

# Non-Extensive Black Hole Thermodynamics Estimate for Power-Law Particle Spectra

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We point out that by considering the Hawking-Bekenstein entropy of Schwarzschild black hole horizons as a non-extensive Tsallis entropy, its additive formal logarithm, coinciding with the Renyi entropy, generates an equation of state with positive heat capacity above a threshold energy. Based on this, the edge of stability is conjectured to be trans-Planckian, i.e. being in the quantum range. From this conjecture an estimate arises for the  $q$ -parameter in the Renyi entropy, ( $q = 1 + 2/\pi^2$ ), also manifested in the canonical power-law distribution of high energy particles ( $q \approx 1.2$  for quark matter).

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## INTRODUCTION

Non-extensive thermodynamics [1] aims at describing dynamically and statistically entangled systems, among others also the quark-gluon plasma at the moment of hadronization. By using a non-Boltzmannian formula for the entropy, the experimentally observed power-law tailed particle spectra can be interpreted as reflecting a canonical ensemble in the non-extensive thermodynamics [2–5]. The theoretical prediction of the power  $\nu$ , related to the Tsallis parameter,  $q = 1 + 1/\nu$ , is however a very difficult problem.

In an earlier work we have fitted several hadron spectra observed at RHIC [6–12] by assuming a blast wave picture and quark recombination [13]. These fits agreed with a common Tsallis parameter of  $q \approx 1.2$  for the quark matter quite remarkably. According to the quark coalescence picture the corresponding value of  $(q-1)$  for mesons should be the half, and for baryons one third of this fit. Unfortunately the experimental data are not yet precise enough to prove or disprove this model. On the other hand, in [14] it was shown that the cut power-law energy distribution may arise from a repeated non-additive combination of small amounts of energy. Starting with a general  $Q^2 = 2E_1E_2(1 - \cos \theta)$  - dependent correction to the addition of the respective kinetic energies of massless partons,  $E_1$  and  $E_2$ , i.e. considering a combination law of the form

$$E_1 \oplus E_2 = E_1 + E_2 + G(Q^2) \quad (1)$$

in the large number limit and neglecting the angular dependence one arrives at a general asymptotic combination law of  $E_1 \oplus E_2 = E_1 + E_2 + aE_1E_2$ . This combination law is equivalent to the addition of formal logarithms [15],

$$L(E) = \frac{1}{a} \ln(1 + aE), \quad (2)$$

leading to

$$L(E_1 \oplus E_2) = L(E_1) + L(E_2). \quad (3)$$

As a consequence the statistical distribution which arises by using such non-additive composition rules is

$$f(E) = \frac{1}{Z} e^{-\beta L(E)} = \frac{1}{Z} (1 + aE)^{-1/(aT)} \quad (4)$$

Here  $a = G'(0)$ , the derivative of the correction function at low  $Q^2$  is non-perturbative, so there is no easy theoretical calculation to predict its value.

On the other hand such an energy-distribution can also be derived as a canonical distribution stemming from the Tsallis ( $S_T$ ) or Renyi-entropy ( $S_R$ ) [16–21]. Both entropy formulas contain a  $q$  parameter and are in fact connected as

$$S_R = \frac{1}{q-1} \ln(1 + (q-1)S_T). \quad (5)$$

The Renyi entropy is defined as

$$S_R = \frac{1}{q-1} \ln \sum_i p_i^q, \quad (6)$$

and in the  $q \rightarrow 1$  limit it coincides with Boltzmann's entropy formula. The canonical energy distribution is derived from maximizing

$$S_R - \beta \sum_i p_i E_i - \alpha \sum_i p_i. \quad (7)$$

Differentiation with respect to  $p_i$  leads to

$$p_i = \frac{1}{Z} \left( 1 + \frac{\beta(E_i - \mu)}{1 - q} \right)^{\frac{1}{q-1}}. \quad (8)$$

Here  $\mu$  can be expressed by  $\alpha$  and the other parameters,  $Z$  is set according to the normalization condition  $\sum_i p_i = 1$ .

The main difference is that while the Renyi entropy is additive for factorizing probabilities, the Tsallis entropy is not. The canonical distribution is the same for both entropies, one being a monotonous function of the other,

but their stability properties – connected to the second derivative – differ as a rule.

In this paper we attempt to obtain an estimate for the  $q$  parameter of the Renyi and Tsallis entropy. Lining up with the conformal field theory – gravity conjecture [22–24] which gave already an estimate for the minimal viscosity of matter [25], now we investigate simple, static and radial Schwarzschild black hole thermodynamics and derive an estimate for the minimal  $q$  parameter which stabilizes their equation of state above an energy considered to be in the quantum gravity (trans-Planckian) range. This is a two step process: First we demonstrate that the interpretation of the Bekenstein-Hawking entropy [26–28] as a non-extensive Tsallis entropy and the use of the additive Renyi entropy instead stabilizes the Schwarzschild black hole horizon equation of state above a given energy. Then we ask the question whether this energy, the inflection point of the  $S(E)$  curve, is at or below the ground state energy of a semiclassical string spanned over the diameter of the horizon,  $2R = 4E$  in Planck units. All over this paper energy, mass and momentum is measured in multiples of the Planck mass,  $M_P$ , while length and time in multiples of the Planck length,  $L_P$ . All equations relating unlike quantities are to be supported by corresponding powers of  $M_P$  and  $L_P$ . The entropy is measured in units of the Boltzmann constant,  $k_B$ . In a  $c = 1$ ,  $k_B = 1$  system one expresses the Planck constant as  $\hbar = L_P M_P$  and Newton's gravity constant as  $G = L_P / M_P$ .

## RENYI ENTROPY FOR BLACK HOLE HORIZON

The Bekenstein-Hawking entropy for simple (static, radial-symmetric) black hole horizons can easily be obtained from Clausius' entropy formula,

$$S = \int \frac{dE}{T}, \quad (9)$$

by using the Unruh temperature [29] associated to the acceleration at the surface of the coordinate singularity (red shift factor corrected surface gravity of the horizon):

$$T = \frac{g}{2\pi}. \quad (10)$$

For the normal gravitational acceleration at Earth's surface this temperature is very small,  $k_B T \approx 10^{-19}$  eV, but in a relativistic heavy ion collision, assuming a stopping from the speed of light to zero in a distance equal to the half Compton wavelength of a proton, meaning a deceleration of  $g = c^2/2\lambda = mc^3/\hbar$ , the Unruh temperature is in the range of  $k_B T = mc^2/2\pi \approx 150$  MeV. Indeed particle spectra stemming from such collisions agree well with thermal model estimates with temperatures of this magnitude. It has already been proposed that such a deceleration could be the general reason behind observing

thermal-like spectra born in sudden, dramatic hadronization events [30, 31].

In general for a metric given by

$$d\tau^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2 \quad (11)$$

with  $t$  and  $r$  being the time and radius coordinates for the far, static observer and  $d\Omega$  the two-dimensional surface angle, the acceleration of a test particle with mass  $m$  follows from the Maupertuis action

$$I = -m \int d\tau. \quad (12)$$

The radial equation of motion derived from the corresponding Lagrangian

$$L = \sqrt{f\dot{t}^2 - \dot{r}^2/f - \dots}, \quad (13)$$

with  $\dot{t} = dt/d\tau$  and  $\dot{r} = dr/d\tau$  and suppressing terms related to angular components, is given by

$$\begin{aligned} f\dot{t} &= K, \\ f\dot{t}^2 - \dot{r}^2/f &= 1 \end{aligned} \quad (14)$$

with an integral of motion related to the energy,  $K$ . Eliminating  $\dot{t}$  from the equations one obtains

$$\dot{r}^2 = K^2 - f(r), \quad (15)$$

whose  $\tau$ -derivative delivers the radial acceleration

$$\ddot{r} = -\frac{1}{2}f'(r) = -g. \quad (16)$$

The corresponding Unruh temperature becomes

$$T = \frac{1}{4\pi}f'(r). \quad (17)$$

Considering now that the internal energy is practically the mass one obtains the entropy

$$S = 4\pi \int \frac{dM}{f'(r)}. \quad (18)$$

This result can be written in a more elegant form by noting that the denominator,  $f'(r)$  – to be evaluated at the condition  $f(r) = 0$  – is a Jacobian for a Dirac-delta constraint. Therefore the above BH-entropy equals to

$$S = 4\pi \iint \delta(f(r, M)) dr dM. \quad (19)$$

This form reminds to a microcanonical shell in the phase space of the variables  $r$  and  $M = E$ .

In the case of the Schwarzschild black hole solution to the Einstein equations one has

$$f(r) = 1 - \frac{2GM}{c^2 r} = 1 - \frac{R}{r}, \quad (20)$$

The horizon condition  $f(r) = 0$  is fulfilled at  $r = R = 2M = 2E$  in Planck units ( $G = 1$ ). The acceleration is  $g = 1/2R = 1/4E$  and the BH-entropy becomes

$$S = 4\pi \int \frac{R}{2} dR = \pi R^2. \quad (21)$$

Since the same result emerges for any  $f(r, M) = 1 - 2GM/r - h(r)$ , linear in  $M$ , the entropy of such simple black hole horizons is proportional to their area.

This result, however, leads to an equation of state,  $S(E)$  which describes an object with negative heat capacity:

$$\begin{aligned} S &= 4\pi E^2, \\ \frac{1}{T} &= S'(E) = 8\pi E, \\ c &= -S''(E) = -8\pi. \end{aligned} \quad (22)$$

Considering now the Hawking-Bekenstein entropy as a Tsallis entropy, since nobody knows whether it were additive if two black holes would be united, one is tempted to use its additive form, the Renyi entropy for defining the equation of state. Using equation (5), one obtains the following:

$$\begin{aligned} S &= \frac{1}{a} \ln(1 + 4\pi a E^2), \\ \frac{1}{T} &= S'(E) = \frac{8\pi E}{1 + 4\pi a E^2}, \\ c &= -S''(E) = 8\pi \frac{4\pi a E^2 - 1}{(1 + 4\pi a E^2)^2} \end{aligned} \quad (23)$$

with  $a = q - 1$ . Figure 1 shows the  $S(E)$  (upper frame), the  $T(E)$  and the  $c(E)$  curves for both the Bekenstein-Hawking and Renyi equation of state for Schwarzschild black holes.

It is straightforward to inspect that above the inflection point in energy,

$$E_0 = \frac{1}{2\sqrt{a\pi}} \quad (24)$$

the heat capacity is positive and the black hole is a thermodynamically stable object. Below this energy the heat capacity continues to be negative, describing an unstable object. The temperature at this energy,

$$T_0 = \frac{1}{8\pi E_0} + \frac{a}{2} E_0 = \frac{1}{2} \sqrt{\frac{a}{\pi}}, \quad (25)$$

is finite, unless one considers the Boltzmann limit by  $a \rightarrow 0$  ( $q \rightarrow 1$ ).

### SEMICLASSICAL ESTIMATE FOR THE BLACK HOLE $q$ PARAMETER

Classical Schwarzschild black hole horizons would be thermodynamically stable if the edge of stability,  $E_0$ , lied

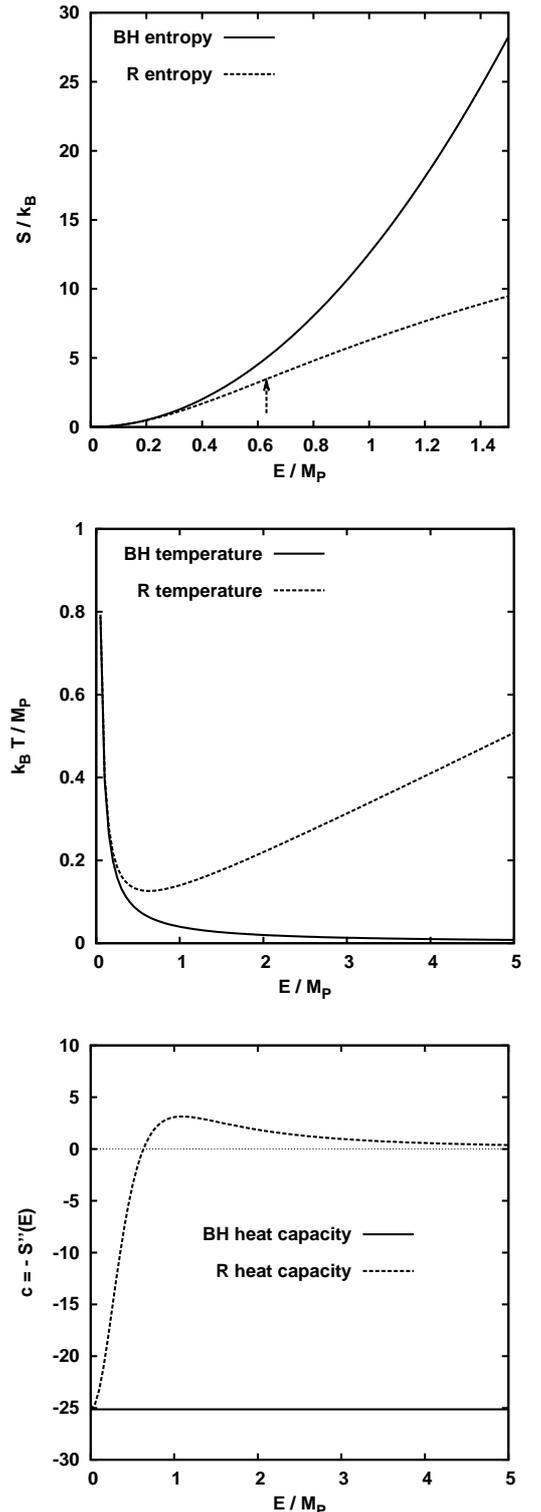


FIG. 1: The Schwarzschild black hole horizon entropy, temperature and heat capacity as a function of the total internal energy (mass inside the horizon). The solid curves correspond to the Bekenstein-Hawking entropy, while the dashed curves to its formal logarithm, the Renyi entropy.

below the quantum mechanical ground state energy. Of course, in the absence of a functioning quantum theory of gravity, one can only have an estimate for this value from semiclassical considerations. The simplest of which is regarding the energy of a string of the length of the diameter,  $2R$ , having a wave number and frequency  $\omega = k = \pi/2R$  (the sinus wave with no intermediate nodes). The corresponding ground state energy is required to be greater than the inflection point of the  $S_R(E)$  curve:

$$\frac{\hbar\omega}{2} = \frac{\pi}{8E_0} \geq E_0. \quad (26)$$

Utilizing the relation (24) one concludes that it is equivalent to the condition

$$a = q - 1 \geq \frac{2}{\pi^2}. \quad (27)$$

It is amazing to note that this estimate,  $q \approx 1.2026$ , how well approximated is by cosmic ray observations ( $q = 11/9$ ) [32, 33] and by the quark coalescence fit to RHIC hadron transverse momentum spectra ( $q \approx 1.2$ ) [13].

In summary, by interpreting the Bekenstein-Hawking entropy as a non-extensive Tsallis entropy of simple black hole horizons, and therefore considering their equation of state based on the Renyi entropy, enables such horizons to be thermodynamically stable above a given energy. Requiring that this energy belongs to a semiclassical ground state of a string stretched over the diameter of the horizon, amazingly an estimate for the  $q$ -parameter of Renyi's and Tsallis' entropy formulas arises. This estimate is close to findings in relativistic heavy ion collisions and in cosmic ray observations. This result does not rely on the AdS – CFT duality, as estimates for the viscosity to entropy density ratio did[34–37].

Of course, we do not suggest that 3-dimensional, gravitational (mini) black holes would form in high energy particle and heavy ion collisions. However, due to the extreme deceleration by the stopping, a Rindler horizon may occur for the newly produced hadrons. This can be, in general, the origin of thermal looking spectra as suggested earlier by Kharzeev[30, 31]. This also can be the mechanism behind the leading non-extensive effect in the power-law spectra, consistent with a statistical, constituent quark matter hadronization picture. It has been recently reported about ultrashort laser pulse experiments producing among other known radiations also a Hawking radiation of photons[38]. The authors emphasize that no black hole formation is needed in the classical sense – the formation of an event horizon suffices to produce features analogue to the Hawking radiation in the spectrum.

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