RESEARCH CENTRE FOR QUANTUM COMPUTING AND SMART SYSTEMS

Quantum Fuzzy Logic

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"As complexity rises, precise statements lose meaning and meaningful statements lose precision" — **Lotfi A. Zadeh**

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Content

FUZZY LOGIC

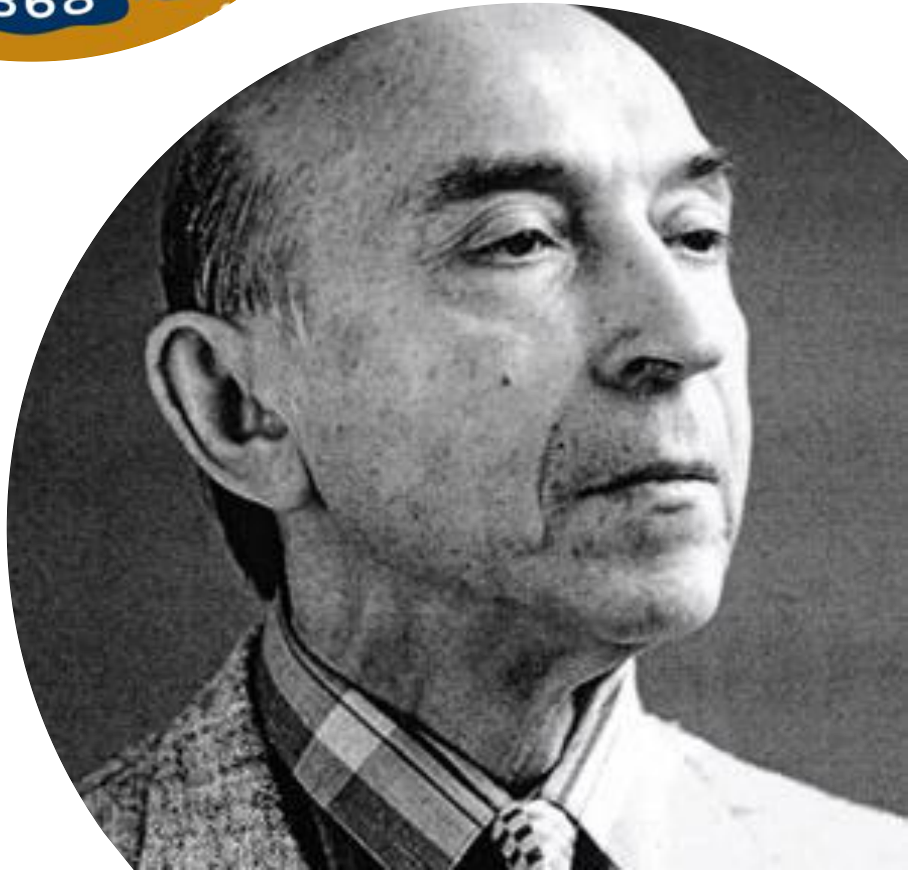
FUZZY SYSTEMS

QUANTUM FUZZY INFERENCE ENGINE

An Introduction on Fuzzy Logic

- Fuzzy Logic was introduced by **Lotfi Zadeh** at UC Berkeley in **1965** due to uncapability of conventional computer logic to represent vague human concepts such as ‘high temperature’

- The human brain can reason with uncertainties, vagueness, and judgments. Computers can only manipulate precise valuations. **Fuzzy logic is an attempt to give computers the capability of dealing with uncertainty.**

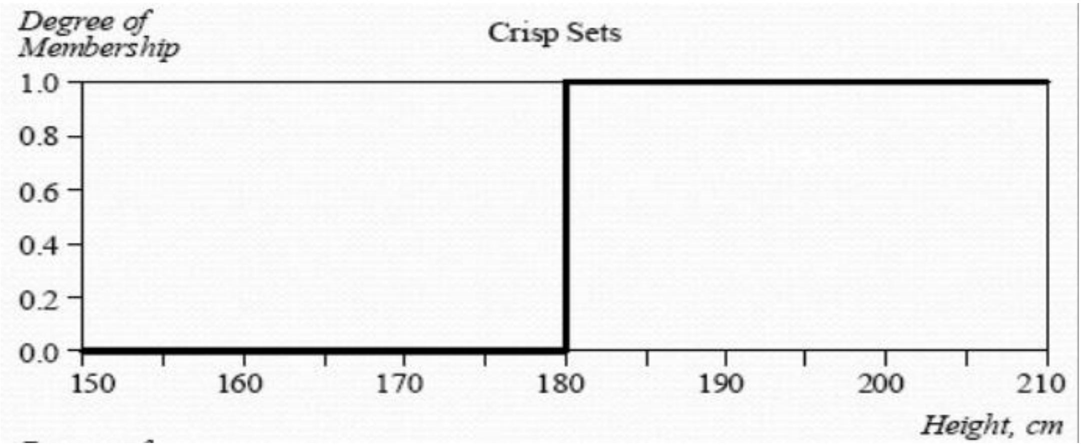


Boolean Logic and Crisp Sets

- In classical logic (Boolean logic) **the values of the variables are the truth values true and false**, usually denoted 1 and 0, respectively.
- The crisp set is defined in such a way as to dichotomize the elements in some given universe of discourse into two groups: **members** and **nonmembers**.
- However, many classification concepts do not exhibit this characteristic: for example, the set of tall people, expensive cars, or sunny days.

Crisp Sets – An Example

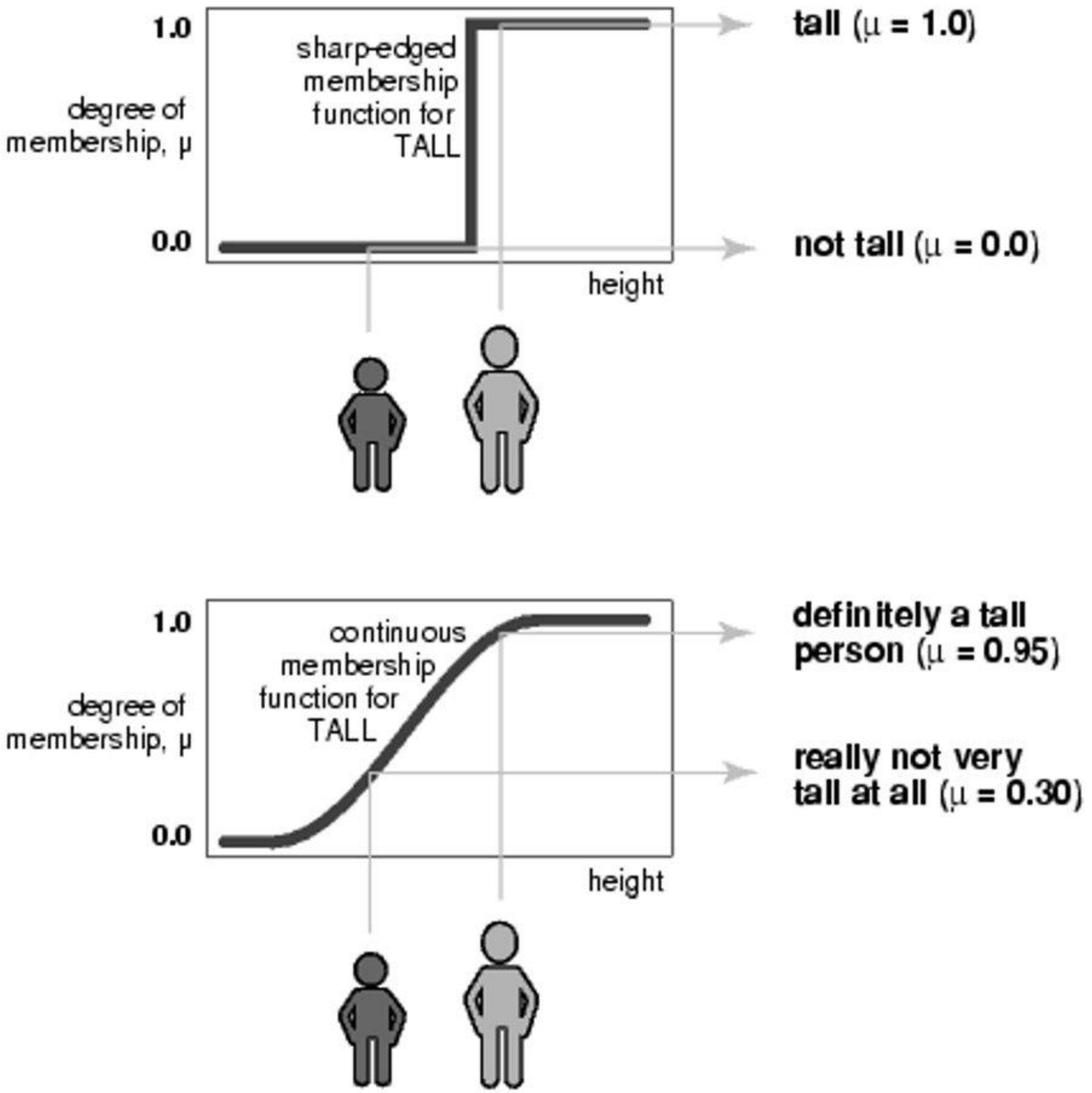
$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$



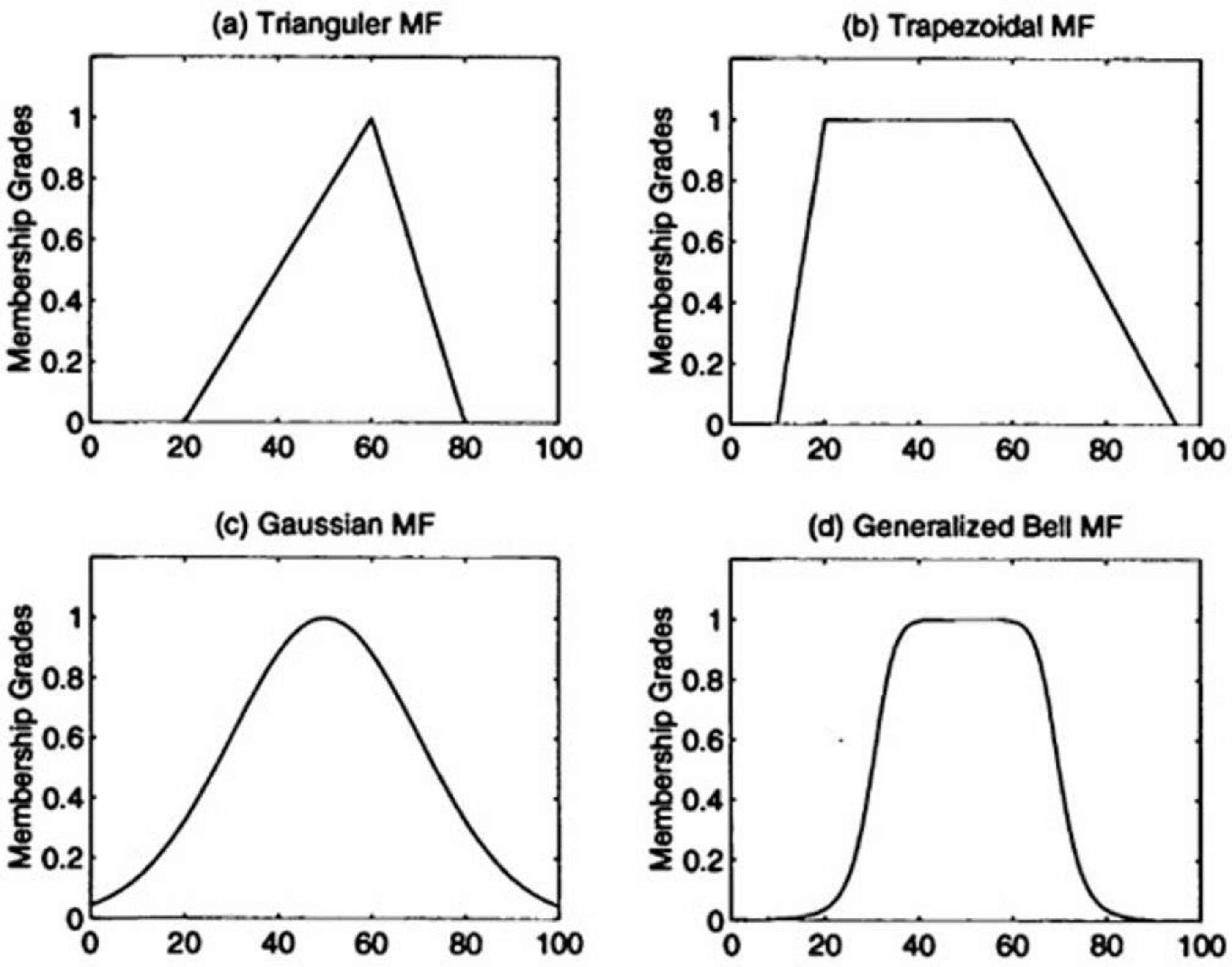
Fuzzy Logic and Fuzzy Sets

- In fuzzy logic, **the truth values of variables are any real number between 0 and 1** both inclusive. It is employed to handle the concept of partial truth, where the truth value may range between completely true and completely false.
- A fuzzy set can be defined mathematically by assigning to each possible element in the universe of discourse a value representing its grade of membership in the fuzzy set.
- For example: a fuzzy set representing our concept of tall might assign a degree of membership of 1 to a person with height 180 cm, 0.6 to a person with height 160cm, 0.1 to a person with height 140cm

Crisp Set vs Fuzzy Set



Membership Functions

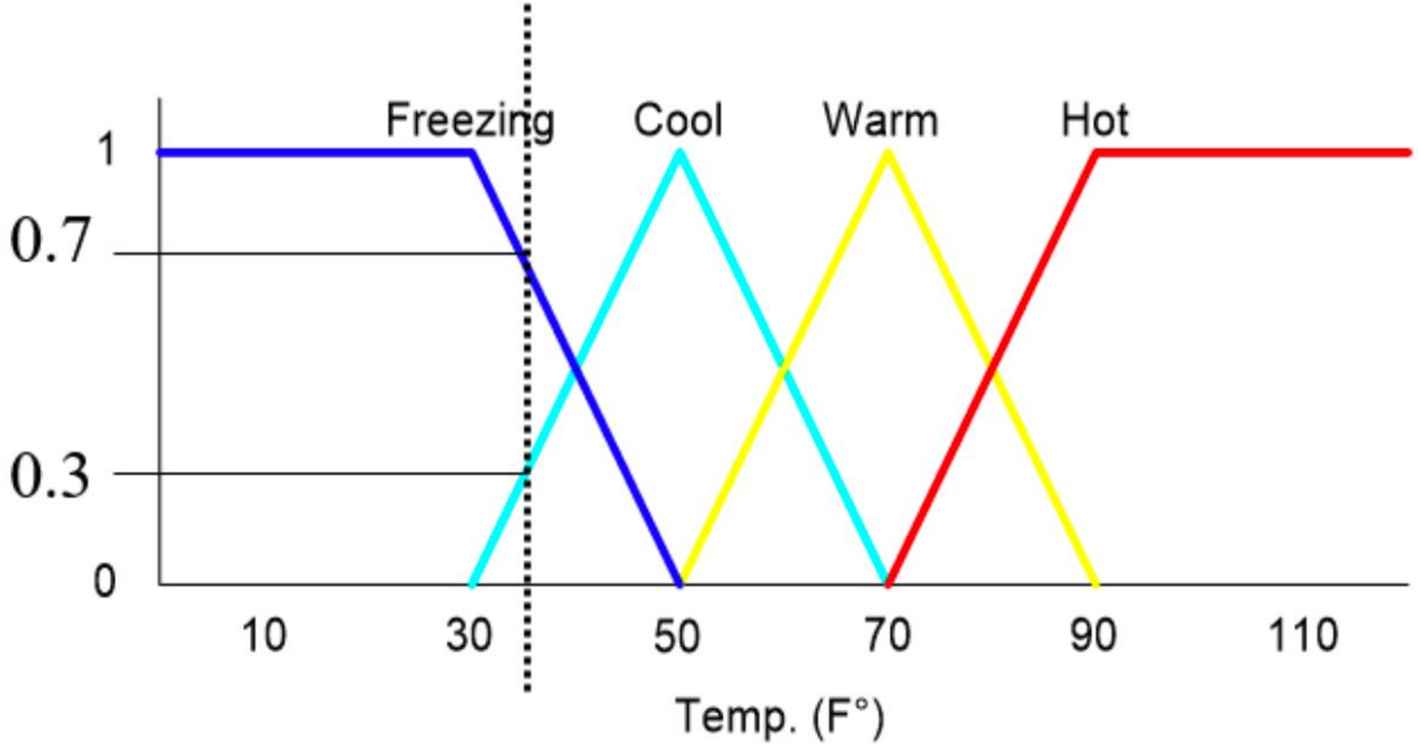


Fuzzy Variables

- Humans say things like "If it is sunny and warm today, I will drive fast"

- Linguistic variables:
  - Temp: {freezing, cool, warm, hot}
  - Cloud Cover: {overcast, partly cloudy, sunny}
  - Speed: {slow, fast}

- Fuzzy linguistic variables (or simply fuzzy variables) are defined in a **universe of discourse** and represented by a collection of fuzzy sets.



$$\mu_F(x = 36^\circ) = 0.7$$

$$\mu_C(x = 36^\circ) = 0.3$$

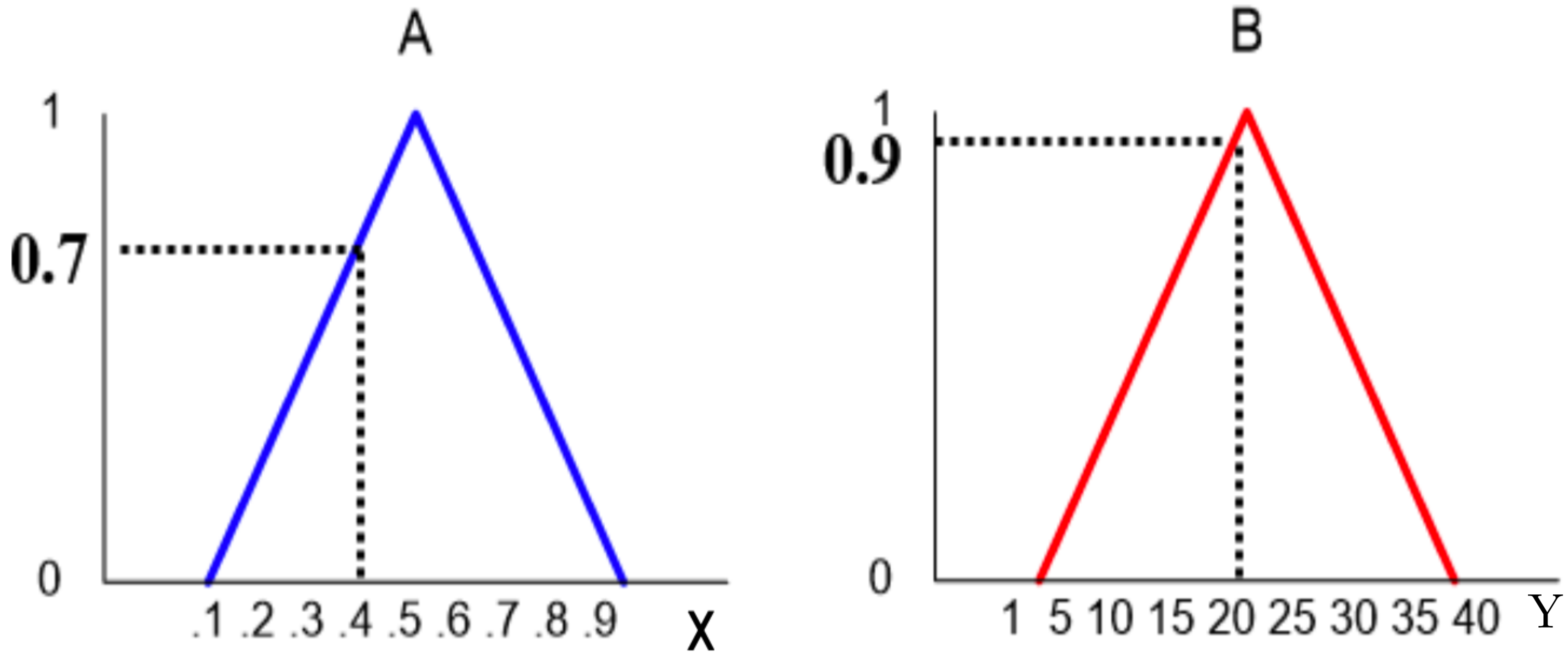
Fuzzy Logic

Classical Conjunction and Disjunction

x	y	$x \wedge y$	$x \vee y$
0	0	0	0
1	0	0	1
0	1	0	1
1	1	1	1

x	$\neg x$
0	1
1	0

Fuzzy Conjunction and Disjunction



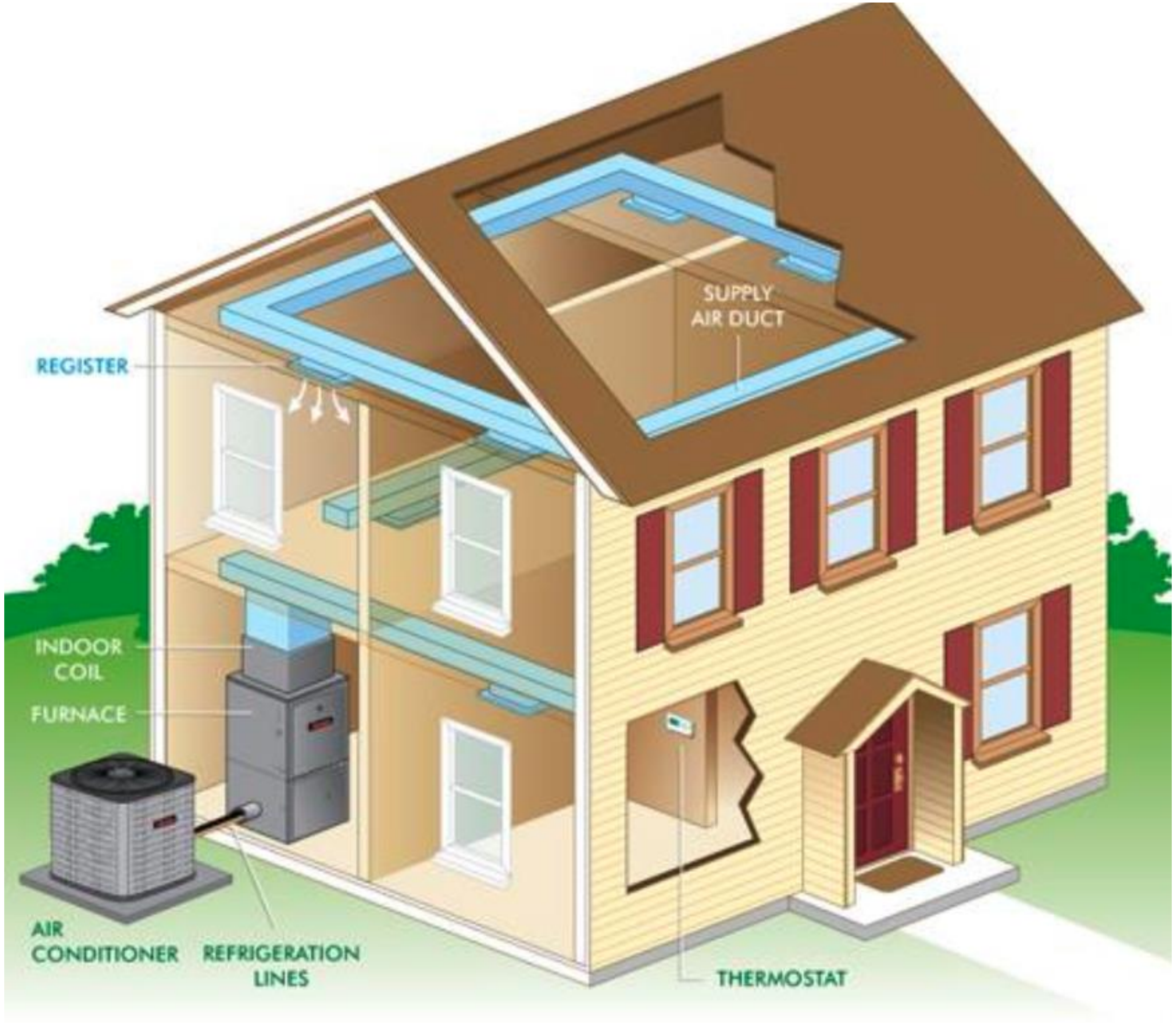
$$\text{X is A and Y is B} \rightarrow \mu_A(X) \cdot \mu_B(Y) = 0.63$$

$$\text{X is A or Y is B} \rightarrow \mu_A(X) + \mu_B(Y) - \mu_A(X) \cdot \mu_B(Y) = 0.97$$

Control Systems

Control Systems in General

- The aim of any control system is to produce a set of desired outputs for a given set of inputs.
- Example: **House temperature control system**
  - A household thermostat takes a temperature input and sends a control signal to a furnace



HOW TO IMPLEMENT A CONTROL SYSTEM?

- Simple solution: **Look-up table**
  - A lookup table is data structure that allows mapping any combinations of input values with an output.
  - It replaces runtime computation with a search in memory

Temperature	Furnace speed
20°C	300 RPM
25°C	250 RPM
10°C	350 RPM
…..	…..

Drawbacks:

- table can get **very long**, especially in situations where there are many inputs
- Controller can give an **uneven response** because it jumps from one table-based value to the next.

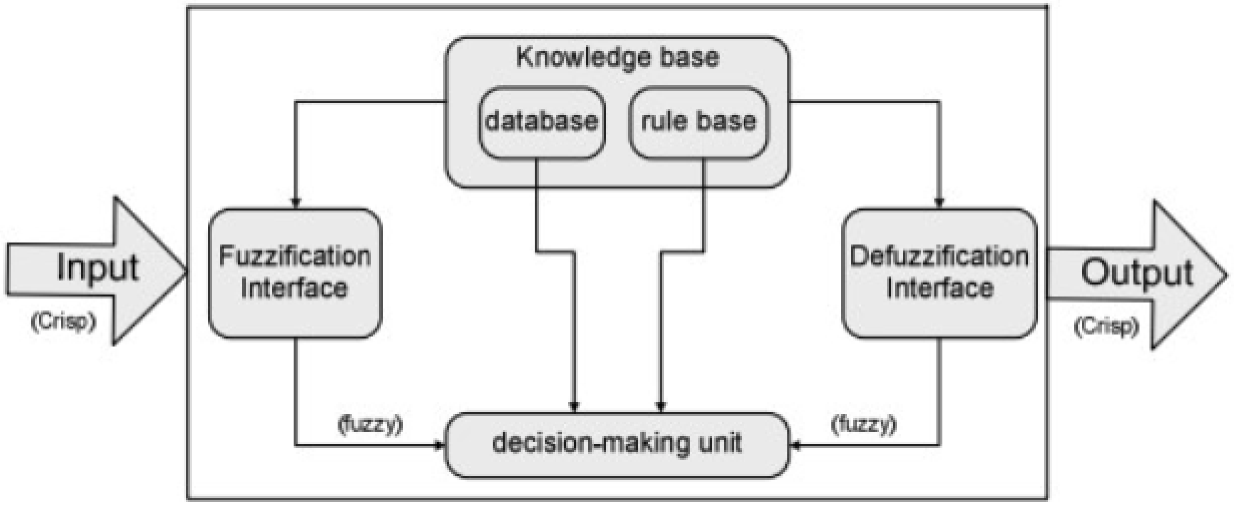
HOW TO IMPLEMENT A CONTROL SYSTEM?

- Alternative solution: **Mathematical formula**
  - The controller execute a mathematical formula – a set of control equations that express the output as a function of the input.
  - Ideally, these equations represent an accurate model of the system behaviour.

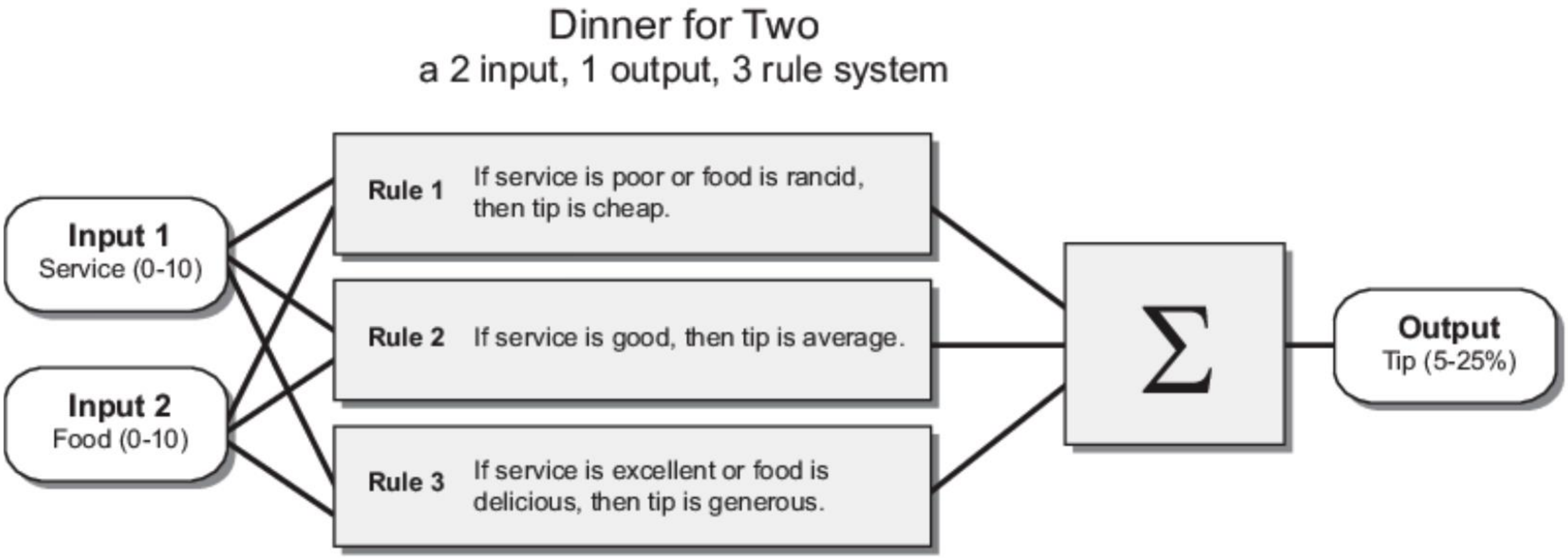
- Drawbacks:
  - The formulas can be **very complex**, and working them out in real-time may be more than an affordable controller (or machine) can manage.
  - It may be difficult or **impossible to derive** a workable mathematical model

Fuzzy Control Systems

- Control system can be modeled as a **Fuzzy Logic Controller** (FLC)
  - FLCs do not have unwieldy memory requirements of look-up tables and have a smoother behaviour.
  - FLCs are not characterized by an expensive computational cost as the formula-based solutions and are simpler to design.



An Example of Fuzzy Controller: Tipper



The input are crisp (non-fuzzy) numbers limited to a specific range

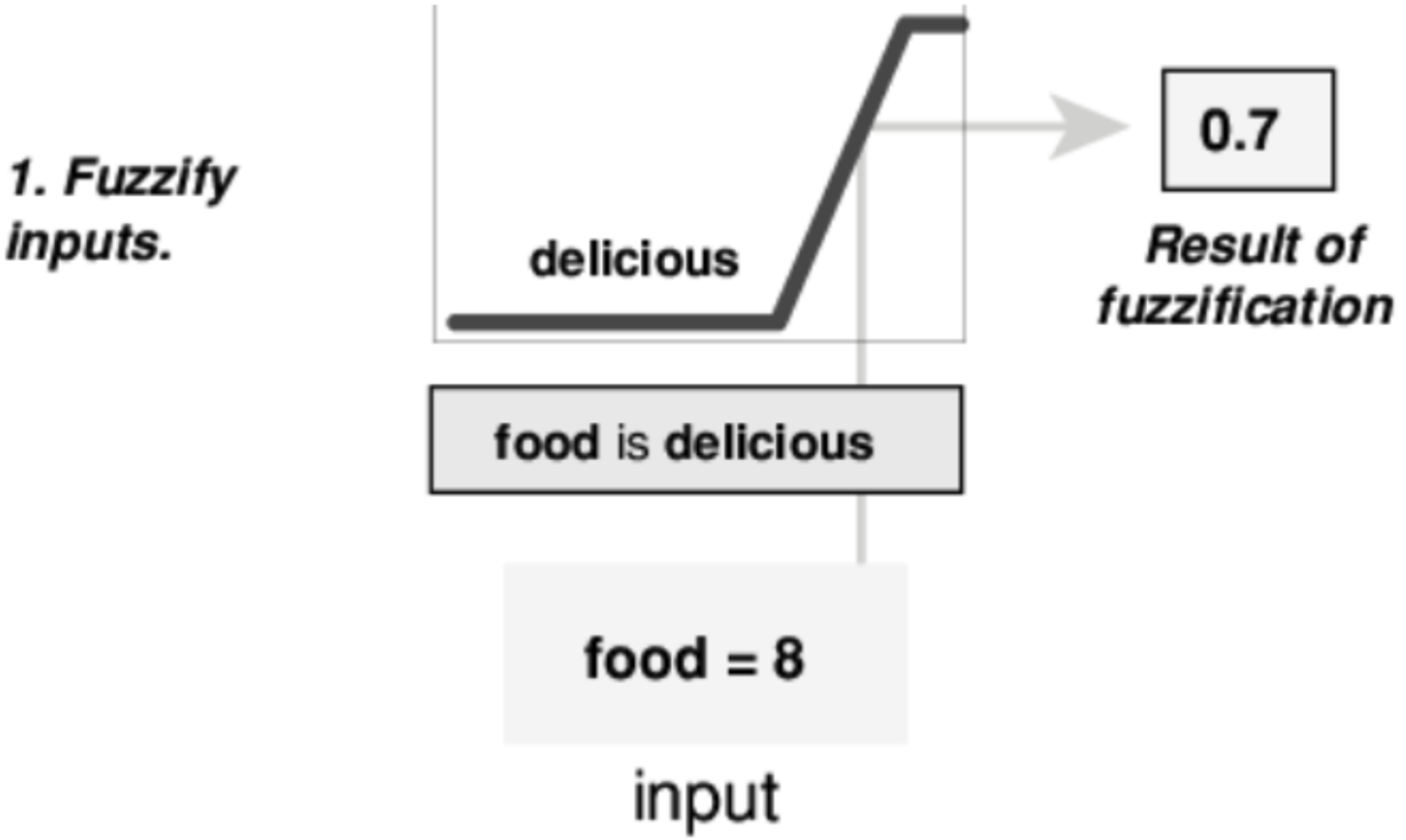
All rules are evaluated using fuzzy reasoning (fuzzy inference)

The results of the rules are combined (defuzzified)

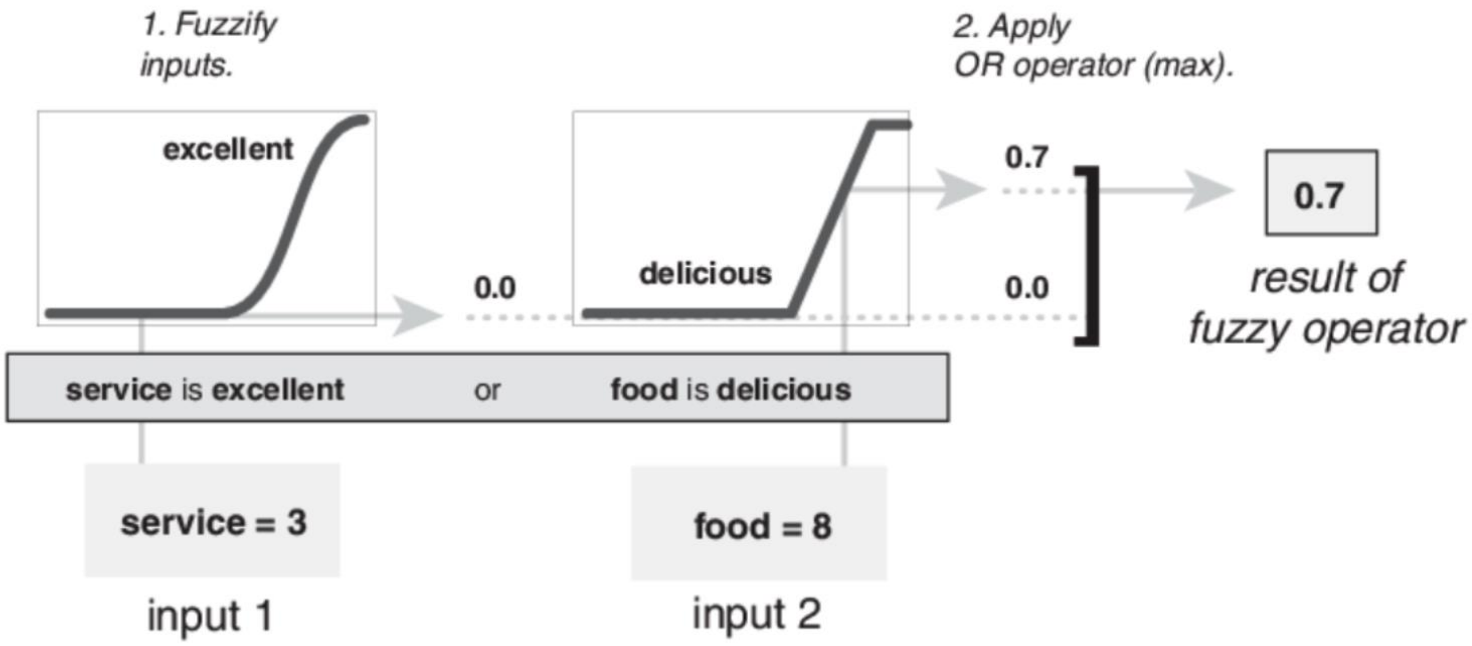
The output is a crisp (non-fuzzy) number.

Tipper Fuzzy Controller: Fuzzification

- The first step is to take the inputs and determine the degree to which they belong to each of the appropriate fuzzy sets via membership functions (*fuzzification*).

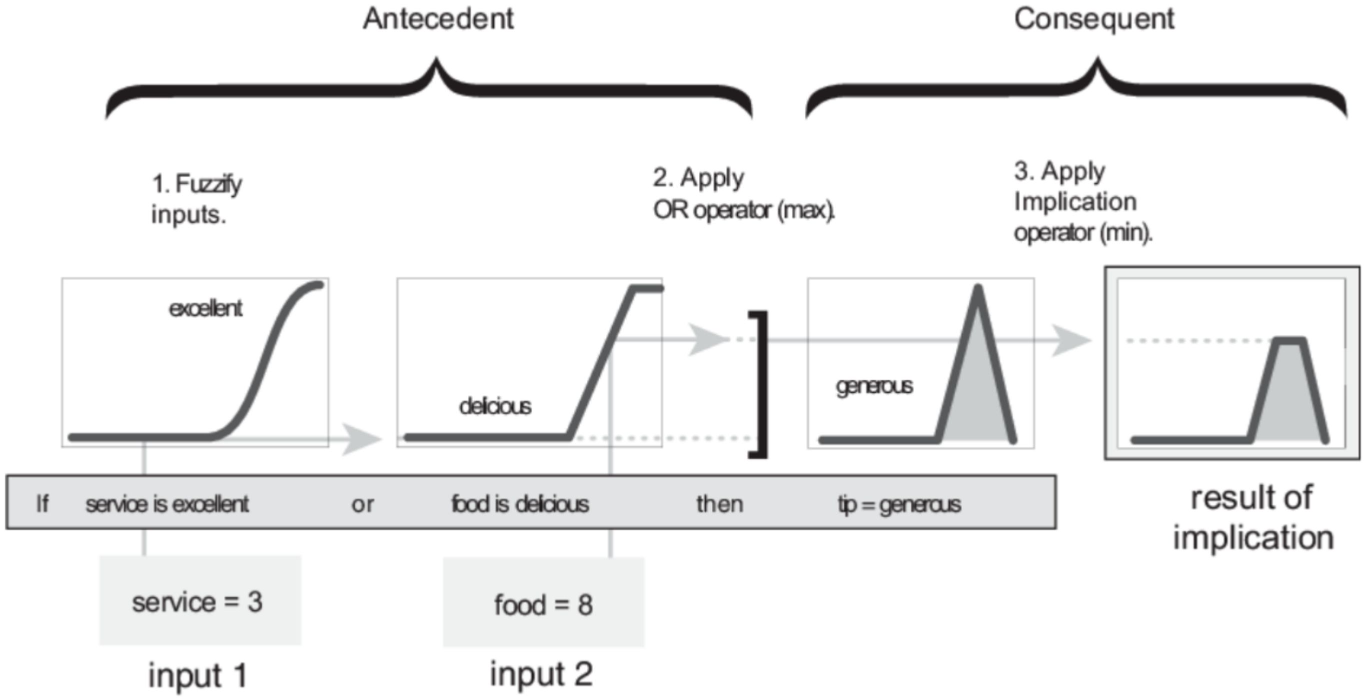


Tipper Fuzzy Controller: Rule Evaluation



- If the antecedent of a rule has more than one part, the fuzzy operator is applied to obtain one number that represents the result of the rule antecedent. The input to the fuzzy operator is two or more membership values from fuzzified input variables. The output is a single truth value.

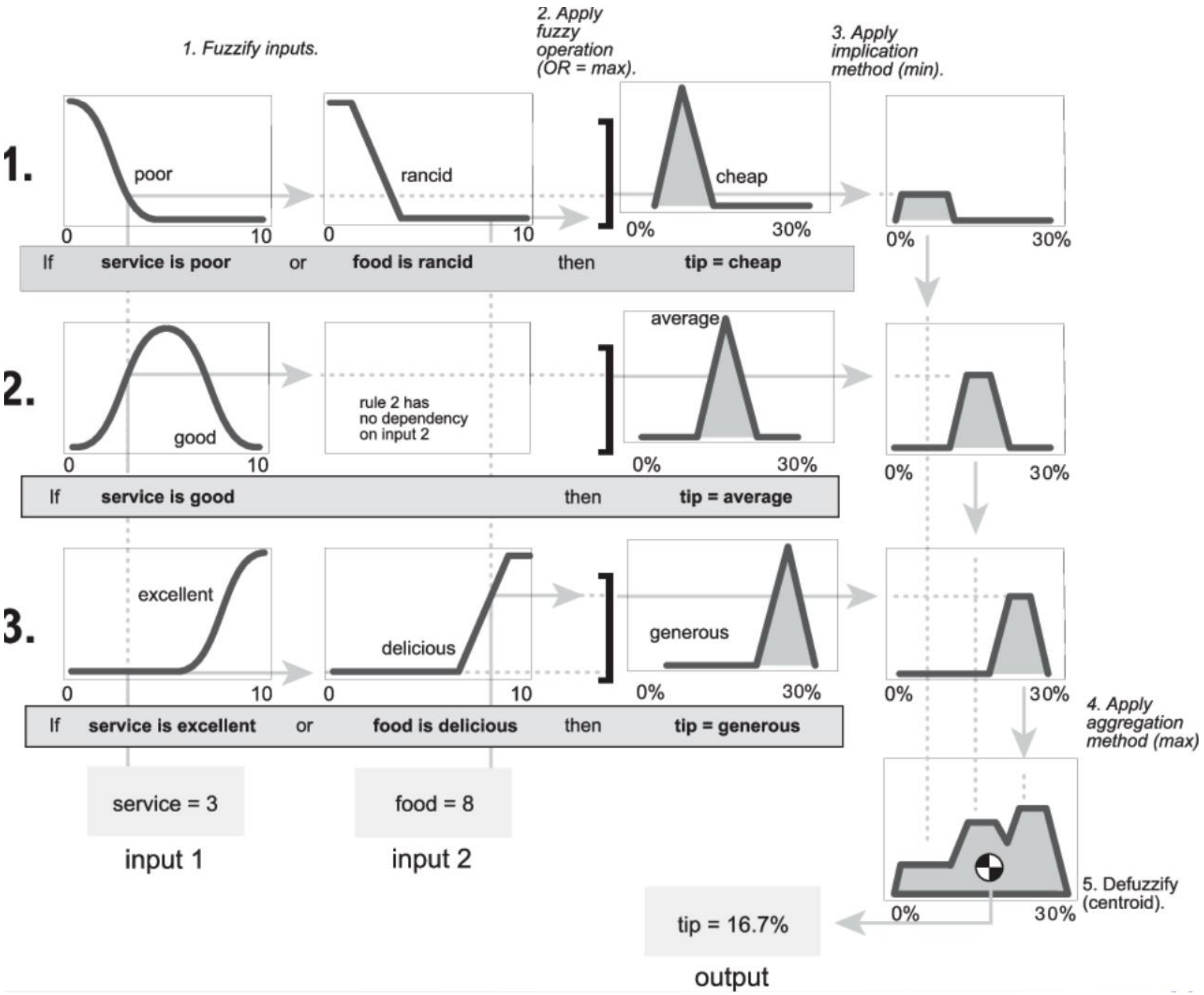
Tipper Fuzzy Controller: Implication



- The input for the implication process is a single number given by the antecedent, and the output is a fuzzy set representing the consequent reshaped.
The implication method typically applies the fuzzy conjunction (**min method**)

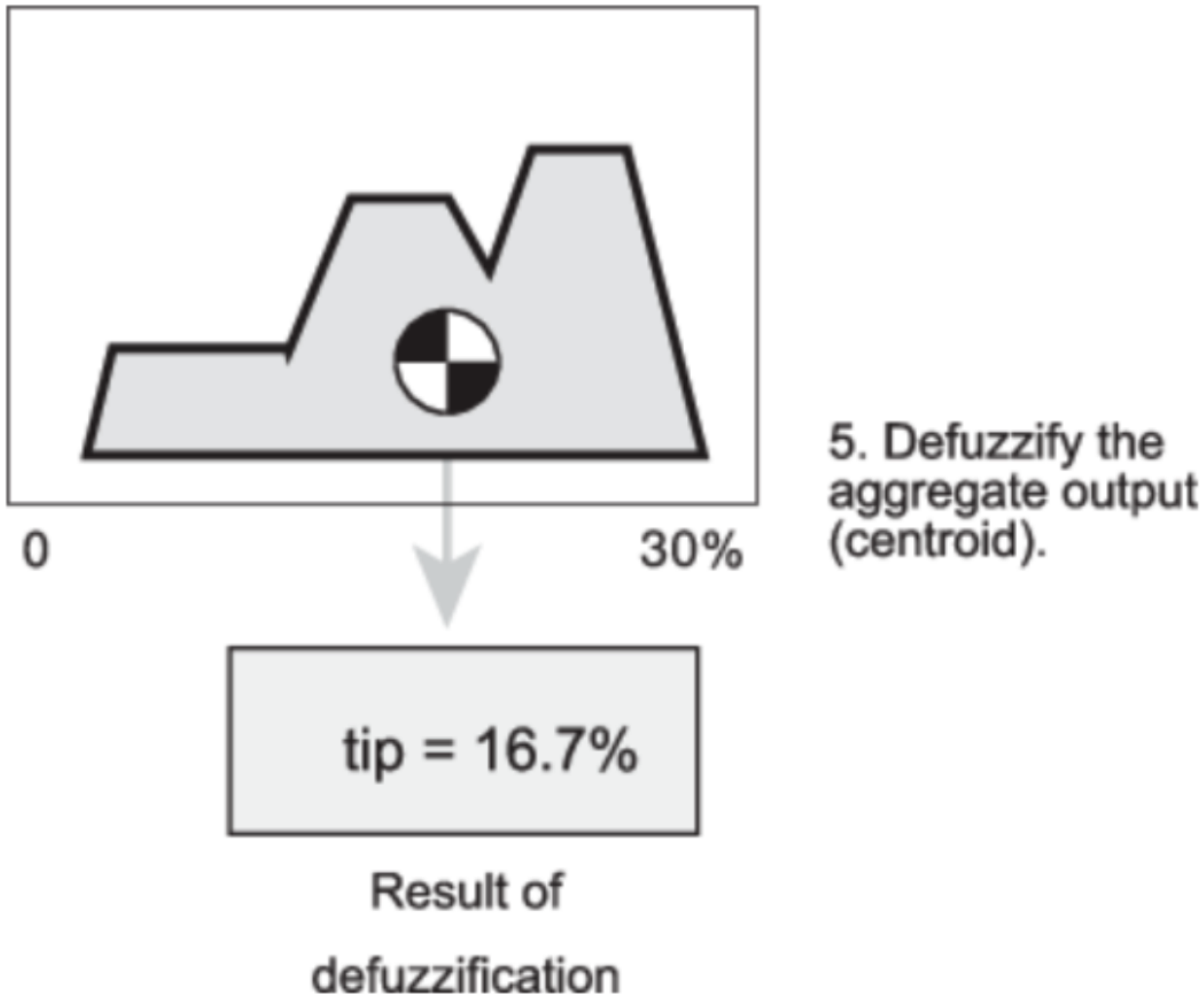
Tipper Fuzzy Controller: Aggregation

- The rule outputs must be combined in some manner.
- Aggregation is the process by which the fuzzy sets that represent the outputs of each rule are combined into a single fuzzy set.
- The output of the aggregation process is one fuzzy set for each output variable.
- The aggregation process typically applies the fuzzy disjunction (**max method**).



Tipper Fuzzy Controller: Defuzzification

- The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.
- There are **different defuzzification methods**: centroid, bisector, middle of maximum (the average of the maximum value of the output set), largest of maximum, and smallest of maximum. Perhaps the most popular defuzzification method is the **centroid calculation**, which returns the center of the area under the aggregate fuzzy set, as shown in the following figure.

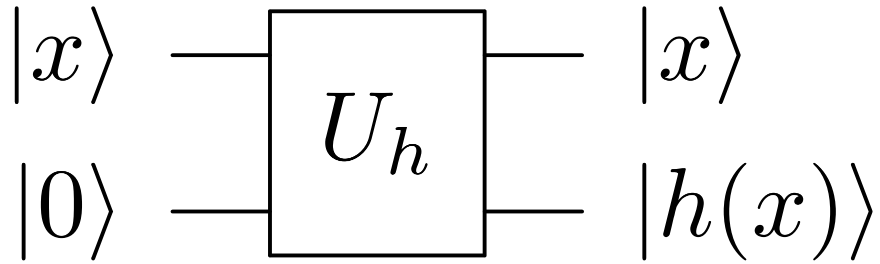


A Quantum Fuzzy Inference Engine

Submitted to IEEE-Transaction on Fuzzy Systems

Quantum Oracles

$$h : \{0,1\}^n \rightarrow \{0,1\} \qquad \rightarrow$$



$$U_h \sum_x |x, \bar{0}\rangle = \sum_x |x, h(x)\rangle$$

Encoding Fuzzified Values in Quantum States

$$X_j = \{T_0^j, T_1^j, \ldots, T_{m_j-1}^j\} \quad j = 1, 2, \ldots N$$

α_i^j membership degree of X_j to T_i^j $\qquad \sum_{i=0}^{m_j-1} \alpha_i^j \leq 1$

$$|\psi_j\rangle = \sum_{j=0}^{m_i-1} \sqrt{\alpha_i^j} \left|T_i^j\right\rangle$$

$$|\psi_0, \psi_1, \ldots \psi_{n-1}\rangle = \sum_{j=0}^{m_0-1} \sum_{k=0}^{m_1-1} \cdots \sum_{l=0}^{m_{n-1}-1} \sqrt{\alpha_j^0 \cdot \alpha_k^1 \cdot \ldots \cdot \alpha_l^{n-1}} \left|T_j^0, T_k^1, \ldots, T_l^{n-1}\right\rangle$$

Computing Fuzzy Rules by Means of a Quantum Oracle

$$|\psi_0, \psi_1, \ldots \psi_{n-1}\rangle = \sum_{j=0}^{m_0-1} \sum_{k=0}^{m_1-1} \cdots \sum_{l=0}^{m_{n-1}-1} \sqrt{\alpha_j^0 \cdot \alpha_k^1 \cdot \ldots \cdot \alpha_l^{n-1}} \, |T_j^0, T_k^1, \ldots, T_l^{n-1}\rangle$$

$$\text{IF } (X_1 \text{ is } T_{s_1}) \text{ and } (X_2 \text{ is } T_{s_2}) \text{ and } \ldots$$
$$\ldots \text{ and } (X_n \text{ is } T_{s_n}) \text{ THEN } Y \text{ is } T_{s_Y}$$

$$Y = \{T_1^Y, T_2^Y, T_3^Y\} \quad \rightarrow \quad \begin{array}{ccc} T_1^Y & \rightarrow & 100 \\ T_2^Y & \rightarrow & 010 \\ T_3^Y & \rightarrow & 001 \end{array}$$

Computing Fuzzy Rules by Means of a Quantum Oracle

$$|\psi'_Y\rangle = \mathcal{O}_f |\psi_0, \psi_1, \ldots, \psi_{n-1}\rangle |\bar{0}\rangle =$$

$$\sum_{i=1}^{m_Y} \sum_{a \in \mathcal{A}_\mathcal{S}^i} \left(\sqrt{F_a} |a\rangle |c_i\rangle\right) + \sum_{a \in \mathcal{A}_\mathcal{S}^0} \left(\sqrt{F_a} |a\rangle |\bar{0}\rangle\right)$$

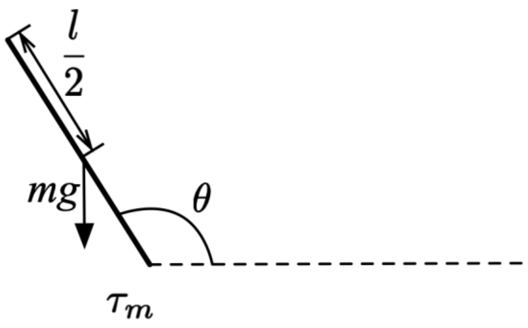
$$\rightarrow$$

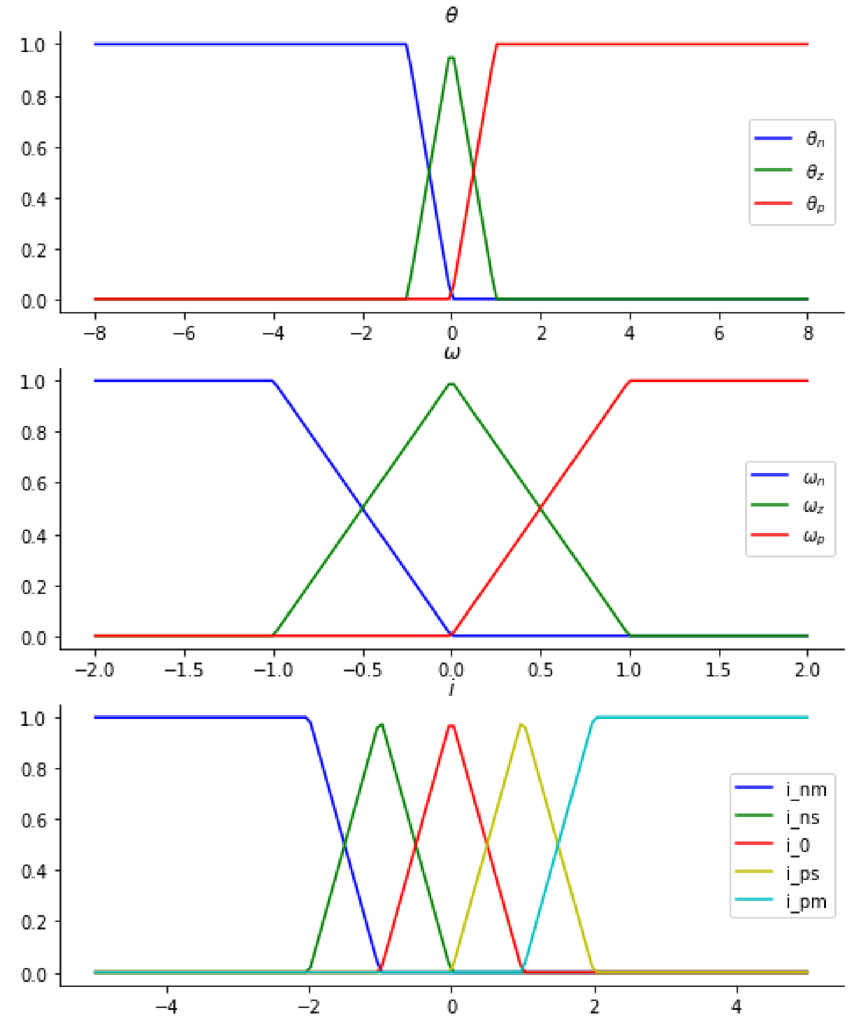
$$P_{c_i} = \langle\psi'_Y| \mathcal{M}_{c_i}^\dagger \mathcal{M}_{c_i} |\psi'_Y\rangle =$$

$$\sum_{a \in \mathcal{A}_\mathcal{S}^i} \sum_{b \in \mathcal{A}_\mathcal{S}^i} \left(\sqrt{F_a} \langle a| \langle c_i|\right) \cdot \left(\sqrt{F_b} |b\rangle |c_i\rangle\right) =$$

$$\sum_{a \in \mathcal{A}_\mathcal{S}^i} \sum_{b \in \mathcal{A}_\mathcal{S}^i} \delta_{a,b} \sqrt{F_a} \sqrt{F_b} \langle a|b\rangle = \sum_{a \in \mathcal{A}_\mathcal{S}^i} F_a$$

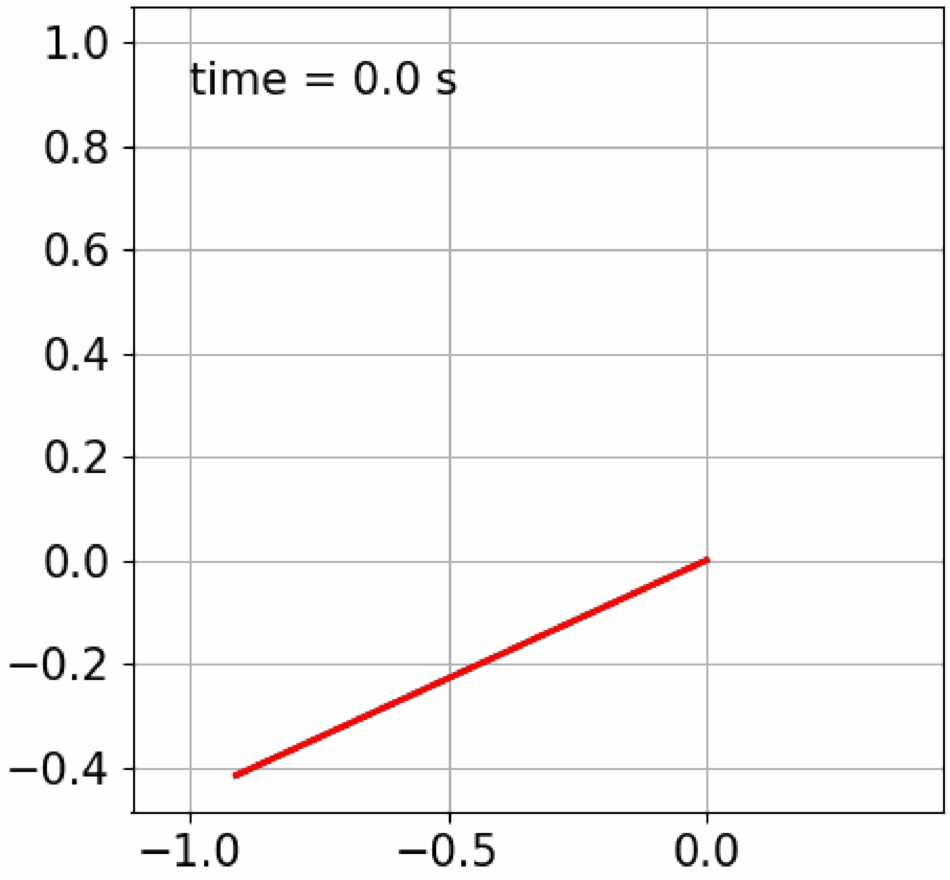
The Inverse Pendulum Control System





if theta is zero and omega is zero then current is zero;
if theta is zero and omega is neg then current is pos_small;
if theta is zero and omega is pos then current is neg_small;
if theta is pos and omega is zero then current is neg_small;
if theta is pos and omega is pos then current is neg_medium;
if theta is pos and omega is neg then current is zero;
if theta is neg and omega is zero then current is pos_small;
if theta is neg and omega is pos then current is zero;
if theta is neg and omega is neg then current is pos_medium.

The Inverse Pendulum Control System



Classical
Quantum