

Cosmology

Lecture 2: Dynamics of the Universe and Thermal History



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CENTRAL EUROPEAN INSTITUTE FOR
COSMOLOGY AND FUNDAMENTAL PHYSICS



EUROPEAN UNION
European Structural and Investment Funds
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MINISTRY OF EDUCATION,
YOUTH AND SPORTS

Friedman-Lemaître-Robertson-Walker (FLRW) spacetime: the geometry of a **homogeneous-isotropic** universe

Conventionally $a = 1$ Today

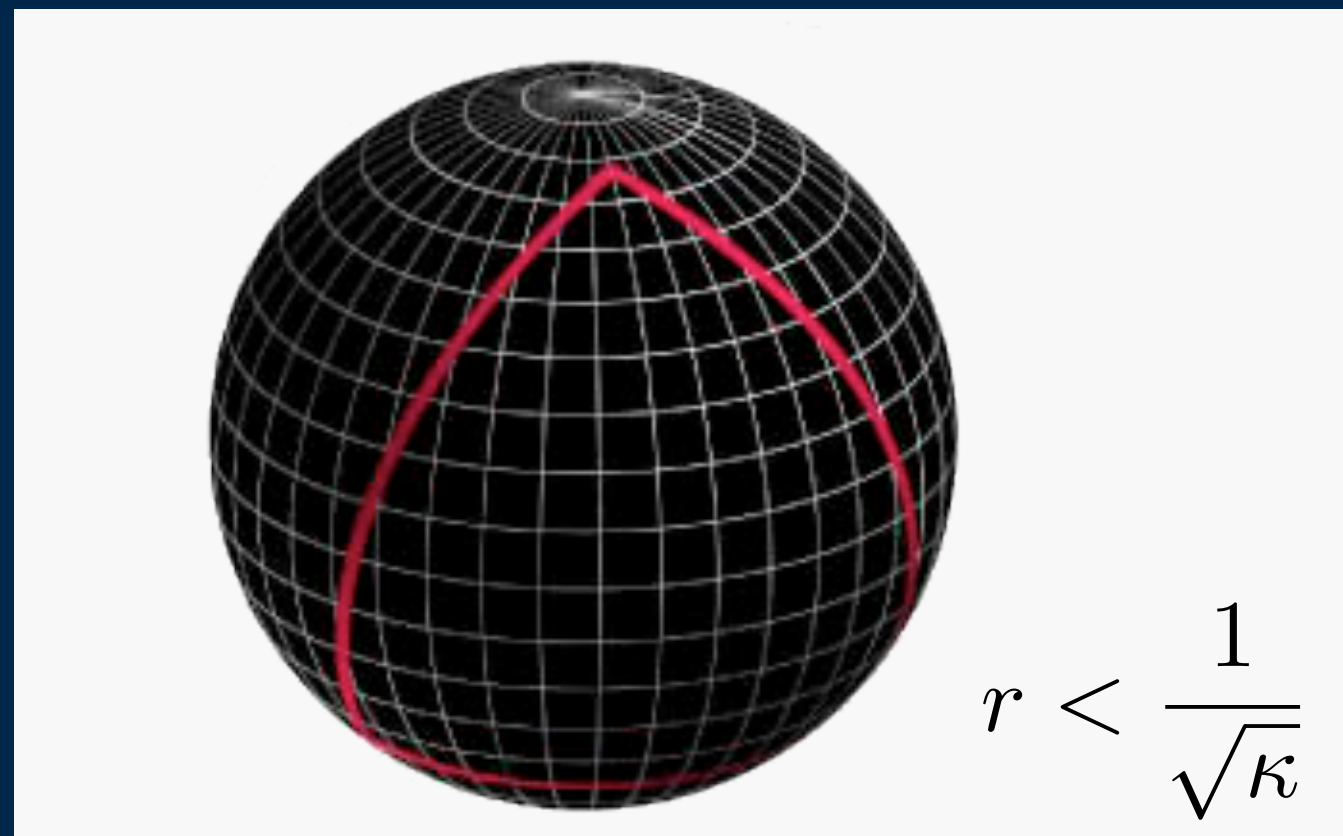
Scale factor – determined by matter content

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad \text{(Spherical coordinates)}$$

Spatial curvature

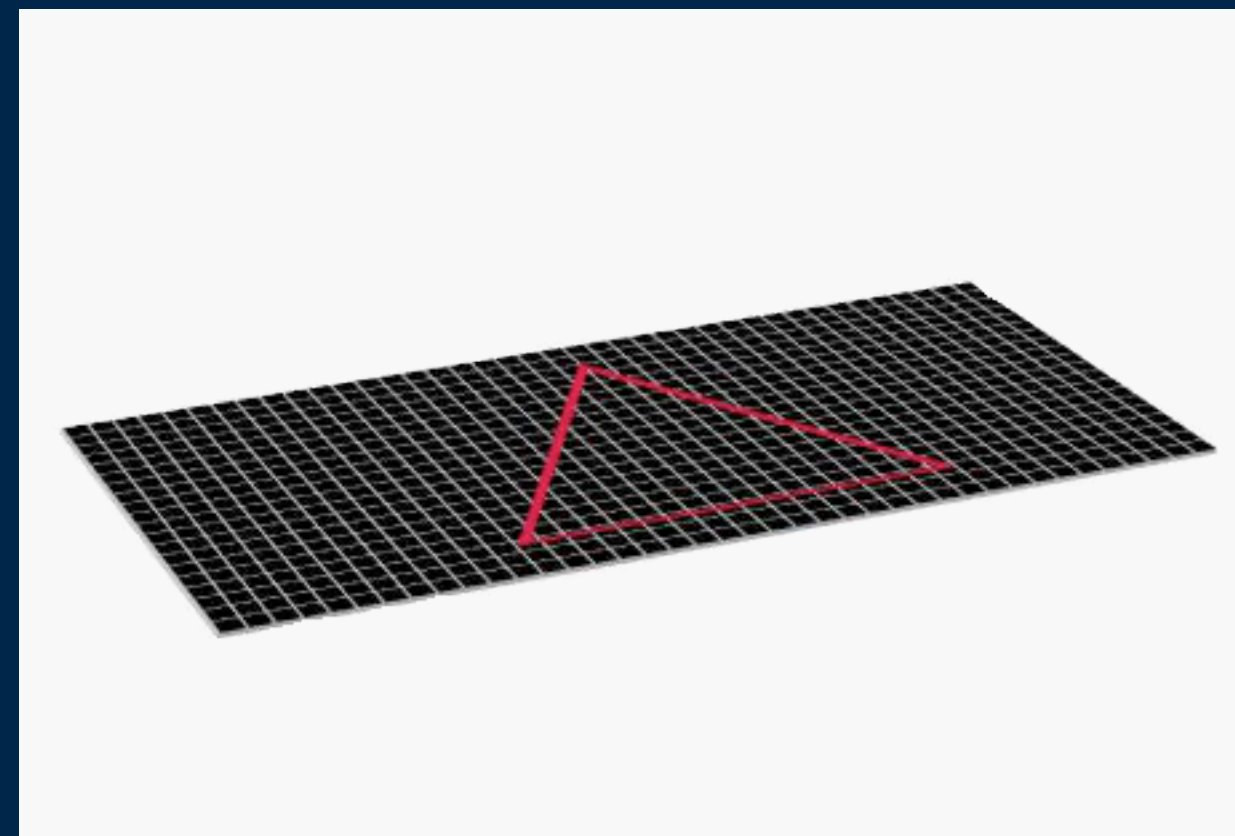
$$\kappa > 0$$

Positively curved (e.g. 3-sphere)



$$\kappa = 0$$

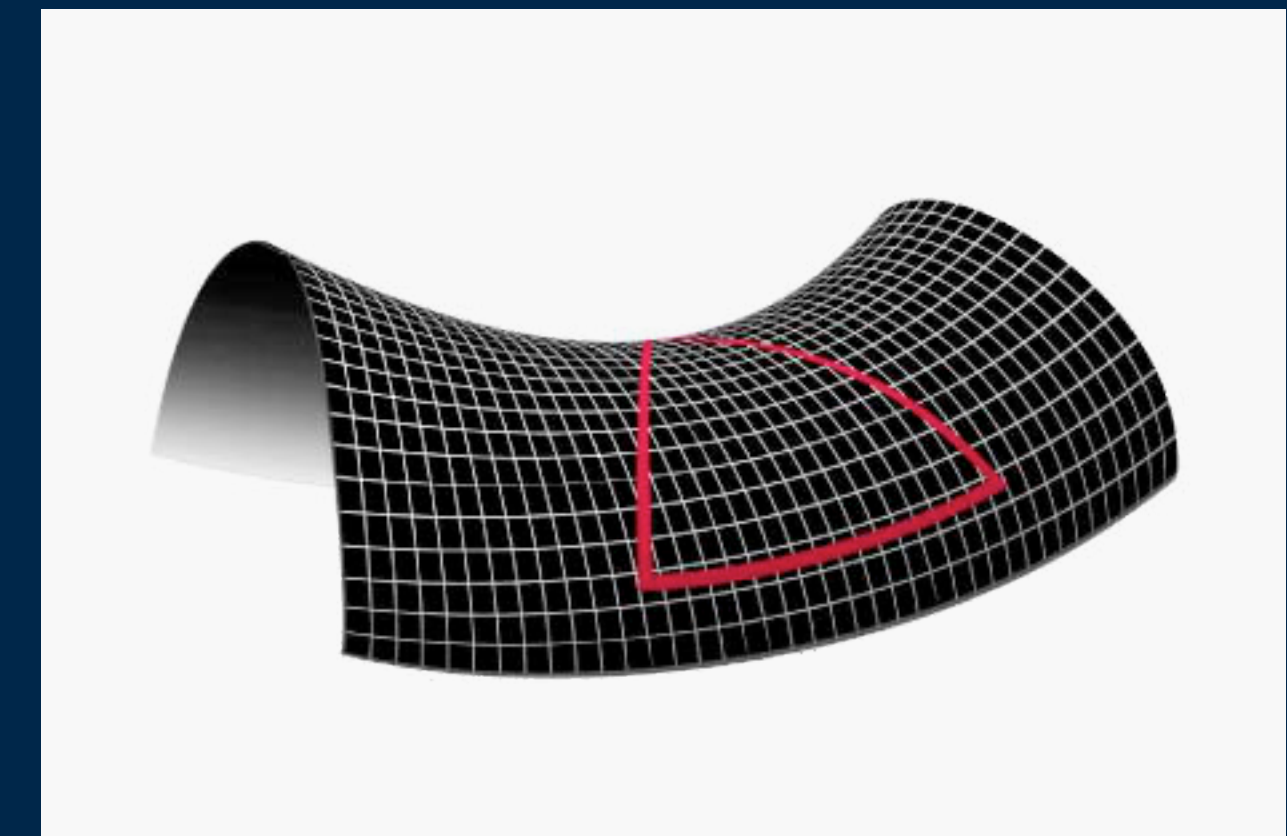
Spatially flat (Euclidean)



Preferred by the data

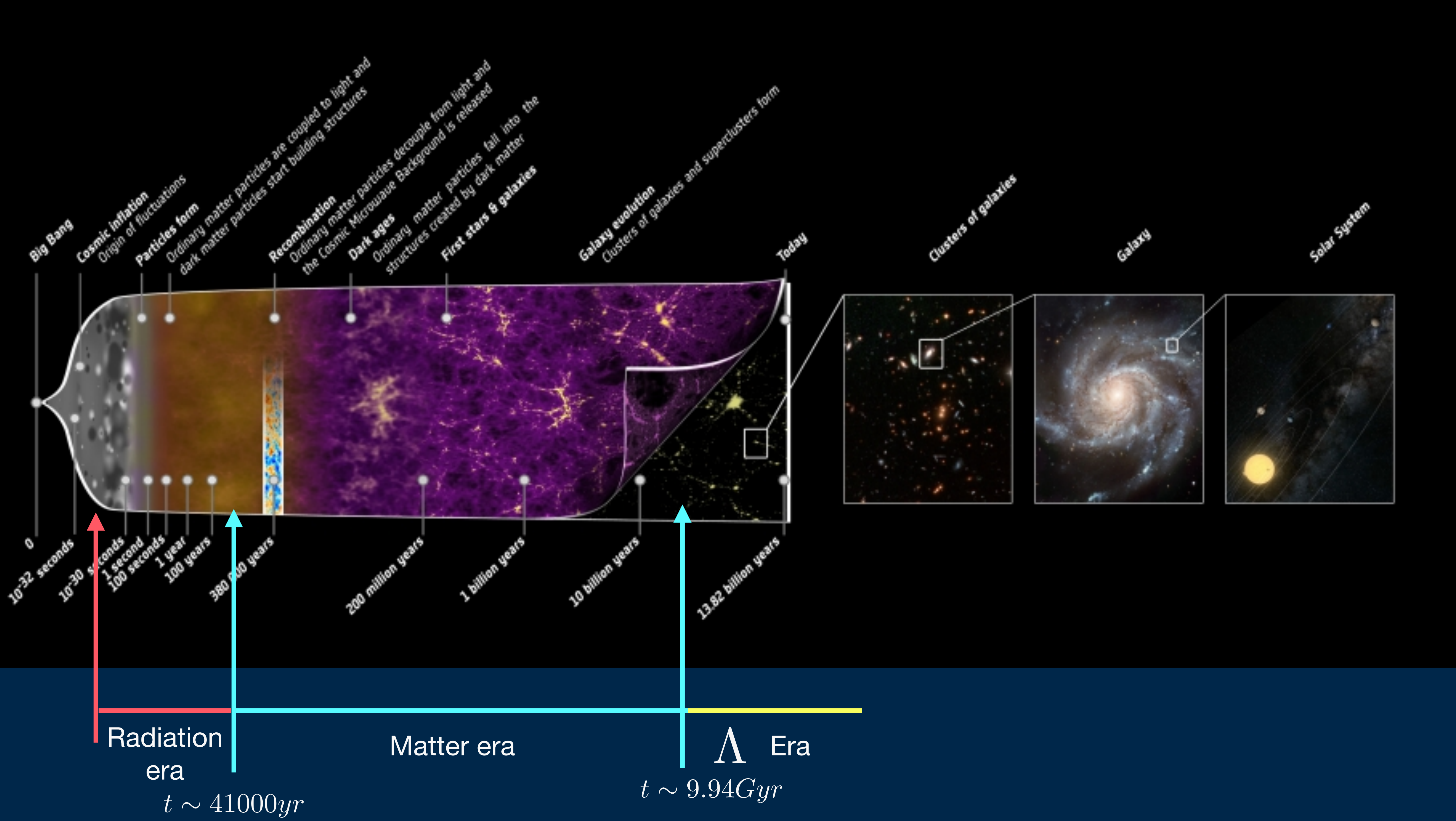
$$\kappa < 0$$

Negatively curved (e.g. hyperbolic 3-space)



Dynamics of homogeneous-isotropic Universe

How to determine $a(t)$



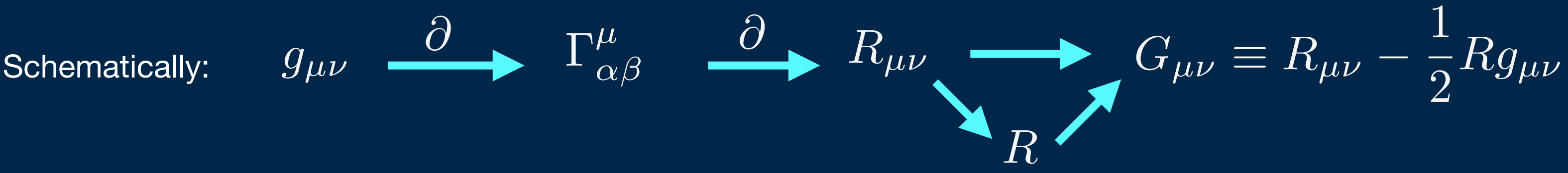
General Relativity — Gravity as geometry of spacetime

Einstein equations

$$\text{Einstein tensor} \rightarrow G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Matter stress-energy tensor

Cosmological constant



For any metric

$$\begin{matrix} \nabla_{\mu} G^{\mu}_{\nu} = 0 \\ \nabla_{\rho} g_{\mu\nu} = 0 \end{matrix} \longrightarrow \text{Compatible with} \quad \nabla_{\mu} T^{\mu}_{\nu} = 0 \longrightarrow \text{Stress-energy conservation}$$

Spatially flat FLRW: $\kappa = 0$

(Cartesian coordinates)

Metric:

$g_{\mu\nu}$

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}$$

Christoffel symbols

$$\Gamma_{\mu\nu}^{\alpha} \equiv \frac{1}{2} g^{\alpha\beta} (\partial_{\mu} g_{\beta\nu} + \partial_{\nu} g_{\beta\mu} - \partial_{\beta} g_{\mu\nu})$$

$$\Gamma_{0j}^i = H \delta^i_j$$

$$\Gamma_{ij}^0 = H g_{ij}$$

Hubble parameter

$$H(t) \equiv \frac{\dot{a}}{a}$$

FLRW

Non-zero components

Ricci curvature

$$R_{\mu\nu} = \partial_{\alpha} \Gamma_{\mu\nu}^{\alpha} - \partial_{\nu} \Gamma_{\alpha\mu}^{\alpha} + \Gamma_{\alpha\lambda}^{\alpha} \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\beta\nu}^{\alpha} \Gamma_{\mu\alpha}^{\beta}$$

$$R_{00} = -3\dot{H} - 3H^2$$

$$R_{ij} = [\dot{H} + 3H^2] g_{ij}$$

Scalar curvature

$$R \equiv g^{\mu\nu} R_{\mu\nu}$$

$$R = 6 [\dot{H} + 2H^2]$$

Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (\text{Includes cosmological constant})$$

$$T_{00} = \bar{\rho}(t)$$

$$T_{ij} = \bar{P}(t)g_{ij} = \bar{P}(t)a^2\gamma_{ij}$$

Friedman eq. $H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \sum_I \bar{\rho}_I - \frac{\kappa}{a^2}$

Energy density = sum over all matter species

Spatial curvature

Raychaudhuri eq. $-2\frac{\ddot{a}}{a} - H^2 = -2\dot{H} - 3H^2 = 8\pi G \sum_I \bar{P}_I - \frac{\kappa}{a^2}$

Pressure = sum over all matter species

Energy conservation

$$\nabla_{\mu} T^{\mu}_{\nu} = 0$$

$$\dot{\bar{\rho}}_I + 3H(\bar{\rho}_I + \bar{P}_I) = 0$$

I Labels species: e.g. baryons, dark matter, photons, neutrinos, cosmological constant, etc

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \sum_I \bar{\rho}_I - \frac{\kappa}{a^2}$$

Conventionally $a = 1$ Today
 $\bar{\rho}_I(a = 1) = \bar{\rho}_{0,I}$

Critical density: $\bar{\rho}_{\text{crit}} \equiv \frac{3H^2}{8\pi G}$

Relative density: $\Omega_I \equiv \frac{\bar{\rho}_I}{\bar{\rho}_{\text{crit}}} = \frac{8\pi G \bar{\rho}_I}{3H^2}$

Today $\Omega_0 \equiv \sum_I \Omega_{0,I} \longrightarrow \kappa = (\Omega_0 - 1) H_0^2 = \Omega_K H_0^2$

Negligible curvature $\Omega_K = -0.0106 \pm 0.0065$ CMB (Planck 2018)

Spatially flat

$$\sum_I \Omega_I = 1$$

Equation of state: a relation between density and pressure

$$\bar{P} = w \bar{\rho}$$

Equation of state parameter

$$\dot{\rho}_I + 3H (\bar{\rho}_I + \bar{P}_I) = 0$$

Solve

$$w = 1$$

Stiff fluid, massless scalar field

$$\rho_{\text{stiff}} \propto a^{-6}$$

$$w = \frac{1}{3}$$

Radiation (photons, massless neutrinos, gravitational waves, ...)

$$\rho_r \propto a^{-4}$$

$$w = 0$$

Pressureless matter, also called dust
(cool baryons, cold dark matter, galaxies, ...)

$$\rho_m \propto a^{-3}$$

$$w = -\frac{1}{3}$$

Spatial Curvature, or cosmic strings

$$\rho_{\text{strings}} \propto a^{-2}$$

$$w = -\frac{2}{3}$$

Domain walls

$$\rho_{\text{walls}} \propto a^{-1}$$

$$w = -1$$

Cosmological constant

$$\rho_{\Lambda} = \frac{\Lambda}{8\pi G} = \text{constant}$$

Conventionally

$$a = 1$$

Today

Simple solutions

Friedman eq. $3 \frac{\dot{a}^2}{a^2} = 8\pi G \bar{\rho} \longrightarrow a(t)$

Today: $3H_0^2 = 8\pi G \bar{\rho}_0$

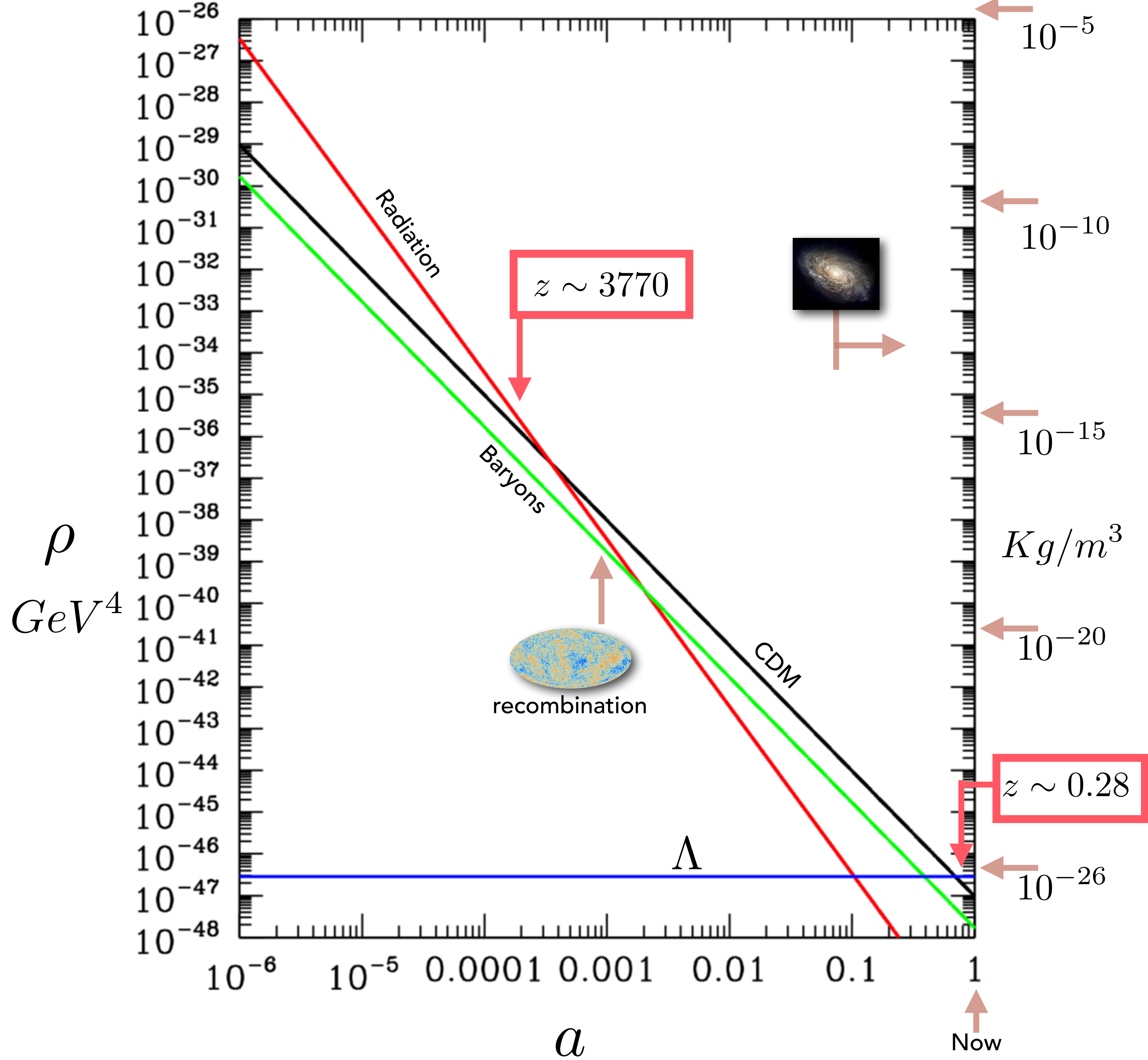
Stiff fluid $w = 1$ $\bar{\rho}_{\text{stiff}} = \bar{\rho}_0 a^{-6}$ $a = (3H_0 t)^{1/3}$

Radiation $w = \frac{1}{3}$ $\bar{\rho}_r = \bar{\rho}_0 a^{-4}$ $a = \sqrt{2H_0 t}$

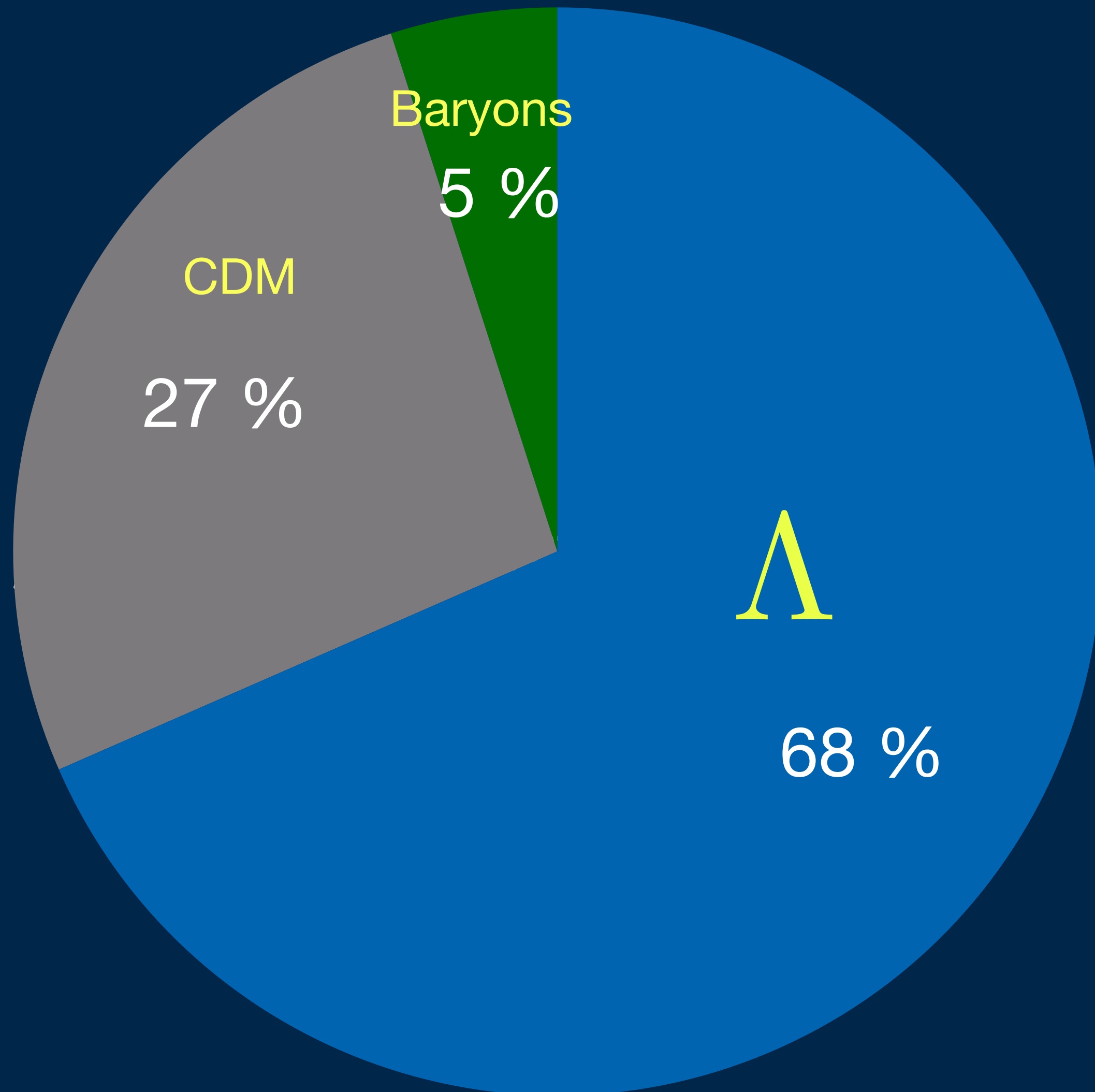
Pressureless matter $w = 0$ $\bar{\rho}_m = \bar{\rho}_0 a^{-3}$ $a = \left(\frac{3}{2}H_0 t\right)^{2/3}$

Cosmological constant $w = -1$ $\bar{\rho}_\Lambda = \bar{\rho}_0$ $a = e^{H_0 t}$

Expansion eras



Material content of the Universe



Hubble constant $\left\{ \begin{array}{l} H_0 = 67.36 \pm 0.54 \text{ CMB (Planck 2018)} \\ H_0 = 73.04 \pm 1.04 \text{ Local (SH0ES 2021)} \end{array} \right\}$ "Hubble tension"

Critical density $\bar{\rho}_{crit,0} = \frac{3H_0^2}{8\pi G} = 1.878 \times 10^{-26} h^2 \text{ kg m}^{-3}$

Negligible curvature $\Omega_K = -0.0106 \pm 0.0065$ CMB (Planck 2018)

Assuming flatness, CMB (Planck 2018)

Matter density $\Omega_{0,m} = 0.3153 \pm 0.0073$

Cosmological constant $\Omega_{0,\Lambda} = 0.6847 \pm 0.0073$

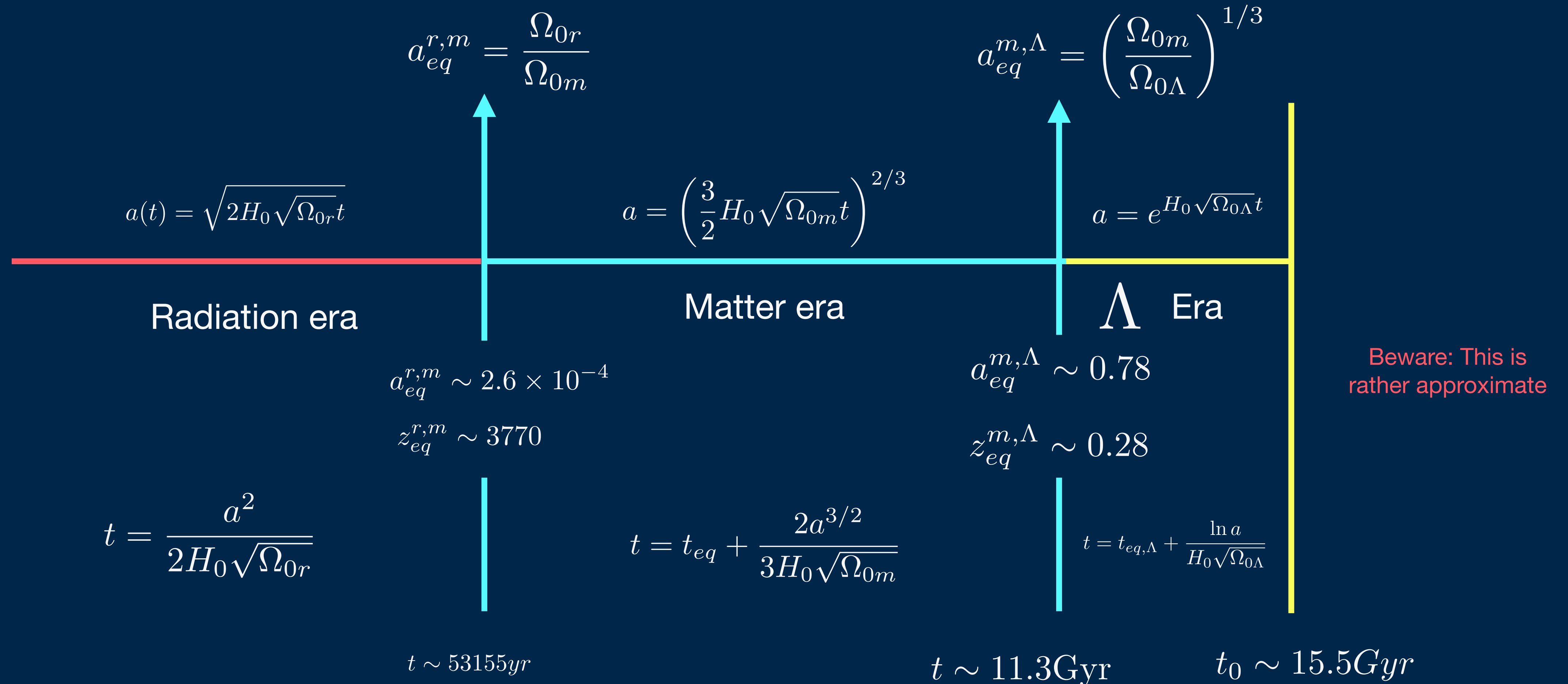
Cold dark matter $\Omega_{0,c} h^2 = 0.1202 \pm 0.0014$
 $\Omega_{0,c} \sim 0.2658$

Baryons $\Omega_{0,b} h^2 = 0.02236 \pm 0.00015$
 $\Omega_{0,b} \sim 0.0494$

Photons $\Omega_{0\gamma} \sim 5.04 \times 10^{-5}$

Neutrinos $3.43 \times 10^{-5} \lesssim \Omega_{0\nu} \lesssim 0.01$

Expansion age of the Universe



Accurate way:

$$t(a_{ref}) = \frac{1}{H_0} \int_0^{a_{ref}} \frac{da}{a \sqrt{\frac{\Omega_{0r}}{a^4} + \frac{\Omega_{0m}}{a^3} + \Omega_{0\Lambda}}}$$

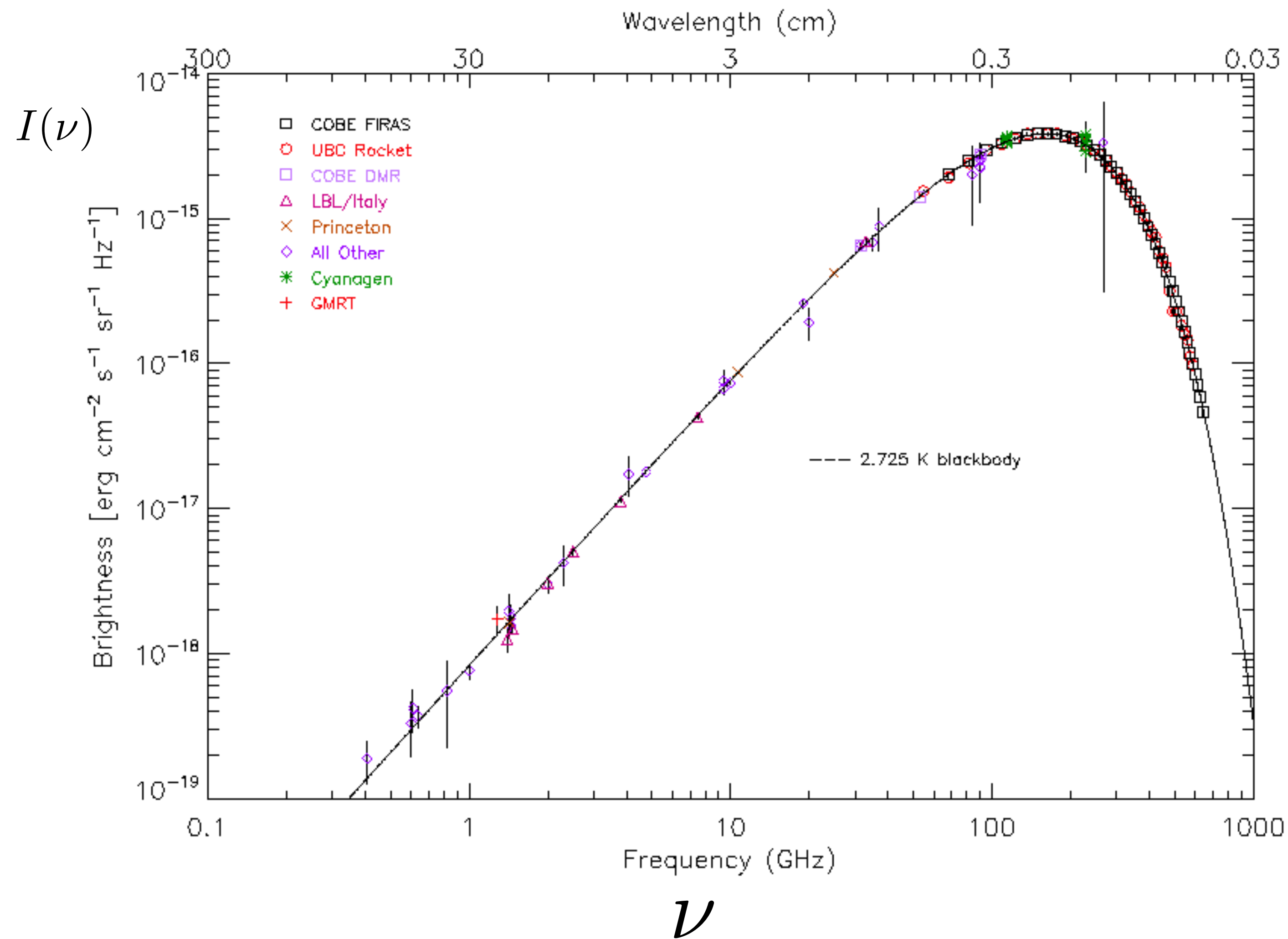
$t \sim 9.94Gyr$

$t_0 \sim 13.205Gyr$

Thermal history of the Universe

Cosmic Microwave Background

$$\bar{T}_{CMB} = 2.7255K$$



Best Black body spectrum in the Universe

$$I(\nu) = \frac{4\pi\nu^3}{e^{2\pi\nu/T} - 1}$$

CMB at thermal equilibrium today

Expansions preserves Black Body spectrum

CMB at thermal equilibrium in the past

+ anything interacting with it.

Today	$z = 0$	$a = 1$	$t_0 \sim 13.7 Gyr$	$T_{cmb} \sim 2.7255 K$	Λ domination	
Distant galaxies	$z = 5$	$a \sim 0.17$	$t \sim 1.15 Gyr$	$T_{cmb} \sim 16 K$	Matter domination	
Photon decoupling	$z \sim 1100$	$a \sim 9 \times 10^{-4}$	$t \sim 350000 yr$	$T_{cmb} \sim 3000 K$		
CMB anisotropies						
Matter-radiation equality	$z \sim 3700$	$a \sim 2.6 \times 10^{-4}$	$t \sim 41000 yr$	$T_{cmb} \sim 10000 K$ $\sim eV$	Radiation domination	
Nucleosynthesis ends	$z \sim 3 \times 10^8$	$a \sim 3 \times 10^{-9}$	$t \sim 200 - 300 s$	$T \sim 0.05 MeV$		
Nuclear reactions efficient: formation of light elements						
e^+e^- annihilation	$z \sim 5 \times 10^9$	$a \sim 2 \times 10^{-10}$	$t \sim 1 s$	$T \sim 0.5 MeV$		
Neutrino decoupling n/p freeze-out	$z \sim 10^{10}$	$a \sim 10^{-10}$	$t \sim 0.2 s$	$T \sim 1 - 2 MeV$		
Quark-gluon phase transition	$z \sim 5 \times 10^{12}$	$a \sim 2 \times 10^{-13}$	$t \sim 10^{-5}$	$T \sim 200 MeV$		
Electroweak unification		$a \sim 2 \times 10^{-16}$	$t \sim 10^{-12} s$	$T \sim 100 GeV$?	
GUT??		$a \sim 10^{-28}$	$t \sim 10^{-36} s$	$T \sim 10^{15} GeV$	Inflation: Scalar field domination $V(\phi) \sim \Lambda_{inf}$	
Quantum gravity??			$t \sim 10^{-43} s$ Graviton decoupling			

Particles at equilibrium

Distribution function: $f(t, \vec{x}, \vec{p}) \rightarrow f(t, p)$

Describes collection of particles of given energy at time t

Physical momentum

$$E(p) = \sqrt{p^2 + m^2}$$

Degrees of freedom

$n = n(T)$ Number density

$$n = \frac{g}{2\pi^2} \int dp \frac{p^2}{e^{E(p)/T} \pm 1}$$

+ Fermions
- Bosons

$\rho = \rho(T)$ Energy density

$$\rho = \frac{g}{2\pi^2} \int \frac{dp p^2}{e^{E(p)/T} \pm 1} E(p)$$

$P = P(T)$ Pressure

$$P = \frac{g}{2\pi^2} \int \frac{dp p^2}{e^{E(p)/T} \pm 1} \frac{p^2}{3E(p)}$$

Remember

$$T = \frac{T_0}{a}$$

1st law of thermodynamics

$$T dS = dE + P dV$$

$S = S(T, V)$ Entropy \uparrow
 Energy $E = \rho V$ = energy density x volume \uparrow
 $\rho = \rho(T)$
 Pressure $P = P(T)$ \uparrow
 Volume $V = a^3$ \uparrow

$$\frac{\partial S}{\partial T} = \frac{V}{T} \frac{\partial \rho}{\partial T}$$

$$\frac{\partial S}{\partial V} = \frac{\rho + P}{T}$$

Consistency: $\frac{\partial}{\partial V} \frac{\partial S}{\partial T} = \frac{\partial}{\partial T} \frac{\partial S}{\partial V} \Rightarrow dP = \frac{\rho + P}{T} dT$

$$dS = d \left[\frac{(\rho + P)V}{T} \right] = 0$$

Entropy is conserved for equilibrium processes

\Rightarrow Entropy density $s = \frac{\rho + P}{T}$

Relativistic species

$$m \ll T$$

$$E(p) \approx p$$

Number density $\longrightarrow n = \frac{gT^3}{2\pi^2} \int_0^\infty \frac{x^2}{e^x \pm 1}$

Energy density $\longrightarrow \rho = \frac{gT^4}{2\pi^2} \int_0^\infty \frac{x^3 dx}{e^x \pm 1}$

Pressure $\longrightarrow P = \frac{1}{3}\rho$

Entropy density $\longrightarrow s = \frac{4}{3} \frac{\rho}{T} \propto n$

$n_B = \frac{g\zeta(3)}{\pi^2} T^3$ Bosons

$n_F = \frac{3}{4} n_B$ Fermions

$\rho_B = \frac{\pi^2 g}{30} T^4$ Bosons

$\rho_F = \frac{7}{8} \rho_B$ Fermions

Relativistic degrees of freedom

- Several relativistic species add together to the radiation energy density — Relevant only during the radiation era
 — Neutrinos are then approximately massless

Effective massless degrees of freedom

$$\rho_r = \frac{\pi^2 g_*}{30} T_\gamma^4$$

$$g_* = \sum_{\text{bosons}, I} g_I \left(\frac{T_I}{T_\gamma} \right)^4 + \frac{7}{8} \sum_{\text{fermions}, I} g_I \left(\frac{T_I}{T_\gamma} \right)^4$$

$$T \gtrsim 300 \text{ GeV}$$

All standard model particles are relativistic:

$$g_* \approx 106.75$$

$$1 \text{ MeV} \leq T \leq 100 \text{ MeV}$$

Relativistic species are: $\gamma, 3 \times \nu, e^\pm$
 ($m_{e^\pm} \sim 0.5 \text{ MeV}$)

$$g_* = 2 + \frac{7}{8} \times 3 \times 2 + \frac{7}{8} \times 2 \times 2 = 10.75$$

$$T \ll 1 \text{ MeV}$$

Relativistic species are: $\gamma, 3 \times \nu$

$$g_* = 2 + \frac{7}{8} \times 3 \times 2 \times \left(\frac{4}{11} \right)^{4/3} \approx 3.36$$

(Neutrinos become non-relativistic inside the matter era)

$N_{eff, \nu} \approx 3.0395$ Due to non-instantaneous neutrino decoupling + QED effects at finite T, see Magnano et al., Phys.Lett. B534, 8 (2002)
 $g_* \sim 3.38$

Neutrino decoupling

Neutrinos at equilibrium with other species

$$\bar{\nu}\nu \leftrightarrow e^+e^-$$

$$\nu e \leftrightarrow \nu e$$

Interaction rate $\Gamma = n\langle\sigma v\rangle \sim T^3(G_F T^2) \sim G_F^2 T^5$

Neutrino decoupling when $\Gamma < H$

(This means that the expansion is faster than the interaction rate and prevents reactions above from occurring efficiently)

$$\frac{\Gamma}{H} \sim \frac{G_F^2 T^5}{\sqrt{8\pi G g_* T^4/3}} \sim \frac{G_F^2}{\sqrt{8\pi G g_* /3}} T^3 \sim \left(\frac{T}{1\text{MeV}}\right)^3$$

Neutrino decoupling when $T < 1\text{MeV}$

Neutrinos maintain the same temperature as the other species but otherwise mind their own business

e^+e^- annihilation — Neutrino temperature

Below $T \sim 0.5MeV$



Heat the photons

Use entropy conservation

$$S_{before} = S_{after}$$

$$\rightarrow s_b a_b^3 = s_a a_a^3$$

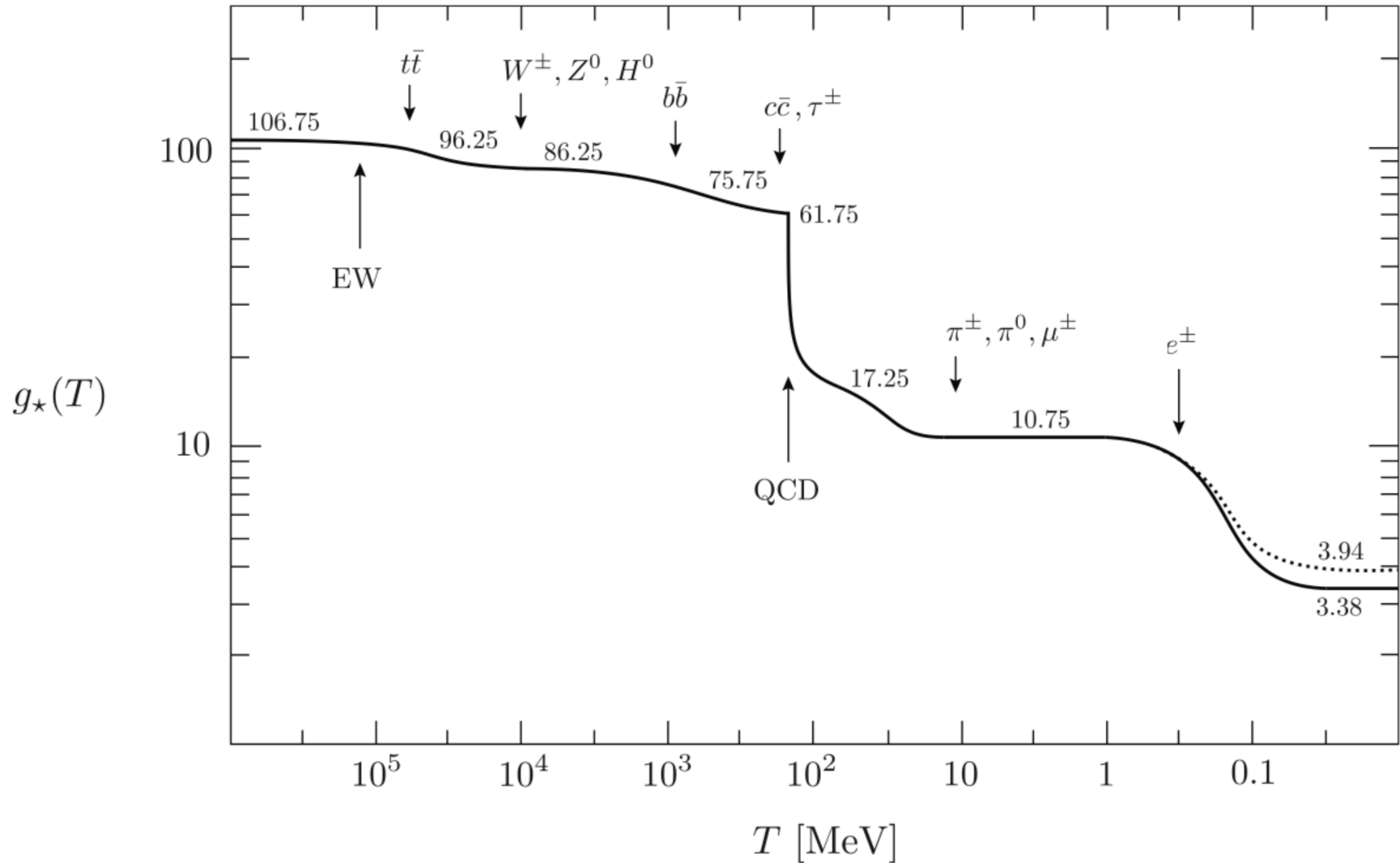
	Before							After		
$s \rightarrow$	γ	ν	$\bar{\nu}$	e^+	e^-		$s \rightarrow$	γ	ν	$\bar{\nu}$
	2	3	3	2	2		2	3	3	

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

$$T_{\nu, \text{today}} \sim 1.94537K$$

Relativistic degrees of freedom

From Julien Baur, PhD Thesis, see <https://inspirehep.net/literature/1765226>



Non-relativistic species

$$m \gg T$$

$$E(p) \approx m$$

Number density \longrightarrow $n = g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T}$

Energy density \longrightarrow $\rho = nm$

Pressure \longrightarrow $P = nT \ll \rho$ ($P \approx 0$)

e.g. massive neutrinos

$$\Omega_\nu = \frac{m_\nu}{94h^2 eV}$$

Gershtein, Zel'dovich (1966)

Marx & Szalay (1972)

Cowsik & McClelland (1972)

$$\text{For } \Omega_\nu < 1 \quad m_\nu \lesssim 45eV$$

Summary

Friedman equation $H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \sum_I \bar{\rho}_I - \frac{\kappa}{a^2}$

Energy conservation $\dot{\bar{\rho}}_I + 3H (\bar{\rho}_I + \bar{P}_I) = 0$

Relative density: $\Omega_I \equiv \frac{\bar{\rho}_I}{\bar{\rho}_{\text{crit}}} = \frac{8\pi G \bar{\rho}_I}{3H^2}$

Today: $\Omega_\Lambda \sim 0.68 \quad \Omega_c \sim 0.27 \quad \Omega_b \sim 0.05$

Expansion age $t(a_{\text{ref}}) = \frac{1}{H_0} \int_0^{a_{\text{ref}}} \frac{da}{a \sqrt{\frac{\Omega_{0r}}{a^4} + \frac{\Omega_{0m}}{a^3} + \Omega_{0\Lambda}}}$

Thermodynamics: entropy conserved

$m \ll T$ Relativistic species: $n \sim T^3 \quad \rho \sim T^4 \quad P = \frac{1}{3}\rho \quad s \sim T^3$

Fermions $\times \frac{3}{4}$ $\times \frac{7}{8}$ relative to bosons