Degenerate fermionic matter at N^3LO

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In collaboration with: T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho and A. Vuorinen

Based on: [2204.11279], [2204.11893]

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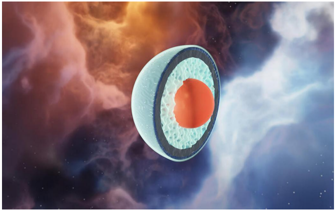
April 29, 2022

Dense strongly interacting matter

- Aim: Understand thermodynamic properties of cold ($T = 0$) and dense ($\mu_B \neq 0$) strongly interacting matter
- Neutron stars:
  - Laboratories for cold ultradense matter
  - Measurements improved e.g. via LIGO, Virgo and NICER
- Application: Neutron-star matter equation of state



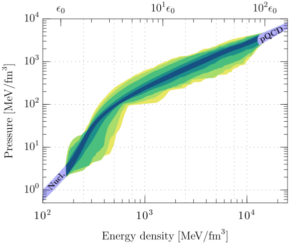
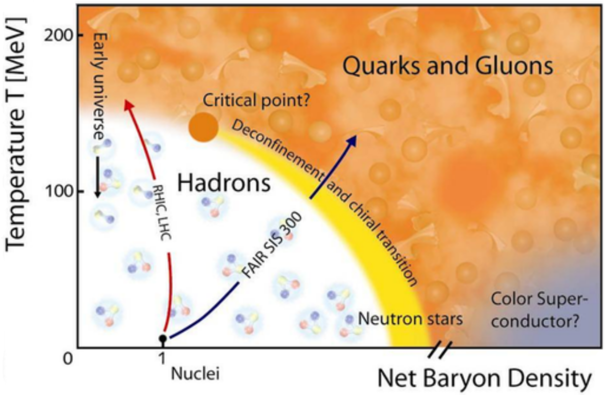


Image: Jyrki Hokkanen, CSC; Annala et al., PRX 12 (2022)

Different phases of QCD matter

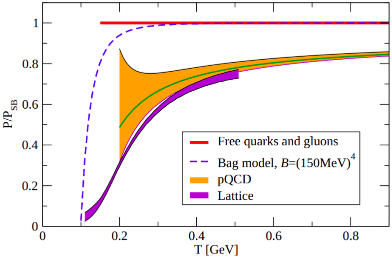


- High temperature: quark-gluon-plasma in early universe and heavy ion collisions
- High baryon density: cold quark matter (possibly) in neutron stars

Image: GSI Darmstadt

Pressure is the most fundamental quantity in bulk thermodynamics

Tools to study QCD pressure

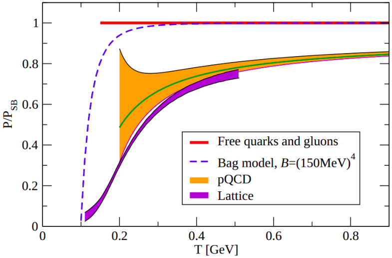


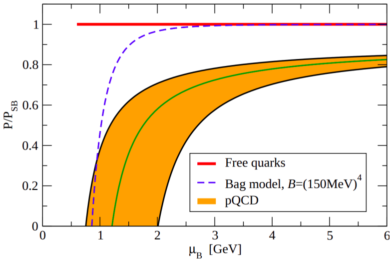
Finite T

- pQCD at high T
- Lattice QCD applicable
- Pressure well understood throughout the T-range

Image: Fraga, Kurkela, Vuorinen, EPJA 52 (2016)

Tools to study QCD pressure





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Finite μ

- pQCD at high μ
- Sign Problem $\Rightarrow$ lattice QCD not applicable

$\Rightarrow$ pQCD only reliable first-principles method!

Image: Fraga, Kurkela, Vuorinen, EPJA 52 (2016)

Framework for calculating dense pQCD pressure

1. Generate Feynman diagrams from partition function:

$$p(\mu) \sim \ln Z = \ln \int \mathcal{D}\overline{\psi}\psi\overline{c}cAe^{-S_{\rm QCD}}$$

$$\overset{\rm pQCD}{=} \text{ sum of connected vacuum diagrams (no ext. legs)}$$

- Imaginary-time formalism
- Euclidean Feynman rules: Information about μ carried by 0-components of 4-momenta: $P^\alpha = (p^0 - i\mu, \mathbf{p})$ (Lorentz symmetry broken)

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Cancellation of IR divergences not so “simple” due to medium induced scales!

IR divergences handled by EFTs

Finite T

Three scales:

1. Hard: T, full-theory diagrams
2. Soft: gT, dimensionally reduced EFT for chromo-electric fields
3. Ultrasoft: $g^2 T$, non-perturbative lattice EFT for chromo-magnetic fields (Linde problem)

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Finite μ

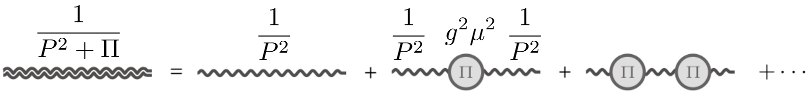
Two scales:

1. Hard: μ, full-theory diagrams
2. Soft: $g\mu$, hard thermal loop (HTL) effective theory for gluon fields

No ultrasoft scale since gluons not thermally excited at $T = 0$
$\Rightarrow$ no Linde problem!

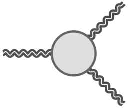
HTL resummation for soft gluons

HTL effective theory [Braaten, Pisarski, NPB 337 (1989)]



$=$ LO HTL gluon self-energy Π acts as a mass $\sim g^2\mu^2$

- HTL self-energy $=$ dominant contribution to a self-energy when external momenta are soft ($P \sim g\mu \ll \mu$)
- Soft gluons propagators with momenta $P \sim g\mu$ must be resummed
- HTL vertex functions must be used as well



$\Rightarrow$ Perturbative series contain terms non-analytic in g (i.e. $\ln g$)

Structure of 3-loop pressure at finite μ

[Freedman, McLerran, PRD 16 (1977)]:

$$p(\mu) = a_0 + a_1 g^2 + a_{2,0} g^4 \ln g + a_{2,1} g^4 + O(g^6)$$

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 = fermion,  = hard gluon,  = soft gluon

- a_0:  (free Fermi pressure)
- a_1: 

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  - IR safe:  $\Rightarrow a_{2,1}$

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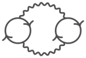
  - 
    IR div.

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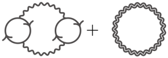
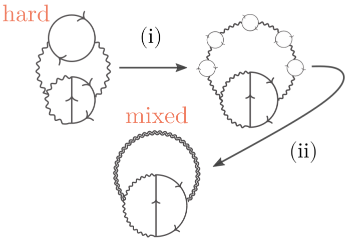
  -  $1/\varepsilon$'s cancel $\Rightarrow \ln g$ from ε/ε terms $\Rightarrow a_{2,0}$ & $a_{2,1}$

    IR div. UV div.

HTL resummation at 4-loop order (N^3LO)



(i) In IR divergent hard diagram one gluon becomes soft: dressed with HTL self-energies

(ii) Sum over the number of SE insertions: HTL-resummed propagator in mixed diagram

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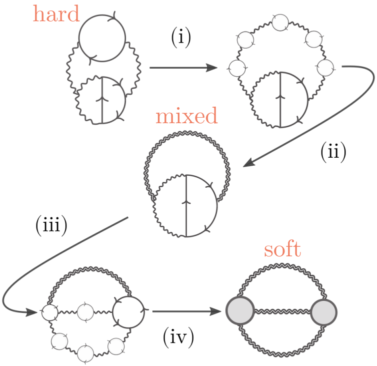


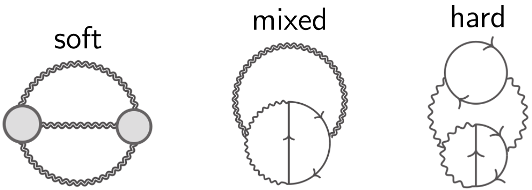
(i) In IR divergent hard diagram one gluon becomes soft: dressed with HTL self-energies

(ii) Sum over the number of SE insertions: HTL-resummed propagator in mixed diagram

(iii) Second gluon becomes soft: more HTL self-energy and vertex insertions

(iv) Fully soft HTL-resummed diagram

State-of-the-art pQCD pressure



- Current state-of-the-art result by [Gorda, Kurkela, Paatelainen, Säppi, Vuorinen, PRL 127 (2021)]: fully soft contributions at order g^6
- Missing from g^6 result: mixed and hard contributions

Quantum electrodynamics: testbed for QCD

- Rest of the talk: QED
- N^3LO correction (e^6) to QED pressure with soft/mixed/hard organization:

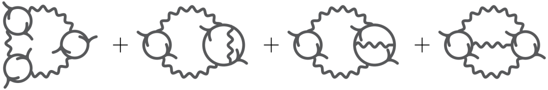
$$\alpha^3 p_3 = \alpha^3 (p_3^s + p_3^m + p_3^h)$$

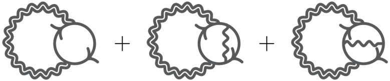
- Different to N^2LO case, end result contains also $\ln^2 \alpha$ term

$$\alpha^3 p_3 = \alpha^3 (a_0 + a_1 \ln \alpha + a_2 \ln^2 \alpha)$$

- Photon does not self-interact $\Rightarrow$ HTL vertex functions vanish
  $\Rightarrow$ no fully soft parts, $p_3^s = 0 \Rightarrow a_2 = 0$
- Complete result given by p_3^m and p_3^h

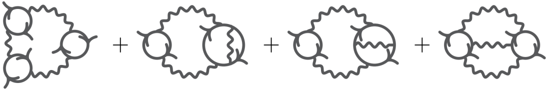
$\alpha^3 p_3^h$ (IR safe): 

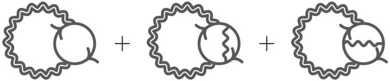
$\alpha^3 p_3^h$ (IR div.): 

$\alpha^3 p_3^m$ (UV div.): 

Contributions to N^{3}LO QED pressure

$\alpha^3 p_3^h$ (IR safe): 

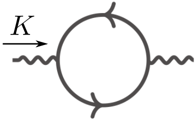
$\alpha^3 p_3^h$ (IR div.): 

$\alpha^3 p_3^m$ (UV div.): 

- Divergences cancel between 2nd and 3rd rows $\Rightarrow$ coefficient for $\alpha^3 \ln \alpha$
- Note: Diagrams on 3rd row contain two-loop self-energy insertions with soft external photons

$\Rightarrow$ Extend LO HTL photon self-energy to NLO

HTL photon self-energy at LO

$\Pi^{\mathrm{LO}}_{\mu\nu}(K) \sim$ 

- External momentum K is soft, $K \sim e\mu \ll \mu$
- HTL limit: First term in $K \ll \mu$ expansion

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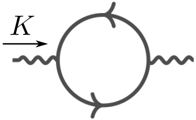

- External momentum K is soft, $K \sim e\mu \ll \mu$
- HTL limit: First term in $K \ll \mu$ expansion
- Broken Lorentz symmetry: $\Pi_{\mu\nu}$ splits into transverse (T) and longitudinal (L) components

$$\Pi_{\mu\nu} = \mathbb{P}^{\mathrm{T}}_{\mu\nu}\Pi_{\mathrm{T}} + \mathbb{P}^{\mathrm{L}}_{\mu\nu}\Pi_{\mathrm{L}}$$

$$\Pi^{\mathrm{LO}}_{\mathrm{T}}(K) = \frac{1}{2}\frac{e^2\mu^2}{\pi^2}\left[\frac{k_0^2}{k^2} + \left(1 - \frac{k_0^2}{k^2}\right)\frac{k^0}{2k}\log\frac{k^0+k}{k^0-k}\right]$$

$$\Pi^{\mathrm{LO}}_{\mathrm{L}}(K) = \frac{e^2\mu^2}{\pi^2}\left(1 - \frac{k_0^2}{k^2}\right)\left[1 - \frac{k^0}{2k}\log\frac{k^0+k}{k^0-k}\right]$$

HTL photon self-energy at NLO

- Ways of extending LO 1-loop result ($e^2\mu^2$) to NLO:
  1. NLO term from small-momentum expansion at 1-loop ($e^4\mu^2$)
  2. LO term from small-momentum expansion at 2-loop ($e^4\mu^2$)
- Contribution 1 (power correction) is straightforward to extend from 1-loop result
- Contribution 2 (2-loop) has to be obtained through a highly non-trivial computation

Computation of 2-loop photon self-energy

Needed: HTL limit (soft external line) of three 2-loop diagrams

$\Pi_{\mu\nu}^{\mathrm{2loop}}(K) \sim$ 

Complicated calculation: Has not been done before at finite μ...

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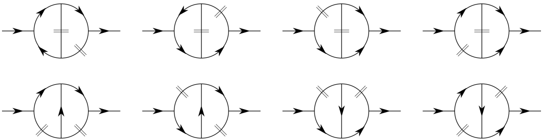
...until now as a part of my master's thesis and a resulting article [K. Seppänen et al., 2204.11279]

Real-time formalism

Framework tailored for calculating *n*-point functions

Different from imaginary-time (Euclidean) formalism:

- Minkowskian signature
- Information about μ carried in propagators by distribution functions (cf. $P^\alpha = (p^0 - i\mu, \mathbf{p})$ in Euclidean case)
- Propagators and self-energies 2×2 matrices where elements have different causality properties $\Rightarrow$ calculations bulky



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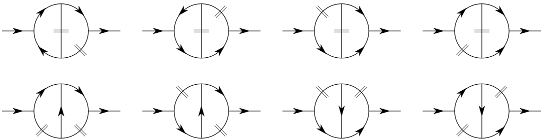


Automation: no problem!

Results

2-loop part of NLO results of order $e^4\mu^2$
[K. Seppänen et al., 2204.11279]:

$$\Pi_{\mathrm{T}}^{\mathrm{2loop}}(K) = -\frac{e^4\mu^2}{8\pi^4}\frac{k^0}{2k}\log\frac{k^0+k}{k^0-k}$$

$$\Pi_{\mathrm{L}}^{\mathrm{2loop}}(K) = -\frac{e^4\mu^2}{8\pi^4}\left\{1+2\left(1-\frac{k_0^2}{k^2}\right)\left[1-\frac{k^0}{2k}\log\frac{k^0+k}{k^0-k}\right]^2\right\}$$

$+O(\varepsilon)$ terms for finite ε/ε contributions in the pressure

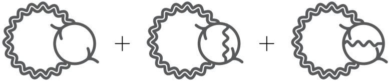
- Finite result since $Z_1 = Z_2$ in QED
- Gauge independent result

Calculation of N^3LO QED pressure

Done in companion paper [K. Seppänen et al., 2204.11893]:

- Mixed diagrams: NLO self-energy $\times$ HTL-resummed prop.

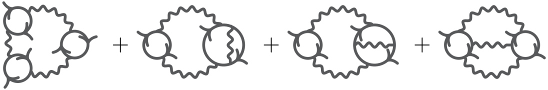
$$\alpha^3 p_3^m =$$



  - Computed fully

- IR sensitive hard 4-loop diagrams:

$$\alpha^3 p_3^{h,\text{IR div.}} =$$



  - Divergences computed, cancel with mixed diagrams
  - Explicit logarithms of renormalization scale $\bar{\Lambda}$ computed
  - First one (leading large-N_f) computed fully

Calculation of N^{3}LO QED pressure

- IR safe hard 4-loop diagrams:

$$\alpha^3 p_3^{h,\text{IR safe}} =$$

  - Explicit logarithms of $\bar{\Lambda}$ computed

$\Rightarrow$ Computed almost complete N^{3}LO result

- Missing: a pure number from hard diagrams subleading in large-N_f limit

Results: N^{3}LO correction to QED pressure

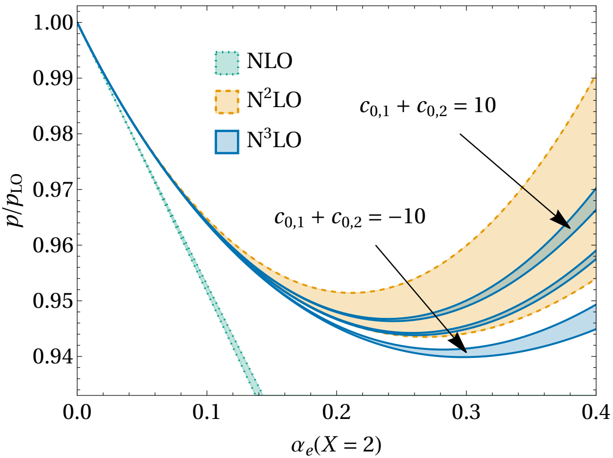
[K. Seppänen et al., 2204.11893] provides update to 45-year-old result:

$$\frac{\alpha^3 p_3}{p_{\rm LO}} = N_f^2 \left(\frac{\alpha}{\pi}\right)^3 \left[a_{3,1} \ln^2 \left(N_f \frac{\alpha}{\pi}\right) + a_{3,2} \ln \left(N_f \frac{\alpha}{\pi}\right) + a_{3,3} \ln \left(N_f \frac{\alpha}{\pi}\right) \ln \frac{\bar{\Lambda}}{2\mu} + a_{3,4} \ln^2 \frac{\bar{\Lambda}}{2\mu} + a_{3,5} \ln \frac{\bar{\Lambda}}{2\mu} + a_{3,6}\right]$$

$a_{3,1}$	0
$a_{3,2}$	$-\frac{5}{4} + \frac{33}{2} N_f^{-1} + \frac{1}{48}\left(7 - 60 N_f^{-1}\right)\pi^2$
$a_{3,3}$	2
$a_{3,4}$	$-\frac{2}{3}$
$a_{3,5}$	$-\frac{79}{9} + \frac{2}{3}\pi^2 + \frac{2}{3}(13 - 8\ln 2)\ln 2 + \delta - \frac{31}{4} N_f^{-1}$
$a_{3,6}$	$1.02270(2) + \left(2.70082 + \frac{1}{2} c_{0,1}\right) N_f^{-1} + \frac{1}{2} c_{0,2} N_f^{-2}$

- $\delta \simeq -0.8563832$ [Vuorinen, PRD 68 (2003)]
- $c_{0,1}$ and $c_{0,2}$ remain unknown (pure numbers from hard diagrams)

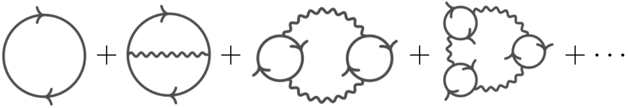
Cold and dense QED pressure up to N^{3}LO



- Physical QED: $N_f = 1$
- Renormalization scale $\bar{\Lambda} = X\mu$, X varied around $X = 2$ by a factor of 2
- Dramatic decrease in $\bar{\Lambda}$ dependence

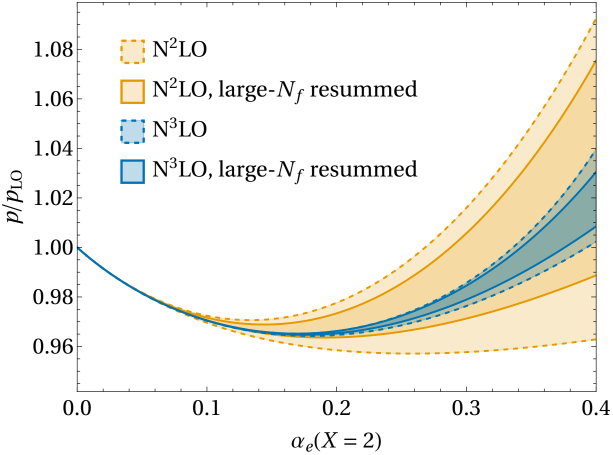
Improve accuracy: Large-N_f resummation

- Try to improve renormalization scale dependence by further resummation
- Resum all leading large-N_f diagrams with full kinematics [Ipp, Rebhan, JHEP 06 (2003)]



- Add the subleading contributions from our calculation

Large-N_f-resummed pressure less sensitive to $\bar{\Lambda}$



Here we have chosen $N_f = 3$ and $c_{0,1} = c_{0,2} = 3$

Next step: QED $\Rightarrow$ QCD

1. Generalize NLO photon self-energy to QCD
   - 3 more 2-loop diagrams, same tools apply to them
   - Fully soft 1-loop digrams
   - Probably not finite result ($Z_1 \neq Z_2$ in QCD)
   - Gauge-(in)dependent?

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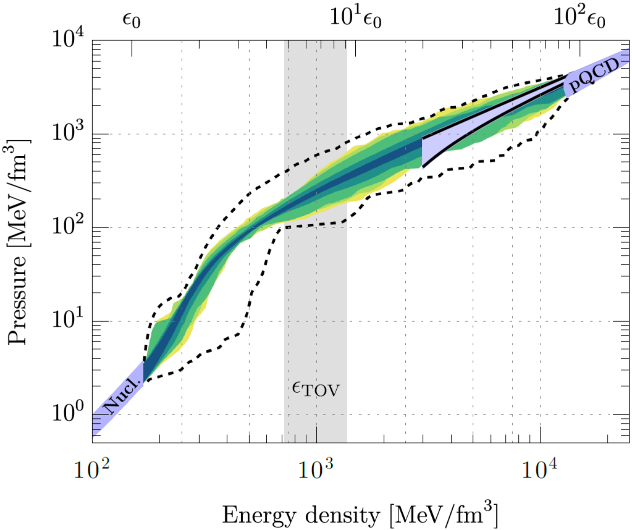
$\Rightarrow$ Accuracy of dense pQCD pressure improved

pQCD result constrains neutron-star equation of state



Thanks! Questions?

[2204.11279]
[2204.11893]

Extra: Speed of sound in QED matter