic point-like source:

$$A_a^i(\boldsymbol{z}) = \frac{1}{2\pi}\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2}\, t^a$$

  - $t^a t^a = C_F = (2N_c^2 - 1)/2N_c \simeq N_c/2$ when $N_c \gg 1$

The dipole kernel (2)

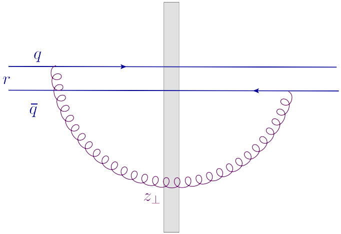
$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2}\right]^2$$

- Colour transparency: $\mathcal{M}_{\boldsymbol{xyz}} \to 0$ when $r = |\boldsymbol{x}-\boldsymbol{y}| \to 0$

- Rapid decrease of the emission probability at large $z_\perp$

$$\mathcal{M}_{\boldsymbol{xyz}} \simeq \frac{r^2}{(\boldsymbol{z}-\boldsymbol{x})^4} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

  - cancellations between self-energy (qq or $\bar{q}\bar{q}$) and exchange ($q\bar{q}$) graphs



The dipole kernel (2)

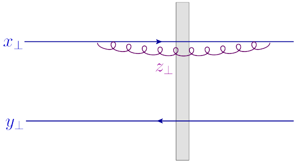
$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2}\right]^2$$

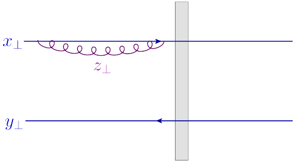
- Colour transparency: $\mathcal{M}_{\boldsymbol{xyz}} \to 0$ when $r = |\boldsymbol{x}-\boldsymbol{y}| \to 0$
- Rapid decrease of the emission probability at large $z_\perp$

$$\mathcal{M}_{\boldsymbol{xyz}} \simeq \frac{r^2}{(\boldsymbol{z}-\boldsymbol{x})^4} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

  - cancellations between self-energy (qq or $\bar{q}\bar{q}$) and exchange ($q\bar{q}$) graphs
- Short-distance poles ($\boldsymbol{z}=\boldsymbol{x}$) cancel between 'crossing' and 'non-crossing'

$$\boldsymbol{z} \to \boldsymbol{x} \implies S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) \to \mathbb{I} \times S_Y(\boldsymbol{x},\boldsymbol{y})$$





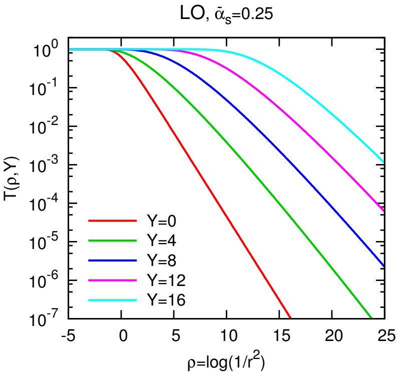
- Non-linear generalization of the BFKL equation for $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}} \big[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}}\big]$$

- Non-linear term T^2: the simultaneous scattering of both daughter dipoles
- When scattering is weak, $T \ll 1$, one recovers the linear BFKL equation
  - exponential increase with Y leading to unitarity violation
- The non-linear term in BK restores unitarity: $T(r, Y) \le 1$ for any r and Y
  - $T = 0$ (no scattering) and $T = 1$ (total absorption) are fixed points
- Saturation momentum $Q_s(Y)$: $T(r, Y) = 0.5$ when $r = 1/Q_s(Y)$
  - $Q_s(Y)$ increases rapidly with Y due to the BFKL dynamics

The saturation front

- Numerical solutions to BK with initial condition from the MV model
- Logarithmic variable $\rho \equiv \ln(1/r^2Q_0^2) \Longrightarrow$ large $\rho \leftrightarrow$ small r



$$T(r, Y=0) = 1 - \mathrm{e}^{-\frac{r^2Q_0^2}{4}\ln\frac{1}{r^2\Lambda^2}}$$

$$T(\rho_s(Y), Y) = 0.5 \quad \text{for} \quad \rho_s(Y) = \lambda_s Y$$

$$T(\rho, Y) \simeq \begin{cases} \mathrm{e}^{-\gamma_s(\rho-\rho_s)} & (\rho > \rho_s) \\ 1 & (\rho \lesssim \rho_s) \end{cases}$$

- Geometric scaling: $T(r, Y) \simeq \left(r^2Q_s^2(Y)\right)^{\gamma_s}$ with $\gamma_s \simeq 0.63$
  - a front which preserves its shape while progressing to larger values of ρ

The saturation front

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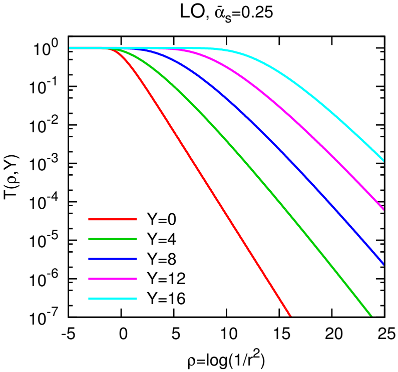

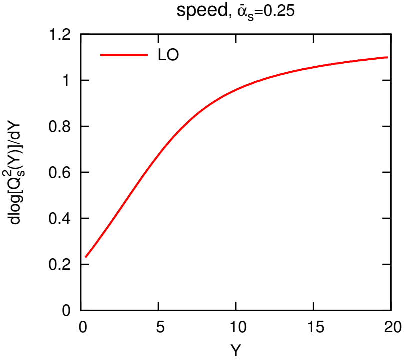

- Saturation exponent: the speed of the saturation front

$$\lambda_s \equiv \frac{\mathrm{d}\rho_s}{\mathrm{d}Y} \simeq 4.88\bar{\alpha} - \frac{1}{2\gamma_s Y}, \qquad Q_s^2(Y) \simeq Q_0^2\, \mathrm{e}^{\lambda_s Y}$$

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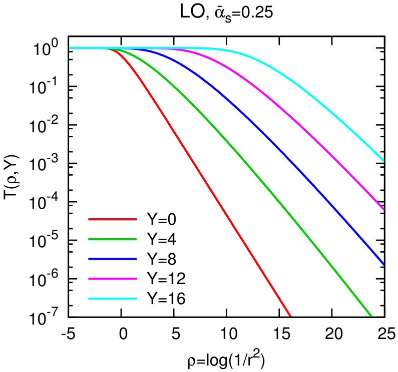

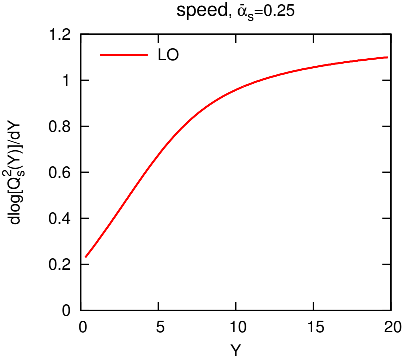

- These properties have been independently established in

  E.I., K. Itakura, L. McLerran, hep-ph/0203137;
  A.H. Mueller, D.N. Triantafyllopoulos, hep-ph/0205167