Gibbs paradox and the Hadron Resonance Gas

Antal Jakovác

BME Technical University Budapest

Outlines

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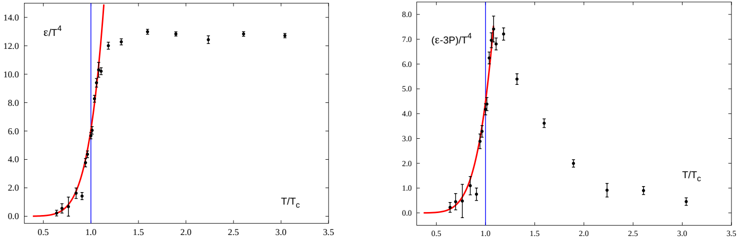
EoS comes from $Z = e^{-\beta F} = \mathrm{Tr}\, e^{-\beta H}$

- exact computation: MC simulations – measurement; we have to understand the result!
- most naive approach: treat plasma as gas of free particles
  - high T: free quarks and gluons $\Rightarrow$ SB limit
  - low T: hadronic degrees of freedom $\Rightarrow$ ε and p at low T.



(F. Karsch, K. Redlich, A. Tawfik, Eur.Phys.J. C29 (2003) 549-556.)

$\Rightarrow$ Seems to work surprisingly well!

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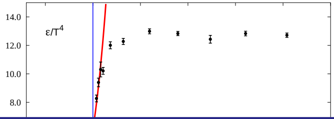

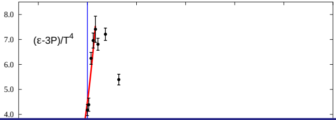

Lesson

- Strong interaction is mostly incorporated in the spectrum!
- remaining interactions are weak

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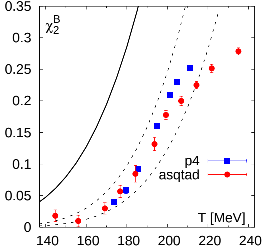
Recent results



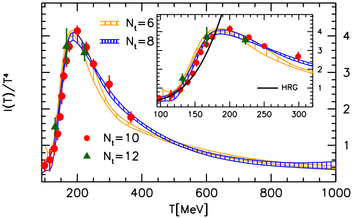
(C. Bernard et al. [MILC], Phys. Rev. D 71, 034504 (2005))

(P. Huovinen, P. Petreczky, Nucl.Phys.A837:26-53,2010)

Baryon number fluctuations:

solid line: naive HRG

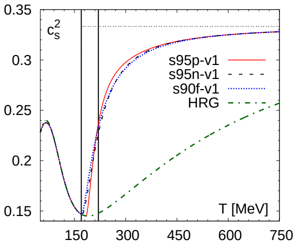
dashed blue lines: hadron masses on lattice; mass cut = 1.8 GeV resp. 2.5 GeV.



(Sz. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti, K.K. Szabo, arXiv:1007.2580v2)

Trace anomaly $I = \varepsilon - 3p$; HRG is not too accurate

Recent results



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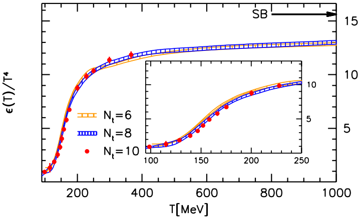
speed of sound minimum is HRG value, for larger T HRG fails.



(Sz. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti, K.K. Szabo, arXiv:1007.2580v2)

energy density: SB limit is not reached at $\approx 5T_c$.

Problems of HRG model

- Recent MC data: not all quantities can be reproduced well
- transport: infinite lifetime $\Rightarrow$ ballistic regime
  $\Rightarrow$ transport coefficients are infinite!
- in energy denstity: too large contribution after $\approx 1 - 1.5\,T_c$!
  - HRG degrees of freedom are not there at hight T? – crossover!
  - more correct: HGR degrees of freedom are resonances with finite (and T-dependent) lifetime
    $\Rightarrow$ a particle with too short lifetime cannot be part of the statistical ensamble

$\Rightarrow$ we have to consider the complete spectrum!

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conceptional problem

statistical physics cannot treat bound state models well!
cf. H-atom problem

H-atom problem

HRG describes bound states of quarks ∼ H-atom problem of QCD

But already the H-atom problem is ill-defined in statistical physics!

Coulomb problem: $E_n = -E_0/n^2$ is the energy level of nth energy level, degeneracy is n^2:

$$Z = \sum_n n^2 e^{-\beta E_n} > \sum_n n^2 \to \infty.$$

Cannot be resolved using a regularization. Eg. take into account states with $n < N$, then the energy expectation value

$$0 < -\langle E\rangle = -\frac{\sum_n^N n^2 E_n e^{-\beta E_n}}{\sum_n^N n^2 e^{-\beta E_n}} < \frac{N e^{\beta E_0}}{N^3} \stackrel{N\to\infty}{\longrightarrow} 0.$$

$\Rightarrow$ at all T, all H-atoms should be at highly excited state !?

Outlines

Not all energy levels are independent!

Gibbs' argumentation: take a classical ideal gas with N independent particles:

$$Z_N = V^N Z_1^N \quad \Rightarrow \quad F = -NT(\ln Z_1 + \ln V) \quad \Rightarrow \quad \text{not extensive!}$$

Solution: particles states that differ only in the value of momenta are indistinguishable:

$$Z_N = \frac{V^N Z_1^N}{N!} \quad \Rightarrow \quad F = -NT(\ln Z_1 + \ln \frac{V}{N}) \quad \Rightarrow \quad \text{extensive!}$$

A degree of freedom (DoF): a bunch of indistinguishable states.

Appears in physical quantities: eg. SB limit $\varepsilon = N_{dof}\frac{\pi^2 T^2}{30}$.

A new problem: what are then those states which are indistinguishable? what are the DoF?

- for sure different DoF: states with different internal quantum numbers (different charges)
- for sure same DoF: states which differ only in their momenta
- states with same charge, but different energy??
  eg. in case of 1s and 2s bound states.

Gibbs takes them different – then states with mass m and $m + dm$ are 2 DoF even in $dm \to 0^+$, but at $dm = 0$ is 1 DoF

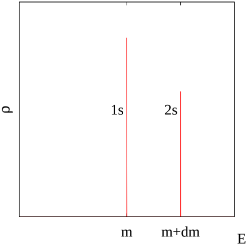
$\Rightarrow$ nonanalytic behaviour in dm.

Quantify: spectral function $\sim$ density of states

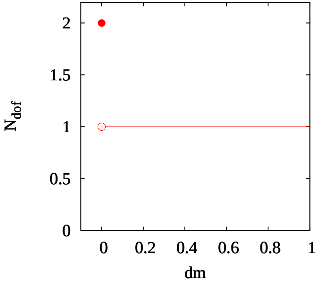
$$\varrho(x) = \langle [A(x), A(0)] \rangle \;\rightarrow\; \varrho(\omega > 0) = \sum_n 2\pi\delta(\omega - E_n)\, |\langle n|A|0\rangle|^2$$

$\Rightarrow$ energy levels appear directly

$\mathbf{k} = 0$:



$\Rightarrow$



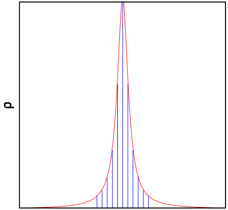
$\Rightarrow$ N_{dof} =number of Dirac-deltas in ϱ

- normalization is irrelevant
- $\varrho \Rightarrow N_{dof}$: nonlinear relation

Outlines

broadening of energy levels

In all real matter the energy levels are not sharp: broadening because of finite T, finite lifetime, or zero mass excitations.
In finite volume: lot of energy levels: in ϱ lot of Dirac-deltas



∞ DoF??

$\Downarrow$

NO: these are multiparticle states...

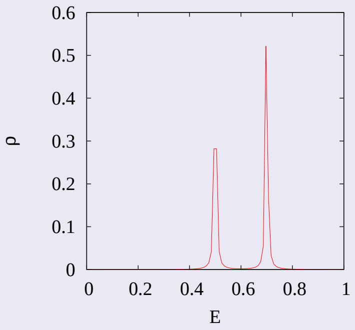
$\Downarrow$

only 1 DoF

BUT: 1s vs. 2s in H-atom: particle content is the same, only the electromagnetic field is in different state, which is a photon coherent multiparticle state.

overlapping states

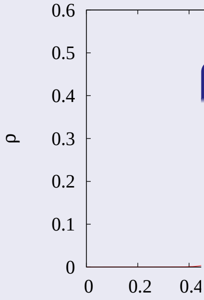
Large separation



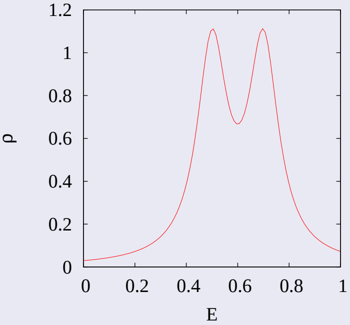
⇒ 2 DoF

overlapping states

Large separation



Overlapping peaks: entanglement



$\Rightarrow$?? DoF

Outlines

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Our goal: contruct the statistical theory of finite lifetime resonances

The drawback of QM statistical treatment: fix the DoF beforehand, and order different wave functions to different DoF. But states with the same quantum numbers can interfere, mix with each other $\Rightarrow$ we should treat them with the same wave function

$$\Downarrow$$

Construct a quadratic field theory which is able to reproduce the observable spectral functions with a single field.

Take a single bosonic degree of freedom with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\Phi(x)\mathcal{K}(i\partial)\Phi(x).$$

The corresponding retarded propagator

$$G_R(p) = \frac{1}{\mathcal{K}(p_0 + i\varepsilon, \mathbf{p})}$$

Spectral function:

$$\varrho(p) = -2\,\mathrm{Im}\,G_R(p) \quad \Leftrightarrow \quad G_R(p) = \int \frac{d\omega}{2\pi} \frac{\varrho(\omega, \mathbf{p})}{p_0 - \omega + i\varepsilon}.$$

$\mathcal{K} \quad \Leftrightarrow \quad \varrho$ relation

- nonlocal in time $\Rightarrow$ no canonical formalism, no imaginary time formalism!
- causal: a theory is causal if $\varrho(x) = 0$ for space-like 4-vectors. Since here we reproduce a given causal spectral function $\Rightarrow$ gives a causal theory

Outlines

- This system is spacetime translation invariant $\Rightarrow$ $\exists$ Noether currents (energy momentum tensor)

$$\hat{T}_{\mu\nu}(k) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\,\Phi(k-p)\mathcal{D}_{\mu\nu}(p)\Phi(p), \qquad D_{\mu\nu}(p) = p_\mu \frac{\partial\mathcal{K}}{\partial p^\nu} - g_{\mu\nu}\mathcal{K}$$

  In case of rel. invariance $\hat{T}_{\mu\nu}$ is symmetric.

- expectation value: $\langle\Phi(k-p)\Phi(p)\rangle = (2\pi)^4\delta(k)\, iG_<(p)$
  $\Rightarrow$ $\left\langle \hat{T}_{\mu\nu}(x)\right\rangle = T_{\mu\nu}$ position independent.

- KMS relation: $\int d^3\mathbf{x}\,\hat{T}_{00} = \hat{H}$ conserved charge $\equiv$ generator of time translations $\Rightarrow$ $e^{-\beta\hat{H}}$ time translation by $-i\beta$
  $\Rightarrow$ KMS relation is still valid.

- renormalization: subtract $T=0$ value
  now: assume spectral function is temperature independent.

Finally

$$\varepsilon = T_{00} = \int_+ \mathcal{DK}(p)n(p_0)\varrho(p)$$

- $\int_+ = \int_0^\infty \frac{dp_0}{2\pi}\int\frac{d^3\mathbf{p}}{(2\pi)^3}$, $\mathcal{DK}(p) = 2p_0^2\frac{\partial\mathcal{K}}{\partial p^2} - \mathcal{K}$
- $\varrho(p)$ spectral function, $n(p_0)$ Bose-Einstein distribution
- $\varrho \quad \Rightarrow \quad \mathcal{K} \quad \Rightarrow \quad \varepsilon$ functional of the spectral function!
- nonlinear function of $\varrho \quad \Rightarrow \quad$ as we expected in the quantum entanglement case
- rescaling invariant: $\varrho \to Z\varrho$, $\mathcal{K} \to \mathcal{K}/Z$
  - $\Rightarrow \quad \varrho\mathcal{DK}$ is invariant
  - $\Rightarrow \quad$ only the energy levels count, not the normalization!

Number of DoF (cf. Williams-Weizsacker formula)

$$N_{dof} = \int_0^\infty \frac{dp_0}{2\pi} \frac{1}{p_0} \mathcal{D}(p) \varrho(p)$$

- not direct physical meaning, but in case of discrete energy levels gives correct result (see later)
- nonlinear in ϱ, rescaling invariant, like ε.

Outlines

Outlines

This is the standard free particle case:

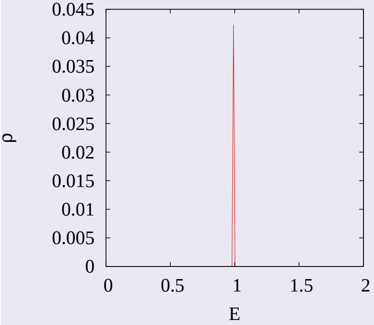
$$\varrho(p_0 > 0) = 2\pi\delta(p^2 - m^2) \quad \Rightarrow \quad \mathcal{L} = \frac{1}{2}\Phi(-\partial^2 - m^2)\Phi$$

- number of DoF: $N_{dof} = 1$ – OK.
- energy density: $\varepsilon = \frac{1}{2\pi^2}\int_m^\infty dp_0\, n(p_0)\, p_0^2 \sqrt{p_0^2 - m^2}$

  – the standard formula.

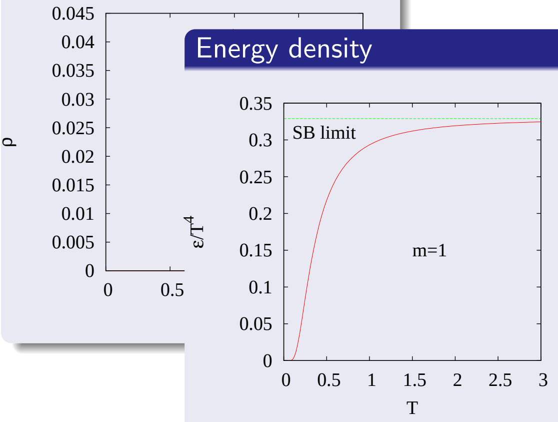
$\Rightarrow$ for one Dirac-delta the usual formulae are reproduced

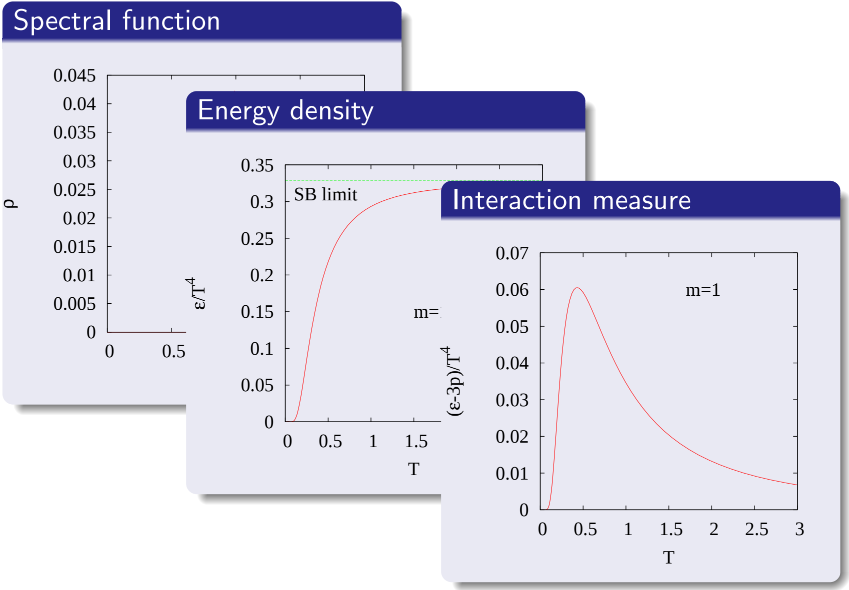
Spectral function



Spectral function

Energy density




Spectral function
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
ρ
0
0.5
Energy density
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
SB limit
ε/T^4
0
0.5
1
1.5
T
Interaction measure
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
m=1
(ε-3p)/T^4
0
0.5
1
1.5
2
2.5
3
T

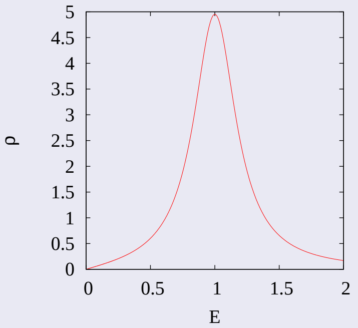
Outlines

Breit-Wigner spectral function:

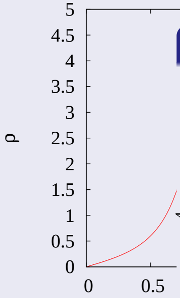
$$\varrho(p) = \frac{4p\Gamma}{(p^2 - m^2)^2 + 4p^2\Gamma^2}$$

- number of DoF analytically computable: $N_{dof} = 1$!
  $\Rightarrow$ irrespective of the lifetime a single particle is 1DoF!
- BUT: it does not mean that n Lorentzians are n DoF, because of nonlinearity!

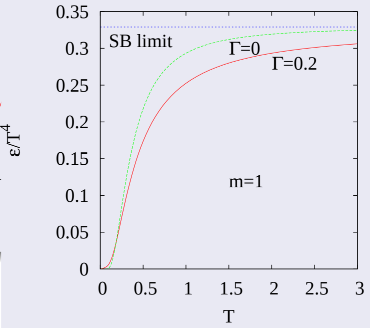
Spectral function



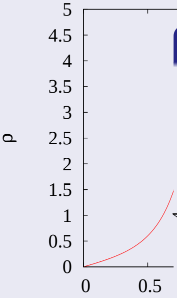
Spectral function



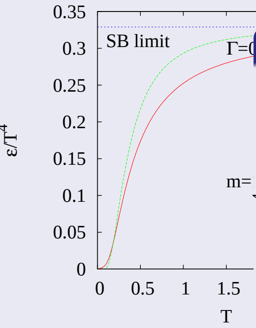
Energy density



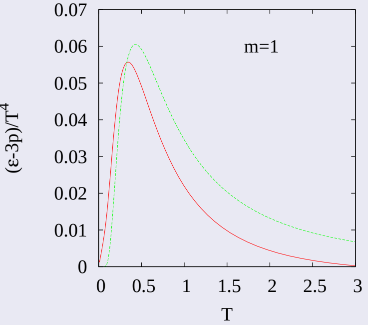
Spectral function



Energy density



Interaction measure



Outlines

1. Introduction
   - Equation of State in QCD
   - Gibbs paradox
   - Interactions
2. Field theory solution
   - Generalized quadratic theory
   - Energy momentum tensor
3. Results
   - Single Dirac-delta
   - Single Lorentzian
   - Two Dirac-deltas
   - Two Lorentzians
4. Conclusion

Two stable particles, with different normalization ($Z_1 + Z_2 = 1$)

$$\varrho(p) = Z_1 2\pi\delta(p^2 - m_1^2) + Z_2 2\pi\delta(p^2 - m_2^2)$$

Computing $N_{dof} = \begin{cases} 2 & \text{if } m_1 \neq m_2 \\ 1 & \text{if } m_1 = m_2 \end{cases}$

Similarly:

$$\varepsilon = \begin{cases} \varepsilon(m_1) + \varepsilon(m_2) & \text{if } m_1 \neq m_2 \\ \varepsilon(m) & \text{if } m_1 = m_2 = m \end{cases}$$

How can have it from an analytical formula?

In formula

$$N=\int_0^\infty dp_0\,\frac{Z_2(p^2-m_1^2)^2+Z_1(p^2-m_2^2)^2}{(Z_2(p^2-m_1^2)+Z_1(p^2-m_2^2))^2}\left[Z_1\delta(p^2-m_1^2)+Z_2\delta(p^2-m_2^2)\right].$$

What is the difference between $m_2=m_1$ and $m_2\to m_1$?

$\Rightarrow$ $p^2\to m_1^2$ and $m_2\to m_1$ are not interchangable!

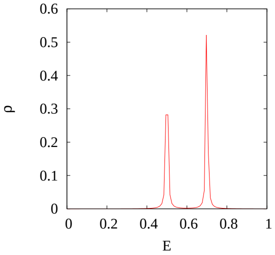
$$X=\frac{Z_2(p^2-m_1^2)^2+Z_1(p^2-m_2^2)^2}{(Z_2(p^2-m_1^2)+Z_1(p^2-m_2^2))^2}\quad\Rightarrow\quad\begin{cases}\lim_{m_2\to m_1}\lim_{p^2\to m_1^2}X=\frac{1}{Z_1}\\ \lim_{p^2\to m_1^2}\lim_{m_2\to m_1}X=1\end{cases}$$

$\Rightarrow$ this is the analytical appearance of Gibbs paradox!

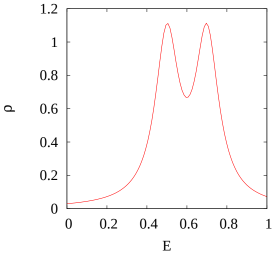
non-linearity of $N[\varrho]$ is important

Outlines

From the lesson of two Dirac-deltas we expect for finite width:

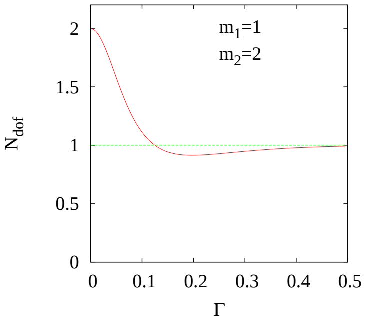


first $p^2 \to m^2$ limit $\Rightarrow$ 2 DoF



first $m_2^2 \to m_1^2$, later $p^2 \to m^2$ limit $\Rightarrow$ 1 DoF

In fact, from direct calculation:



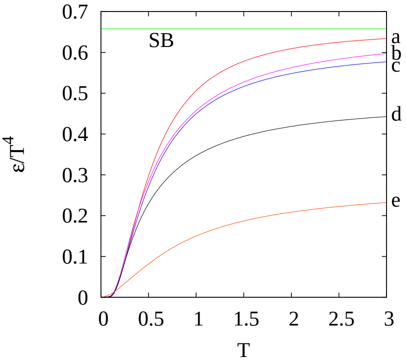
This is the smoothed version of Gibbs paradox

Lesson

indistinguishability of particles is a dynamical question!

thermodynamics

What is the effect to the thermodynamics?



- a: 2 Dirac-delta ($\Gamma_1 = 0, \Gamma_2 = 0.5$)
- b: add $\Gamma_1 = 0$ and $\Gamma_2 = 0.5$ independently (no entanglement)
- c: $\Gamma_1 = 0$, $\Gamma_2 = 0.5$ but few states in between
- d: $\Gamma_1 = 0$, $\Gamma_2 = 0.5$ with higher state density in between
- e: $\Gamma_1 = 0.5$, $\Gamma_2 = 0.5$

thermodynamics

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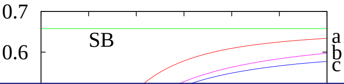

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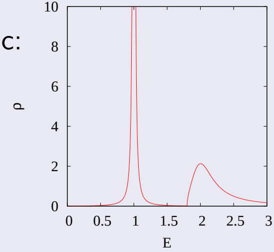
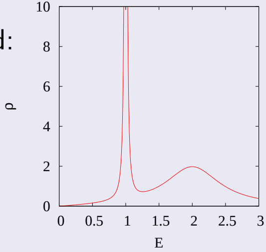
Importance of states between the peaks

c:



d:



thermodynamics

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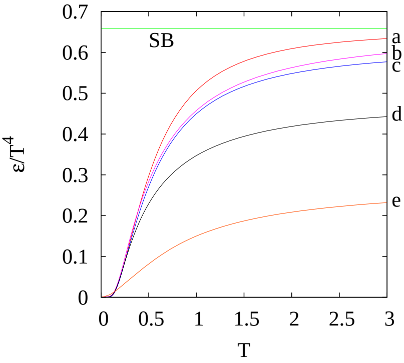

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- e: $\Gamma_1 = 0.5$, $\Gamma_2 = 0.5$

Most important region for entanglement: between the two peaks!

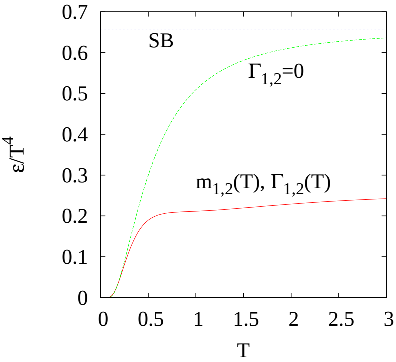
Temperature dependent parameters

In real plasma mass and width are T-dependent quantities:

$$m^2 = m_0^2 + \# T^2, \quad \Gamma^2 = \Gamma_0^2 + \# T^2,$$

the coefficients are $\mathcal{O}(1)$ numbers.

Typical result:



- for small T: same curves
- for large T: does not reach SB limit

Outlines

The most important points that modify the stable bound state gas results:

- states with same quantum numbers can mix $\Rightarrow$ they should be described by the same field
- entanglement of the states (or number of DoF) is a dynamical question
- with T-dependent mass and width no SB limit