Elementary and advanced black hole physics: a modern practitioner's guide.

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Authorship credits

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OUTLOOK

- Lesson 0: Some preliminary mathematical concepts
- Lesson I: Towards the Schwarzschild solution
- Lesson II: Features of the Schwarzschild metric
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- Lesson VIII: Black holes beyond GR
- Lesson IX: Black holes in the metric-affine approach

Code colour for difficulty of subject.

- Green: elementary (BsC)
- Yellow: medium (MsC/early-PhD)
- Red: advanced (late-PhD/Postdoc)

RECOMMENDED BOOKS

- M. P. Hobson, “General Relativity”. My very first choice for a first take on black hole space-times.
- T. Padmanabhan, “Gravitation”. A complementary choice to Hobson’s including topics off the mainstream.
- R. M. Wald, “General Relativity”. An old but never outdated reference from a well posed mathematical formulation of physical principles.
- S. Chandrasekhar, “The mathematical theory of black holes”. Old classics die hard.
- P. S. Joshi, “Gravitational collapse”. The title is informative enough.
- T. Ortín, “Gravity and Strings”, Chapters 7, 8, 9. Ortin’s stringy viewpoints are never to be dismissed.
- M. Maggiore, “Gravitational wave: volumes 1 and 2”. A gargantuan effort to summarize theory and observations on GWs from black holes.
- M. Visser, “Lorenztian wormholes”. For a detailed account on wormholes.
- E. Curiel: Singularities and black holes, Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/entries/spacetime-singularities/ . A full conceptual take on space-time singularities to reconcile yourself with philosophers.
- E. N. Saridakis (Ed.), “Modified gravity and Cosmology: an updated by the CANTATA Network”. An update on the phenomenology of modified gravity including many applications.

LESSON 0: SOME PRELIMINARY MATHEMATICAL CONCEPTS

Main definitions

- In a differentiable manifold $\mathcal{M}$ one can define vectors

$$v = \sum_{\mu=1}^{n} v^{\mu} x_{\mu} \tag{1}$$

where x_{μ} is a coordinate basis.

- In particular, the components of the tangent vector to a curve γ are written as

$$T^{\mu} = \frac{dx^{\mu}}{dt} \tag{2}$$

- A **covariant derivative** is defined as a derivative satisfying certain properties (linearity, Leibnitz rule, commutativity with contraction, torsion-free). For a tensor field it reads explicitly

$$\nabla_a T^{b_1\ldots b_k}_{c_1\ldots c_l} = \partial_a T^{b_1\ldots b_k}_{c_1\ldots c_l} + \sum_i C^{b_i}_{ad} T^{b_1\ldots d \ldots b_k}_{c_1\ldots c_l} - \sum_j C^{d}_{aj} T^{b_1\ldots b_l}_{c_1\ldots d\ldots c_l} \tag{3}$$

For a vector field we have

$$\nabla_a t^b = \partial_a t^b + \Gamma^b_{ac} t^c \tag{4}$$

where Γ^b_{ac} are the components of the affine connection.

- Interpretation: there is no way of determining whether a tangent vector at p is "the same" as a tangent vector at q. We need an additional notion of "parallel transport" of vectors from p to q along a curve joining these points. **The covariant derivative**.

Geodesics

- A **geodesic** is a curve whose tangent vector is parallel-transported along itself. That is, $t^a\nabla_a v^b = 0$. It is the "straightest" possible curve. In a coordinate system: $t^a\partial_a v^b + t^a\Gamma^b_{ac}v^c = 0$, which in terms of components reads

$$\frac{dv^\nu}{d\lambda} + \sum_{\mu,\gamma} t^\mu \Gamma^\nu_{\mu\gamma} v^\gamma = 0 \tag{5}$$

Choosing $v^\nu = \frac{dx^\mu}{d\lambda}$, where λ parameterizes the curve, one has

$$\frac{d^2x^\nu}{d\lambda^2} + \sum_{\mu,\gamma} \Gamma^\nu_{\mu\gamma} \frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{6}$$

- How can we fix the connection?. We bring in the **metric tensor**, which is defined via

$$ds^2 = \sum_{\mu,\nu=1}^{n} g_{\mu\nu} dx^\mu dx^\nu \tag{7}$$

- If we demand the inner product $g_{ab}v^a w^b$ to remain unchanged as we parallel-transport them along the curve, one gets the **metric-connection compatibility condition**

$$\nabla_a g_{bc} = 0 \tag{8}$$

Exercise: From this condition derive the form of the **Christoffel symbols** Γ^c_{ad} (hint: use the definition of covariant derivative and cycle three times the indexes):

$$\Gamma^c_{ad} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \tag{9}$$

Basic geometrical objects

- A **spacetime** is a pair $(\mathcal{M}, g_{\mu\nu})$, where $\mathcal{M}$ is a connected 4-dimensional Hausdorr C^∞ manifold with a Lorentz metric $g_{\mu\nu}$ defined on $\mathcal{M}$.
- Riemann tensor
$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\nu\beta} \tag{10}$$
- Ricci tensor (assumed to be symmetric)
$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} \tag{11}$$
- Curvature scalar
$$R \equiv g^{\mu\nu} R_{\mu\nu} \tag{12}$$
- Einstein-Hilbert action (minimal coupling)
$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu}, \psi_m) \tag{13}$$
$\kappa^2 = 8\pi$ (in $G = c = 1$ units); $g = \det(g_{\mu\nu})$, L_m: matter Lagrangian; ψ_m: matter fields;
- Variational principle:
$$\delta\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x(\delta\sqrt{-g}R + \sqrt{g}\delta R) + \int d^4x\delta(\sqrt{-g}\mathcal{L}_m) \tag{14}$$
Exercise: Derive the Einstein equations (hint: integrate partial derivatives by parts):
$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu};\, T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g^{\mu\nu}} \tag{15}$$
- Second-order differential equations: two integration constants. Physical interpretation.

Some concepts to keep in mind for black hole physics - I

- The **chronological future (past)** of a event $p \in \mathcal{M}$, dubbed as $I^+(p)$, is the set of all $q \in \mathcal{M}$, such that there is a smooth future (past)-directed non-degenerate timelike curve from p to q.
- **Null and time-like vectors**. Let $u^\mu = dx^\mu/d\lambda$ be the tangent vector to a geodesic curve $\gamma^\mu = x^\mu(\lambda)$. This vector satisfies:

$$g^{\mu\nu} u_\mu u_\nu = -k \tag{16}$$

  where $k = 0$ is a null vector (region), $k = 1$ is a time-like vector, and $k = -1$ is a space-like vector. *Trajectories of photons and particles*.
- A subset $\mathcal{S} \subset \mathcal{M}$ is said to be **achronal** if there does not exist a pair of events $p, q \in \mathcal{S}$ such that it can be connected by causal curves.
- Let $\mathcal{S}$ be a spacelike three-dimensional manifold. If every inextendible non-spacelike curve in $\mathcal{M}$ intersects $\mathcal{S}$, then $\mathcal{S}$ is said to be a **Cauchy surface**.
  It is believed that *all physically reasonable spacetimes are globally hyperbolic*, but there is no proof of this.
- $\mathcal{M}$ is said to be **globally hyperbolic** if it admits a global Cauchy surface.

Some concepts to keep in mind for black hole physics - II

- **Event horizon**: To formulate a black hole as the intuitive notion of a "region of no scape", or a region of space-time where gravity is so strong that any particle or light ray entering into that region can never space from it. This becomes equivalent to an infinite redshift surface.
  Let $(\mathcal{M}, g_{\mu\nu})$ be an asymptotically flat space-time. It is said to contain a black hole if $\mathcal{M}$ is not contained in J^- (the causal past of null infinity). The black hole region, B, is defined as $B = [\mathcal{M} - J^-]$ and the boundary of B in $\mathcal{M}$, that is, $H = J^- \frown \mathcal{M}$, is called the **event horizon**.
- **Trapped surface**. A closed trapped surface is a 2-dimensional embedded submanifold S (aka a surface) such that the two families of light rays emerging ortogonally from S towards the future converge (that is, they are "trapped").
  Locally, the two families of light rays can be represented by a pair of future-directed affine-parameterized geodesic null vector fields $k_{\pm}^{\mu}$. The infinitesimal variation of the area of S is measured by the respective expansions:

$$\theta_{\pm} = \nabla_{\mu} k_{\pm}^{\mu} \tag{17}$$

  Therefore, a closed trapped surface satisfies $\theta_{\pm} < 0$.

LESSON I: TOWARDS THE SCHWARZSCHILD SOLUTION

Static, spherically symmetric metric - I

- A **static** space-time:
  - All the metric components $g_{\mu\nu}$ are independent of x^0 (aka "time").
  - The line element ds^2 is invariant under the transformation $x^0 \to -x^0$.

  Bonus track: A space-time satisfying the first but not the second feature is called **stationary** (e.g. Kerr).
- Let us begin with an **isotropic** metric (not yet static). That is, it depends on rotational invariants $\vec{x}\dot{\vec{x}} = r^2$, $d\vec{x} \cdot d\vec{x}$ and so on.

$$ds^2 = A(t,r)dt^2 - B(t,r)dt\vec{x} \cdot d\vec{x} - C(t,r)(\vec{x} \cdot d\vec{x})^2 - D(t,r)d\vec{x}^2 \qquad (18)$$

- Go to spherical polar coordinates

$$\begin{aligned} x^1 &= r \sin\theta \cos\phi & (19)\\ x^2 &= r \sin\theta \sin\phi & (20)\\ x^3 &= \cos\theta & (21) \end{aligned}$$

we obtain (Exercise : show this is true)

$$ds^2 = A(t,r)dt^2 - B(t,r)rdtdr - C(t,r)r^2dr^2 - D(t,r)(r^2 d\theta^2 + r^2 \sin^2\theta^2 d\phi^2) \qquad (22)$$

- Redefining A, B, C, D we can write

$$ds^2 = A(t,r)dt^2 - B(t,r)dtdr - C(t,r)dr^2 - D(t,r)(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \qquad (23)$$

Static, spherically symmetric metric - II

- Introduce a new radial coordinate $\bar{r}^2 = D(t,r)$ to write (Exercise : There is here an unstated assumption. Can you spot it?.):

$$ds^2 = A(t,\bar{r})dt^2 - B(t,\bar{r})drdr - C(t,\bar{r})d\bar{r}^2 - \bar{r}^2 d\Omega^2 \tag{24}$$

- Introduce a new "time" coordinate $d\bar{t} = \Phi(t,\bar{r})[A(t,\bar{r})dt - \frac{1}{2}B(t,\bar{r})d\bar{r}]$ and square it to write

$$Adt^2 - Bdtd\bar{r} = \frac{1}{A\Phi^2}d\bar{t}^2 - \frac{B^2}{4A}d\bar{r}^2 \tag{25}$$

- Now define the new functions $\bar{A} = \frac{1}{A\Phi^2}, \bar{B} = C + \frac{B^2}{4A}$ and remove bars everywhere to write

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega^2 \tag{26}$$

This is the *most general isotropic metric* one can build.

- For the *most general static metric* one just needs to neglect the time dependence:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2 \tag{27}$$

Exercise : Verify all these steps.

Derivation of the Schwarzschild metric - I

- The Schwarzschild metric corresponds to a static, spherically symmetric, vacuum solution of Einstein's field equations, $G_{\mu\nu}=0$. Because in vacuum $R=0$, it suffixes to solve $R_{\mu\nu}=0$.
- How?. Step 0: We will need the covariant and contravariant components of the metric tensor (remember that $g_{\mu\nu}g^{\nu\alpha}=\delta_{\mu}^{\alpha}$):

$$\begin{aligned} g_{00} &= A(r); g_{11}=-B(r); g_{22}=-r^2; g_{33}=-r^2\sin^2\theta; && (28)\\ g^{00} &= \frac{1}{A(r)}; g_{11}=\frac{-1}{B(r)}; g_{22}=\frac{-1}{r^2}; g_{33}=\frac{-1}{(r^2\sin^2\theta)} && (29) \end{aligned}$$

- Step 1: Compute the Christoffel symbols:

$$\Gamma^{\sigma}_{\mu\nu}=\frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\rho\mu}+\partial_{\mu}g_{\rho\nu}-\partial_{\rho}g_{\mu\nu}) \tag{30}$$

The non-vanishing ones are:

$$\begin{aligned} \Gamma^0_{01} &= \frac{A'}{2A}; \Gamma^1_{22}=-\frac{r}{B}; \Gamma^1_{00}=\frac{A'}{2A}; \Gamma^1_{11}=\frac{B'2}{2B}; \Gamma^1_{33}=-\frac{r\sin^2\theta}{B} && (31)\\ \Gamma^2_{12} &= \frac{1}{r}; \Gamma^2_{33}=-\sin\theta\cos\theta; \Gamma^3_{13}=\frac{1}{r}; \Gamma^3_{23}=\cot\theta && (32) \end{aligned}$$

Derivation of the Schwarzschild metric - II

- Step 2: Compute the Ricci tensor

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\nu\mu} - \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\alpha\mu} \tag{33}$$

The non-vanishing ones are:

$$R_{00} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} \tag{34}$$

$$R_{11} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} \tag{35}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \tag{36}$$

$$R_{33} = R_{22}\sin^2\theta \tag{37}$$

- Step 3: Solve the field equations $R_{\mu\nu} = 0$. Combine first

$$\frac{B}{A}R_{00} + R_{11} = 0 \to A'B + AB' = 0 \to (AB)' = 0 \to AB = \alpha \tag{38}$$

and then use $R_{22} = 0$ to find $A + rA' = \alpha$ which upon integration yields $rA = \alpha(r+k)$, that is

$$A(r) = \alpha\left(1 + \frac{k}{r}\right); B(r) = \left(1 - \frac{k}{r}\right)^{-1} \tag{39}$$

Derivation of the Schwarzschild metric - III

- Step 4: Interpret physically the solution.
- The weak-field limit (i.e. consistence with Newtonian gravity) demands that, asymptotically

$$\frac{A(r)}{c^2} = 1 + \frac{2\Phi}{c^2} \tag{40}$$

where Φ is the Newtonian potential. For a spherically symmetric mass M we have $\Phi = -\frac{GM}{r}$, therefore $k = -\frac{2GM}{c^2}$ and $\alpha = c^2$.
- This way we have arrived to the canonical form of the Schwarzschild metric (1917)

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{41}$$

which describes the space-time geometry outside a spherical body of total mass M. The latter represents the mass seen by an asymptotic observer orbiting the body following Newton's law, and it is consistent with global mathematical definitions of the mass enclosed in the sphere.

Exercise: Could you check that this is the case with the Arnowitt-Deser-Misner (ADM) definition of the mass?.
- **Birkhoff's theorem**: The space-time geometry outside *any* spherically symmetric matter (vacuum) distribution is Schwarzschild.
- Indeed, **Uniqueness theorems**[1]: Schwarzschild geometry is the only vacuum, static, spherically symmetric solution of GR.

[1] W. Israel, Phys. Rev. **164**, 1776 (1967).

LESSON II: FEATURES OF THE SCHWARZSCHILD METRIC

Event horizon

- Schwarzschild metric in $G = c = 1$ units

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{42}$$

- The function $A(r) = 1 - 2M/r$ has a simple monotonic behavior:

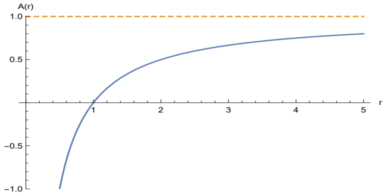


Figure: Schwarzschild solution for $M = 1/2$.

- Potential difficulties at two points where the metric components become zero or infinity (actually both). The first one is the *Schwarzschild radius*:

$$r_S = \frac{2GM}{c^2} \tag{43}$$

For the Sun, $r_S \approx 2.95$km; for a proton, $r_S \approx 2.5 \times 10^{-54}$m.

- At this point $g^{rr} = g_{tt} = 0$ blows up. "Singular" behaviour?. Interpretation?.

Freedom of coordinates: Eddington-Filkenstein coordinates

- It can be removed out of a coordinate basis (remember that the metric is a tensor). One suitable choice is the so-called (advanced) **Eddington-Finkelstein coordinates, 1958**:

$$v \equiv ct + r + 2M \log\left|\frac{r}{2M} - 1\right| \to dv = cdt + \frac{r}{r-2M}dr \tag{44}$$

when allows to cast Schwarzschild metric into the form

$$ds^2 = \left(1 - \frac{2M}{r}\right)dv^2 - 2dvdr - r^2 d\Omega^2 \tag{45}$$

In these coordinates the metric is regular at $r = 2M$, while the zero of g_{tt} is interpreted as an infinite redshift as seen by an external observer (see next slide).

Exercise: Using this system of coordinates, can you tell me what would happen to an observer crossing $r = 2M$? (hint: use the fact that $r < 2M$ is a space-like region, because $A(r)$ changes its sign there and the metric changes its signature).

- Bottom line: coordinate singularities are not physical singularities!.
- The metric is singular at $r = 0$ as well. Is there any change of coordinate allowing to remove this singularity?. To prove that there is not, one compute objects that do not depend on the choice of coordinates. For Schwarzschild, $R = R_{\mu\nu}R^{\mu\nu} = 0$, but the Kretchsmann scalar behaves as

$$K \equiv R^{\alpha}{}_{\mu\nu\beta}R_{\alpha}{}^{\mu\nu\beta} = \frac{48M^2}{r^6} \tag{46}$$

which blows up at $r = 0$ and, therefore, this is a truly singular behaviour. Curvature singularities.

Event horizon as an infinite redshift function

- Let us two observers, one acting as the emitter of two photons and the other as the receiver of them.
- In the frame of the emitter, the first photon is emitted at a time t_E and the second at a time $t'_E = t_E + \delta$, which on its frame are received by the receiver at times t_E and $t'_R = t_R + \delta$, since the equality $t'_E - t_E = t'_R - t_R$ must hold or, in other words, $\Delta t_R = \Delta t_E$.
- The proper times measured on each frame are related by the equation

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 \Rightarrow \Delta\tau = \left(1 - \frac{2M}{r}\right) \Delta t \tag{47}$$

- Therefore, the rate of periods they experiment between each pulse is given by (using the equality $\Delta t_R = \Delta t_E$)

$$\left(\frac{1 - 2M/r_E}{1 - 2M/r_R}\right)^{1/2} = \frac{\Delta\tau_E}{\Delta\tau_R} = \frac{\nu_R}{\nu_E} \tag{48}$$

- Let us define the redshift as $1 + z = \nu_E/\nu_R$ and assume the emitter to be located at the horizon, $r_E \to r_S$, and the receiver at asymptotic infinity, $r_R \to \infty$, so that the redshift goes as

$$z \sim \left(\frac{r_E}{r_E - 2M}\right)^{1/2} \to \infty \tag{49}$$

  i.e. an infinite redshift implying an infinite time to cross the horizon as seen by the asymptotic observer.

Interior solution

- The Schwarzschild metric describes *only* the exterior of the spherically symmetric body of mass M. What about its interior. What is it filled of?.
- Complement Einstein's equation with a perfect fluid for $r < r_s$:

$$T_{\mu\nu} = -(\rho + P)u_\mu u_\nu + p g_{\mu\nu} \tag{50}$$

Assuming spherical symmetry, $ds^2 = A(r)dt^2 - B(r)dt^2 - r^2 d\Omega^2$ one can solve Einstein's equations as

$$\frac{A'(r)}{A(r)} = -\frac{2P'}{\rho + P} \tag{51}$$

$$B(r) = (1 - 2m(r)/r)^{-1}; m(r) = 4\pi \int_0^r \rho(\tilde{r})\tilde{r}^2 d\tilde{r} \tag{52}$$

$$P_r = -\frac{\rho + P}{r^2}\left[4\pi P r^3 + m(r)\right] B(r) \tag{53}$$

These are just the Oppenheimer-Tolman-Volkof equations of hydrostatic equilibrium for any spherical symmetric body filled with a perfect fluid (provided an equation of state, $P = P(\rho)$ is given). It is suitable for any star, but the interior fluid of Schwarzschild metric is not known.
Exercise : Can you tell me whether a delta-type source at the center of the Schwarzschild black hole would be a suitable candidate to generate this geometry[2].

[2] Hint: check T. Ortin, "Gravity and Strings" book.

Maximal extensions - I

- Exercise : Can you introduce new (retarded) Eddington-Filkenstein coordinates to deal with the white hole horizon?.
- It is possible to introduce a new set of coordinate such that the resulting solution (combining advanced + retarded EF coordinates) is valid for every $r > 0$ (i.e. everywhere outside the central singularity). These are the Kruskal-Szekeres coordinates

$$\begin{aligned} T &= \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right); R = \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right); r > 2M \\ T &= \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \cos\left(\frac{t}{4M}\right); R = \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right); 0 < r < 2M \end{aligned}$$

- The line element becomes

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} (dT^2 - dR^2) + r^2(u,v) d\Omega^2 \tag{54}$$

where the radial function is implicitly defined by the relation

$$T^2 - R^2 = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}}; T^2 - R^2 < 1 \tag{55}$$

Maximal extensions - II

- The Kruskal extension splits the extended space-time into four regions.

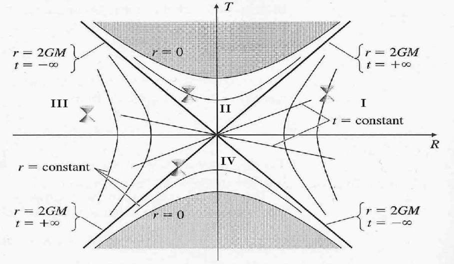


Figure: Kruskal-Szekeres maximal extension of Schwarzschild geometry. Light rays are 45 degrees straight lines and physical observers are contained in their light cones.

- From the asymptotic regions I ($-R < T < +R$) and III ($+R < T < -R$) signals can be transmitted to region II ($|R| < T < \sqrt{1+R}$) crossing the future event horizon, but they are not causally connected.
- Any signal/particle in line II unavoidably end up hitting the central singularity $r = 0$ in finite proper time. This is the black hole region.
- Any signal departing from region IV ($-\sqrt{1+R} < T < -|R|$) must have arisen at $r = 0$ at a finite past time, and unavoidably exits crossing the past event horizon towards either region I or III. This is the white hole region.

Geodesic behaviour

- A geodesic curve $\gamma^\mu = x^\mu(\lambda)$ with tangent vector $u^\mu = \frac{dx^\mu}{d\lambda}$ and affine parameter λ satisfies

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{56}$$

- By spherical symmetry there are two conserved quantities: $L = r^2 d\varphi/d\lambda$ (angular momentum per unit mass) and $E = Adt/d\lambda$ (total energy per unit mass).
- Rewrite the geodesic equation in terms of the geodesic tangent vector

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - A(r)\left(-\kappa + \frac{L^2}{r^2}\right) \tag{57}$$

where $k = 0(-1)$ for null (time-like) geodesics. Null geodesics represent light rays (aka transmission of information), while time-like geodesics are associated to physical observers (aka physical particles particles). $k = +1$ for space-like observers, i.e., hypothetical tachyon-like particles.
- This is as a single differential equation akin to that of a classical particle in a one dimensional potential of the form

$$V(r) = A(r)\left(\kappa + \frac{L^2}{r^2}\right) \tag{58}$$

Exercise : Can you classify when a given geodesic will be able to cross $r = 2M$ (hint: those maxima of the potential seem to be hiding something...)[3].

[3] You can help yourself by checking S. Chandrasekhar, "The mathematical theory of black holes".

- For the Schwarzschild black hole the effective potential reads

$$E^2 = \dot{r}^2 + \left(\frac{L^2}{r^2} - k\right)\left(1 - \frac{2m}{r}\right) \tag{59}$$

- For a circular orbit $V'_{eff} = 0$, which amounts to

$$V'_{eff}(r) = \frac{2M}{r^2}\left(\frac{L^2}{r^2} - k\right) - \frac{2L^2}{r^3}\left(1 - \frac{2M}{r}\right) = 0 \tag{60}$$

which is satisfied at a radius

$$\frac{1}{r_m} = \frac{1}{6M}\left[1 \pm \sqrt{1 + 12\left(\frac{M}{L}\right)^2 k}\right] \tag{61}$$

which will be unstable (stable) if $V'' > 0$ ($V'' < 0$). For a photon, $k = 0$, this radius reads $r_{ps} = 3M$, which corresponds to a maximum of the effective potential and is dubbed as the *photonsphere*. It is the uttermost relevant feature of a black hole for GWs and shadows.

- For a massive particle $k = -1$ the innermost unstable circular orbit (ISCO) exists if $1 \geq 12(M/L)^2$ and lies always beyond its photonsphere at $r = 6M$.
- There are no stable circular orbits in Schwarzschild geometry. Exercise : What about the Reissner-Nordström geometry?.

Gravitational lensing - I

- Suppose a photon that departs from asymptotic infinity and is deflected by the black hole at some radius $r_0 > r_{ps}$. At this point we have $E^2 = V(r_0)$, that is

$$b(r_0) = \frac{1}{r_0}\sqrt{1-2M/r_0} \tag{62}$$

where $b \equiv L/E$ is the impact parameter.

- Replacing the definition of angular momentum $L = r^2 d\varphi/d\lambda$ in the geodesic equation one finds

$$\frac{d\phi}{dr} = \left[\frac{r^4}{b^2} - r^2\left(1-\frac{2M}{r}\right)\right]^{-1/2} \tag{63}$$

whose integration defines the angular light deflection angle (the angle between the asymptotic incoming and outgoing trajectories) as

$$\alpha(r_0) = 2\int_{r_0}^{\infty}\left[\frac{r^4}{b^2} - r^2\left(1-\frac{2M}{r}\right)\right]^{-1/2} dr - \pi \tag{64}$$

- In the weak field limit, via the change of variable $u = 1/r$ this expression becomes

$$\alpha = \int_0^{u_0}\left[u_0^2(1-2Mu_0) - u^2(1-2Mu)\right]^{-1/2} du - \pi \approx \frac{4M}{b} \tag{65}$$

which is the angular deflection in the standard Schwarzschild spacetime.

Gravitational lensing - II

- **Exercise**: Prove that, for a generic spherically symmetric line element

$$ds^2 = -A(r)dt^2 + B^{-1}(r)dr + C(r)d\Omega^2 \tag{66}$$

from the geodesic equation the deflection angle is obtained as

$$\alpha = 2\int_{r_0}^{\infty} r\left[\sqrt{\frac{B(r)}{C(r)[C(r)/A(r)] - b^2}}\right]^{1/2} - \pi \tag{67}$$

where the impact parameter is defined as $b^2 = C(r_0)/A(r_0)$[4].

- In the strong-field regimen one can show that the deflection angle in that limit reads[5].

$$\alpha = 4\sqrt{r_0/s}F(\varphi, m) - \pi \tag{68}$$

with the definitions

$$s = \sqrt{(r_0 - 2M)(r_0 + 6M)}; m = (s - r_0 + 6M)/s \tag{69}$$

$$\varphi = \arcsin\sqrt{2s/(3r_0 - 6M + s)} \tag{70}$$

- Light rays whose impact parameter approaches $b > b_0$ experience larger deflections. When $\alpha > 2\pi$, this means that they can turn n loops around the black hole before leaving it (for $b = b_c$ it would turn infinite times while approaching $r = r_0$). For $b < b_c$ the photon is captured by the black hole.

[4] Hint: check the book: S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[5] V. Bozza, *Gen Rel. Grav.* **42** (2010) 2269.

The black hole lens

- A lens equation is the relation between the positions of the relativistic images and the geometry (relative position of source, observer and lens) parameters of the system from the observer's viewpoint.

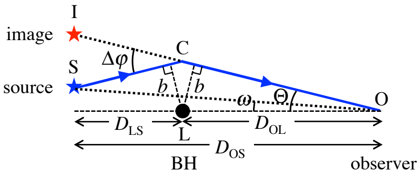


Figure: Lens diagram, where S, L, O, and I are the positions of the source, lens object, observer, and image, respectively.

- Assuming both source and lens to be far away from the black hole so that the gravitational fields there are weak enough to be described by a flat metric, the lens equation is given by[6]

$$\tan\omega = \tan\theta - \frac{D_{LS}}{D_{OS}}\left[\tan(\Delta(\varphi)-\theta)+\tan(\Theta)\right] \tag{71}$$

where ω and Θ correspond to the lens/source and the lens/observer angular separation, while D_{LS} and D_{OS} stand from the distance between lens and source, and observer and source.

[6] K. S. Virbhadra, G.F.R. Ellis, Schwarzschild black hole lensing, Phys. Rev. D **62** (2000) 084003.

Strong gravitational lensing

- In the strong gravitational lensing one goes to the strong deflection limit source, assuming lens and observer to be highly aligned, i.e., $\omega \ll 1$ and $\Theta \ll 1$ (and $(\Delta\varphi_n - \Theta) \ll 1$, where $\Delta\varphi_n \equiv \Delta - 2\pi n$). Using that in the lens geometry $b \simeq D_{OL}\Theta$ one gets the deflection angle
$$\Delta\varphi(\Theta) = -a_1 \log\left(\frac{D_{OL}\Theta}{b_c} - 1\right) + a_2 \,. \tag{72}$$
where the strong deflection coefficients a_1 and a_2 are functions of r_0 for every black hole.
- The relativistic images correspond to $\Delta\varphi(\Theta) = 2\pi n$, which yields
$$\Theta_n^0 = \frac{b_c}{D_{OL}}\left[1 + \exp\left(\frac{a_2 - 2n\pi}{a_1}\right)\right] \tag{73}$$
where Θ_n^0 is the angle of the nth relativistic image. Due to the exponential contribution the first relativistic image, Θ_1^0, is the brightest one, while the others are greatly demagnified.
- Three other observables (assuming that the first relativistic image Θ_1^0 can be resolved from the others:) the position of the relativistic images except the first one, Θ_∞^0, and the two quantities
$$s \equiv \Theta_1^0 - \Theta_\infty^0 = \Theta_\infty^0 \exp\left(\frac{a_2 - 2\pi}{a_1}\right) \tag{74}$$
$$R = \exp(2\pi/a_1) \tag{75}$$
which are the angular separation between the first image and all the others, and the ratio between the flux of the first image and all the others. The latter defines a more convenient observable, $R_m = 2.5\log_{10} R$, which is the relative magnification of the images.

Observables of strong lensing

Figure: Under construction

The dynamical Vaidya metric - I

- Consider the formation of a black hole from an ingoing flux of pressureless neutral matter following null radial geodesics towards the interior. Energy-momentum tensor:

$$T_{\mu\nu} = \rho_{in} l_\mu l_\nu \tag{76}$$

where ρ_{in} is the energy density of the ingoing stream and l_μ represents a null radial vector $l_\mu l^\mu = 0$ ("radiation"). So that we have to solve $R_{\mu\nu} = \kappa^2 \rho_{in} l_\mu l_\nu$.

- This process represents the dynamical generation/perturbation of a black hole with spherical symmetry from a pure radiation field. This problem is more easily dealt with using the following coordinate system:

$$ds^2 = g_{ab} dx^a dx^b + r^2 d\Omega^2 \tag{77}$$

where (x_0, x_1) is the coordinates of the two-spaces with $(\theta, \phi) =$constant, and $r(x^a)$ measures the area of the two spheres of $x^a =$ constant.

- The Einstein tensor is computed as

$$
\begin{aligned}
{}^4G_{ab} &= -\frac{1}{r^2}[2rr_{;ab} + (1 - 2r\Box r - r'^a r_{,a})g_{ab}] && (78) \\
G_{\theta\theta} &= \sin^2\theta G_{\phi\phi} = r\Box r - \frac{1}{2}r^2 R && (79)
\end{aligned}
$$

where $\Box\psi = g^{ab}\nabla_a\nabla_b\psi$.

The dynamical Vaidya metric - II

- For a null presureless fluid ($T=0$), introducing $A=r^{\prime a} r_{\prime a}=1-2m/r$ these equations can be combined as

$$r_{;ab}-\frac{m}{r^2}g_{ab} = -4\pi r\rho_{in} l_a l_b \quad (80)$$

$$m_{,a} = 4\pi r^2 T_a^b r_{,b} \quad (81)$$

- To solve them, let us introduce the (Vaidya-type) line element

$$ds^2=-Ae^{2\psi}dv^2+2e^{\psi}dvdr+r^2d\Omega^2 \quad (82)$$

where $A(v,r)=1-\frac{2m(v)}{r}$ and $\psi(v,r)$ are the metric functions and the null vector is normalized as $l_a=-\partial_a v$.
- The $v-r$ component of (80) component yields $\psi_r=0\to\psi=\psi(v)$. Thus redefining $dV=e^{\psi(v)}dv$ we can eliminate this metric component from the line element.
- Integration of the $v-v$ component of (80) yields $m_v=\frac{\kappa^2 r^2}{2}\rho_{in}$, which is consistent with the $r-r$ component of (81) since the latter implies $m_r=0$.
- Introducing the **luminosity function** $L(v)=\frac{\kappa^2 r^2}{2}\rho_{in}$ one finally finds

$$ds^2=-\left(1-\frac{2\int_{v_1}^{v2}L(v)dv}{r}\right)dv^2+2dvdr+r^2d\Omega^2 \quad (83)$$

which is the **Vaidya metric**. For a Dirac-delta pulse $L(v)=m_0\delta(v-v_0)$ (with $v_0\in[v_1,v_2]$), the integration yields $m(v)=m_0$, which causes the event horizon to grow until the flux ceases.

Gravitational collapse - I

- To describe gravitational collapse we consider comoving coordinates (t, r, θ, φ) (i.e. locally moving with the matter) as

$$ds^2 = -e^{2\nu(r,t)}dr^2 + e^{2\psi(r,t)}dr^2 + R^2(t,r)d\Omega^2 \tag{84}$$

where the function $R(t,r)$ is the physical radius of the collapsing shell at a given time t and comoving radius r. At the initial time t_i, we can write $R(t_i, r) = r$.

- Assuming an anisotropic fluid with stress-energy tensor

$$\rho = -T^t{}_t; p_r = T^r{}_r; p_\perp = T^\theta{}_\theta = T^\varphi{}_\varphi \tag{85}$$

satisfying the dominant energy condition, which amounts to $\rho \geq 0$, $\rho + p_r \geq 0$, $\rho + p_\perp \geq 0$, $|p_r| \leq \rho$, $|p_\perp| \leq \rho$.

- The Einstein equations in this case yield a complicated system of equations depending on ψ, ν, R and their temporal and spatial derivatives, which can be recast as

$$\rho = \frac{F'}{R^2 R'}; p_r = \frac{\dot{F}}{R^2 \dot{R}}; \nu' = \frac{2(p_\theta - p_r)R' - p_r' R}{(\rho + p_r)R}; R'\frac{\dot{G}}{G} + \dot{R}\frac{G'}{G} = 0 \tag{86}$$

via the definitions

$$G(t,r) = e^{-2\psi}R'^2; H(t,r) = e^{2\nu}\dot{R}^2 \to F(t,r) = R(1 + H - G) \tag{87}$$

Gravitational collapse - II

- The first equation is integrated as

$$F(t,r)=\int dRR^2(t,R)\rho(t,R)dR \qquad (88)$$

  which is interpreted as the total mass inside a shell of comoving radius r at a time t. The DEC translates in $F>0$, while the condition $F(r,r=0)$ is needed to ensure the regularity of the density at the center of the shell.
- Two classes of possible singularities:
  - $R'=0$ are shell-crossing singularities, which happen when outer shells collapse faster than their neighbouring inner shells, and are removable by coordinate changes.
  - $R=0$ is the shell-focusing singularity, caused by the matter shells collapsing to a zero physical radius
- We have six independent variables $\rho, p_r, p_\theta, \psi, \nu, R$ and four equations, to evolve with initial conditions $\rho(t_i,r), p_r(t_i,r), p_\perp(t_i,r)$ together with the velocity profile $\psi(t_i,r)$. Close the system with equations of state $p_r=p_r(\rho), p_\perp=p_\perp(\rho)$.
- The goal of the subsequent analysis is to determine whether the time needed for the shells to hit the shell-focusing singularity is prior to the formation of the event horizon or not. In the latter case the end-state of the gravitational collapse is a black hole, in the former it is a naked singularity instead (**forbidden by the cosmic censorship?**).

Gravitational collapse - IV: naked singularities

- A naked singularity occurs when there are nonspace-like geodesics emanating out of it, therefore rendering it visible to external observers.
- Naked singularities are abhorrent since they undermine the predictability and determinism of our physical theories.
- Usually this question is sweep under the carpet by invoking the **cosmic censorship** (Penrose 1969) in either its **weak form**: *Singularities are always hidden from asymptotic observers behind event horizons* or on its **strong form**: *The space-time cannot be extended beyond its Cauchy horizon* (to preserve determinism, given the instability of Cauchy horizons, as explained below).
- The strong CC has been disproved by explicit counter-examples, recently developed in the literature.
- As for the weak CC, one thus build explicit collapse model and focus on whether the formation of a trapped surface occurs before or after the singularity appears yielding a black hole in the first case or a naked singularity in the former, which is therefore visible to external observers even for a while $\Rightarrow$ explicit such models do exist in the literature.
- While inhomogeneouties are frequently invoked in order to delay the collapse, alternatively bounces by quantum matter/gravity effects are also able to do the job.

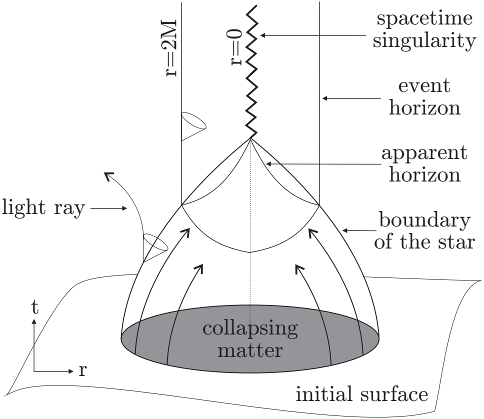


Figure: Dynamical evolution of a homogeneous spherical dust cloud collapse, as described by the Oppenheimer-Snyder-Datt solution.

Perturbations and stability

Figure: Under construction

LESSON III: ELECTRICALLY CHARGED BLACK HOLES. THE REISSNER-NORDSTRÖM METRIC

The Reissner-Nordström black hole - I

- We look now for the metric outside a static spherically symmetric charged body. A (electric) Maxwell field is described by the Lagrangian density

$$\mathcal{L}_m = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{89}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor of the vector potential A_μ.

- Einstein equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ with

$$T_{\mu\nu} = -\frac{1}{4} \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \tag{90}$$

- The electromagnetic field satisfies the field equations

$$\nabla_\mu F^{\mu\nu} = 0 \tag{91}$$

and the Bianchi identities hold $\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0$.

- (Electro)-static, static, spherically symmetric solutions satisfy $A^\mu = (\phi(r), 0, 0, 0)$ so that the only non-vanishing component of the field strength tensor is $F_{tr} \equiv E(r) \neq 0$.

The Reissner-Nordström black hole - II

- For the line element
$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2 \tag{92}$$
Maxwell field equations become
$$\nabla_\mu F^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0 \to \partial_r(\sqrt{AB}r^2 F^{tr}) = 0 \tag{93}$$
and are immediately integrated as $E(r) = Q/(r^2\sqrt{AB})$, where Q is an integration constant associated to the electric charge. In flat (Minkowski) space-time, $A = B = 1$, and we recover standard's Coulomb field.
- Integration of the field equations: first note that $T^t{}_t = T^r{}_r$ which entails $A = B^{-1}$, and one write the remaining Einstein equations as
$$\frac{d}{dr}(rA(r) - r) = -8\pi Q^2/r^2 \tag{94}$$
$$\frac{d^2}{dr^2}(rA(r)) = 16\pi Q^2/r^3 \tag{95}$$
which form a compatible set with Maxwell equations via $\nabla^\mu G_{\mu\nu} = 0$.
- Integration and interpretation of constants yields:
$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} + r^2 d\Omega^2 \tag{96}$$
which is the Reissner-Nordström solution (1916-1921) in Schwarzschild coordinates.

Properties of the Reissner-Nordström black hole - I

- The Reissner-Nordström solution has *two* coordinate singularities at $g_{tt} = g^{rr} = 0$ as

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} \tag{97}$$

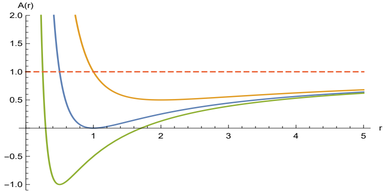


Figure: The Reissner-Nordstöm black hole $M = 1$ and $Q^2 = 1/2, 1, 2$.

- These zeros of are interpreted as horizons, yielding three scenarios:
  - $M^2 > Q^2$: There are two horizons and three regions. The external is the event horizon and the internal is called the **Cauchy horizon**. The external region is the asymptotic one, in the intermediate t and r change their nature (as in the Schwarzschild one) and in the innermost the time-like character of the region is restored.
  - $M = Q^2$: Event and Cauchy horizon meet at $r = M$. This is a **extremal black hole**.
  - $M < Q^2$: No horizon and t is timelike. A **time-like naked singularity**.

Properties of the Reissner-Nordström black hole - II

- **Mass inflation**: An instability of the inner horizon caused by counterstreaming fluxes of ingoing and outgoing fluxes crossing the inner horizon. It can thus be argued that the Cauchy horizon is the true singularity in a Reissner-Nordström black hole.
  - E. Poisson and W. Israel, “Inner-horizon instability and mass inflation in black holes,” Phys. Rev. Lett. **63** (1989), 1663
  - A. Ori, “Inner structure of a charged black hole: An exact mass-inflation solution,” Phys. Rev. Lett. **67** (1991) 789.
- **Cosmic censorship conjecture**: In a generic stellar evolution, any singularity are hidden behind an event horizon, so the overcharged Reissner-Nordström solution is (classically) forbidden.
- **The sources problem**: Since Birkhoff’s theorem tells us that the external solution to a Schwarzschild/Reissner-Nordström black hole is the same regardless of the distribution of the matter behind their event horizons, is it possible to have all the matter piled up at the center (where it will end up eventually in the Sch case given the space-like nature of its singularity) generating the geometry?. The answer is **NO**, since it is mathematically inconsistent to have a delta-source type at the center containing all the mass and charge of the system while at the same time being a solution of the Einstein-Maxwell equations everywhere.

Thermodynamics - I

- Consider a black hole which undergoes a small pertubation and settles down into a new stationary state with small changes to its mass dM, charge δQ, angular momentum δJ, and horizon area dS
- The black hole horizon has a surface gravity defined by κ, $k^a \nabla_a k^b = \kappa k^b$, where k^a is a Killing vector for a stationary black hole is constant. For a static, spherically symmetric black hole the surface gravity reads

$$\kappa = \frac{1}{2} \frac{\partial g_{tt}}{\partial r} \Big|_{r=r_h} \tag{98}$$

- The four laws of black hole mechanics:
    - **Zeroth law**: κ is constant over the horizon.
    - **First law**: The perturbed black hole quantities satisfy

$$dM = \frac{\kappa}{2\pi} dA + \Phi dQ + \Omega dJ \tag{99}$$

      where $\Phi = \int_{r_h}^{\infty} E(r) dr$ the electric potential at the horizon and Ω the angular velocity (when the extension to non-vanishing rotating is implemented).
    - **Second law**: $dA \geq 0$ as long as the null energy condition holds for any observer crossing the event horizon.
    - **Third law** (conjectured): No set of finite physical processes can reduce κ down to zero.

Thermodynamics - II

- The four laws have the same formal form as those of standard thermodynamics provided that one identifies the surface gravity with the temperature as $T = \kappa/2\pi$ and with A playing the role of the entropy.
- However, black holes are, well, black, so how could they have a temperature?.
- Bekenstein argued in 1973 that to preserve the second law of thermodynamics there should be an entropy associated to the black hole horizon as $A/\hbar$ so that a generalized second law would read $dS_{gen} \geq 0$.
- In 1974 Hawking announced that black holes are not black: vacuum fluctuations of the quantum fields in the vinicity of the horizon yields particles which propagate to asymptotic infinity and others that get trapped in the event horizon, resulting in a black body radiation with temperature $T = \hbar k/2\pi$. Isolated black holes evaporate with time!.
- For a Schwarzschild black hole, the evaporation time scales as $t \sim M^3/\hbar$, which for a one solar mass black hole amounts to $\sim 10^{67}$ years to evaporate!.

- For Reissner-Nordstöm black hole, the temperature can be computed as (Exercise [easy] verify it; [hint]: you need to use the horizon equation to obtain the final solution):

$$T = \frac{1 - 8\pi r^2 T^0{}_0(r_h)}{4\pi r_h} = \frac{1 - 8\pi Q^2/r_h^2}{4\pi r_h} \tag{100}$$

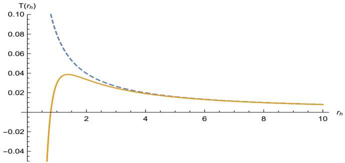


Figure: Temperature for Schwarzschild (dashed) and Reissner-Nordström (solid).

- There are further thermodynamical quantities useful in the characterization of black holes. For instance, the specific heat

$$C_Q = \left.\frac{\partial M}{\partial T}\right|_Q \tag{101}$$

yields stable/unstable solutions depending on its positive/negative sign.

Exercise : Obtain C_Q for Schwarzschild and Reissner-Nordström BHs. Are there any unstable solution? [hint: remember that M and T are functions of r_h. Use the chain rule!].

- Last stages of evaporation process, reliable of neglecting back-reaction, information loss problem?.

The information loss problem

- A central tenet to quantum mechanics is **unitarity**: a complete quantum system in a pure state remains in a pure state. In other words, information must be preserved from a quantum mechanics perspective.
- If a pure quantum state enters a black hole, the transformation of that state into the mixed state of into the mixed state of Hawking radiation violates the unitarity and then destroys information about the original quantum state.
- Proposed solutions:
  - Information is unavoidably lost - hence quantum mechanics is wrong?.
  - Information is gradually and somewhat leaked out via the evaporation process.
  - A black hole remnant stores all the missing information.
  - Information never entered the black hole (firewall paradigm).

- Motivated by the late-time acceleration of the Universe, whose simplest implementation is via the addition of a cosmological constant term to Hilbert's action:

$$S_{EH\Lambda} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R-2\Lambda) \tag{102}$$

so that Einstein's equations read $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}$.

- Spherically symmetric electrovacuum solutions are easily found as

$$f(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \frac{r^2}{l^2} \tag{103}$$

where the cosmological constant length is defined as $\Lambda = -3/l^2$, characterizing asymptotically anti-de Sitter spaces ($l > 0$), de Sitter ones ($l < 0$) and Minkowski otherwise ($l = 0$).

- In this case the mass function (solution of $f(r_h) = 0$ reads as

$$M(r_h) = \frac{r_h}{2}\left(1 + \frac{r_h^2}{l^2}\right) + \frac{Q^2}{2r_h} \tag{104}$$

and the temperature as

$$T = \frac{3r_h^4 + l^2 r_h^2 - l^2 Q^2}{l^2 r_h^3} \tag{105}$$

- The interest in the de Sitter branch comes from the fact that, in addition to the solutions having two horizons or a single extreme (degenerate) one, one may also find solutions with up to three horizons. In all cases, the largest such horizon is known as a **cosmological horizon** rather than the event one.
- The interest in the anti-de Sitter branch comes from the behaviour of the temperature function, where a relative minimum can be found in some cases.

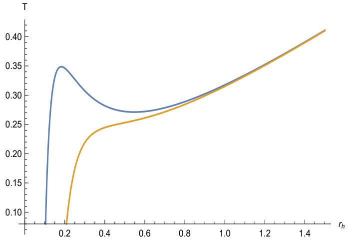


Figure: Temperature for RN-AdS black holes for $l = 1$ and $Q = 1/10$ (blue) and $Q = 1/5$ (orange).

- Its interpretation can be revealed by having a glance at the specific heat at constant charge:

$$C_Q \equiv \frac{\partial M}{\partial T}\Big|_Q = \frac{\partial M}{\partial r_h}\Big|_Q \left(\frac{\partial T}{\partial r_h}\right)^{-1}\Big|_Q \tag{106}$$

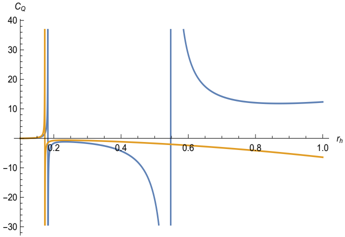


Figure: Specific heat for RN-AdS black holes with $q = 1/10$ and $l = 1$ (blue), and its comparison with the standard RN case, $l \to \infty$ (orange).

- A new branch of stability in $C_Q > 0$ is found for $r_h > r_{min}$. The first-order transition between the RN-AdS black hole and the thermal AdS space is known as the Hawking-Page transition.

Higher dimensions and topological black holes - I

- In higher $D=(n+1)$ dimensions one needs to generalize the spherically symmetric line element as
$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega_{D-2}^2 \tag{107}$$
where $d\Omega_{D-2}^2 = d\theta_1^2 + \sum_{i=1}^{D-2}\prod_{j=1}^{i-2}\sin^2\theta_j d\theta_1^2$ is the metric on the unit $(D-2)$ sphere.
- The corresponding electrovacuum solution is given by
$$f(r) = 1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2(n-2)}} + l^2 r^2 \tag{108}$$
where now the cosmological constant has been rescaled as $\Lambda = -\frac{n(n-1)}{2l^2}$.
- The integration constants m and q are related to the physical (ADM) mass and charge as
$$M = \frac{(n-1)\omega_{n-1}m}{16\pi G}; Q = \sqrt{2(n-1)(n-2)}\frac{\omega_{n-1}}{8\pi G}q \tag{109}$$
where ω_{n-1} is the volume of the $(n-1)$ sphere.
- As usual, horizons are found at the locations of $f(r_h)=0$ and, in particular, the solutions will described a charged black hole with a non-singular horizon at $r=r_+$ if
$$\frac{n}{n-2}r^{2n-2} + l^2 r_h^{2n-4} \geq q^2 l^2 \tag{110}$$

Higher dimensions and topological black holes - II

- The laws of black hole thermodynamics can be formulated in the usual way, with the electric potential given by $\Phi = \sqrt{\frac{n-1}{2(n-2)}} \frac{q}{r^{n-2}}$.
- For instance, the temperature can be obtained as

$$T = \frac{nr^{2n-2} + (n-2)l^2 r^{2n-4} - (n-2)q^2 l^2}{4\pi l^2 r^{2n-3}} \tag{111}$$

- However, the obtained solution is NOT the most general one. Indeed, one can generalize spherical symmetry to

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 \sigma_{ij} dx^i dx^j \tag{112}$$

  where $\sigma_{ij} dx^i dx^j$ represents the line element of a $(n-1)$ dimensional hypersurface with constant curvature $(n-1)(n-2)k$.
- Solutions with negative constant or vanishing curvature are known as **topological black holes**.
- Since without loss of generality one may take $k = 1, -1, 0$, the corresponding electrovacuum solution reads

$$f(r) = k - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2(n-2)}} + l^2 r^2 \tag{113}$$

  where $k = 0$ represents the usual spherical topology for the horizons of the asymptotically flat case.
- Due to the different horizon structures, these topological black holes behave in many aspects quite different from their spherically symmetric counterparts.

The BTZ solution - I

- The $2+1$-dimensional Einstein-Λ system admits an exact static, spherically symmetric solution as
$$ds^2 = -(N^{\perp})^2 dt^2 + f^{-2} dx^2 + x^2 (d\phi + N^{\phi} dt)^2 \tag{114}$$
with the definitions
$$N^{\perp} = f = \left(-M + \frac{x^2}{l^2} + \frac{J^2}{4x^2}\right)^{1/2} ; N^{\phi} = -\frac{J^2}{2x^2} \tag{115}$$
where M is the ADM mass of the system, $l = -\Lambda^{-2}$ is the AdS length and J is the angular momentum.
- It can be alternatively rewritten as
$$ds^2 = \left(M - \frac{x^2}{l^2}\right) dt^2 + f^{-2} dx^2 + x^2 d\phi^2 - J dt d\phi \tag{116}$$
- Ergohorizons are given by the solutions of $g_{tt} = 0$:
$$x_{erg} = l M^{1/2} \tag{117}$$
- The Killing horizons of this geometry are given by the zeroes of $g^{xx} = f^2$, which are found as
$$x_{\pm}^2 = \frac{Ml^2}{2}\left(1 \pm \left[1 - \left(\frac{J}{Ml}\right)^2\right]\right) \tag{118}$$
corresponding to the event and inner horizons, respectively.

The BTZ solution - II

- Five classes of configurations:
  1. For $M > 0$ and $|J| \leq Ml$: a spectrum of black holes (covered by an ergohorizon).
  2. For $M > 0$ and $|J| = Ml$: extreme black holes.
  3. For $M > 0$ and $|J| < Ml$: naked singularities.
  4. For $M = J = 0$: a massless black hole, corresponding to purely AdS space
  5. The $M = -1, J = 0$ state corresponds to a regular de Sitter core, disconnected from the black hole spectrum by a mass gap.
- The BTZ can be generalized to the Einstein-Maxwell-Λ system as

$$ds^2 = \left(M - \frac{x^2}{l^2} + \frac{Q^2}{2}\log x\right) dt^2 + f^{-2}dx^2 + x^2 d\phi^2 - Jdtd\phi \tag{119}$$

  with $f = \left(-M + \frac{x^2}{l^2} + \frac{J^2}{4x^2} - \frac{Q^2}{2}\log x\right)^{1/2}$ and Q the electric charge.
- The ergohorizons are found by solving $g_{tt} = 0$, which amounts to

$$x^2 - \frac{Q^2 l^2}{2}\log x - l^2 M = 0 \rightarrow x_{\pm} = \pm\frac{lQ}{2} PL\left[\frac{-4e^{-4M/Q^2}}{l^2 Q^2}\right]^{1/2} \tag{120}$$

  where PL denotes the principal solution of Lambert's W function. This represents two real solutions depending on the combination of the values of $\{l, Q, M\}$.
- Unfortunately, in the charged case the regular BTZ solution is no longer present.

Multicenter solutions - I

- The Majumdar-Papapetrou family is a particular class of solutions solving Einstein equations
$$G_{\mu\nu} = \kappa^2 (T^m_{\mu\nu} + T^{em}_{\mu\nu}) \ , \tag{121}$$
where $T^{em}_{\mu\nu}$ corresponds to an electromagnetic field and $T^m_{\mu\nu} = \rho\, u_\mu u_\nu$ to a pressureless (time-like) fluid.
- The matter field equations read
$$\nabla_\mu F^{\mu\nu} = 4\pi J^\nu \ , \tag{122}$$
where the charge density $J^\nu = \rho_e u^\nu$.
- The MP class assumes the following relation between the metric and the electrostatic potential $A_\mu = \delta^t_\mu \phi$
$$g_{tt} = \left(C \pm \frac{\phi}{\sqrt{2}} \right)^2 \ . \tag{123}$$
with C some constant. Enforcing this condition amounts to $\rho_e = \rho$, which is known as an extreme counterpoised dust.
- In Cartesian coordinates the MP metric can be expressed as
$$ds^2 = -\frac{1}{U(x,y,z)^2} dt^2 + U(x,y,z)^2 d\vec{x} \cdot d\vec{x} \ , \tag{124}$$
where $U = \pm \frac{\sqrt{2}}{\phi}$, while the matter field equations yield $U = -4\pi\rho\, U^3$.

Multicenter solutions - II

- Any collection of extreme Reissner-Nordström black hole solutions ($m^2 = q^2$) located at will is a particular MP solution without any need to impose additional symmetries. For this reason, these configurations are sometimes called *multicenter* solutions.
- Finding additional solutions of interest typically involve the spherical symmetry restriction:

$$ds^2 = -U^{-2}(R)dt^2 + U^2(dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)) \, , \tag{125}$$

so that all functional dependencies are expressed in terms of R.
- A further coordinate change brings the line element into standard Schwarzschild form:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \, , \tag{126}$$

via the identifications $r = RU(R)$, $A(t) = U^{-2}(R)$ and $B(r)^{-1/2} = 1 + \frac{R}{U}dU(R)/dR$.
- Either under (125) or (126), there are two paths to solve the MP field equations: i) to assume a functional form for $U(R)$ and solve the matter field equation to find the energy density $\rho(R)$ or ii) a function $\rho(R)$ for the inner region is set and resort to numerical methods to resolve the corresponding differential equation in order to get $U(R)$.
- Example of the first path are the so-called Bonnor stars defined by two regions as

$$U^E = 1 + \frac{m}{r}, r \geq r_0 \tag{127}$$

$$U^I = 1 + \frac{m}{r_0} + \frac{m(r_0^2 - r^2)}{2r_0 r_0^3}, 0 \leq r \leq r_0 \, , \tag{128}$$

where the exterior solution, U^E corresponds to an extreme Reissner-Norström black hole, which is matched to the interior solution at a certain $r = r_0$.

Bonnor-Vaidya solutions

- The dynamical Vaidya solution can be generalized to account for the presence of a charge, yielding the Bonnor-Vaidya solution as

$$ds^2 = -\left(1 - \frac{r_S(v)}{r} + \frac{r_q^2(v)}{r^2}\right)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{129}$$

where now the mass radius $r_S(v)/2 = M + \int L(v)$ and charge radius $r_q^2(v) = \kappa^2 q^2(v)/(8\pi)$ are related to the density of the ingoing stream of particles as

$$\rho_{in} r^2 = \frac{2}{\kappa^2}\left(L(v) - \frac{\kappa^2 q q_v}{r}\right) \tag{130}$$

- In this system, $L(v)$ and $q(v)$ are free functions, whose dependence on v reflects the presence of the charged stream of null particles.
- For vanishing charge, $q(v) = 0$, one gets the usual Vaidya solution.

Properties of the RN black hole - mass inflation

- Recall that the energy-momentum associated with various massless test fields diverges at a null hypersurface of the Cauchy horizon, triggering the development of an instability[7].
- Model the infinitely blue-shifted radiation by an ingoing spherically symmetric stream of massless particles, and add another flux of outgoing infinitely red-shifted massless particles to model a piece of the ingoing field which has been backscattered by the hole's curvature to become outgoing.
- Construct an exact mass inflation solution by matching two patches of Bonnor-Vaidya solutions (at constant q) outside (region I) and inside (region II) the inner horizon, and through an outgoing null "thin layer", using suitable double null coordinates $\{U, V\}$.
- Three matching conditions arise at and across the layer. These result into an expression for the mass of the thin layer $\Delta m(u) = m_2(u) - m_1(u) \propto R'(u)$, where the function $R(u)$ is the value of r along S, i.e., $R(\lambda) = r(V = u, U = 0)$.
- Computation of this quantity yields the result:

$$\Delta m(u) \propto v_1^{-p} \exp(k_0 v_1) \tag{131}$$

  where $k_0 = (2r_0)^{-1}(e^2/r_0^2 - 1)$, v_1 is the advanced time in region I, while $p \geq 12$ comes from the mass contribution associated with the radiative tail dominating the late-time behaviour of realistic perturbations.
- Since the local curvature $K \propto m^2$ this entails its divergence at $v_1 \to \infty$, i.e., the interaction between such fluxes makes the development of *mass inflation*. This curvature-singularity, however, is weak, since the tidal forces (to be integrated twice) scale with the proper time as $\tau^2 |\log(\tau)|^{-p}$ as $\tau \to 0$.

[7] A. Ori, Phys. Rev. Lett. **67** (1991), 789-792.

LESSON IV: STATIONARY AXISYMMETRIC SOLUTIONS. THE KERR(-NEWMAN) BLACK HOLE

Stationary axisymmetric solutions

- Stationary metric outside a rotating massive body.
- Metric coefficients are independent of the timelike coordinate $t(\equiv x^0)$ and the azimutal angle $\phi(\equiv x^3)$, that is $g_{\mu\nu} = g_{\mu\nu}(x^1, x^2)$. Besides, the rotating body is invariant under simultaneous inversion $t \to -t, \phi \to -\phi$, thus $g_{01} = g_{02} = g_{13} = g_{23} = 0$. Thus the metric becomes

$$ds^2 = g_{00}dt^2 + 2g_{03}dtd\phi + g_{33}d\phi^2 + [g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_22(dx^2)^2] \qquad (132)$$

  This can be rewritten as

$$ds^2 = Adt^2 - B(d\phi - \omega dt)^2 - Cdr^2 - Dr^2 \qquad (133)$$

- The Kerr metric is the geometry of a space-time outside a rotating matter distribution, that is, an axisymmetric solution of vacuum Einstein equations $R_{\mu\nu} = 0$. BUT, there are less equations than unknown variables!. GR equations are insufficient to determine all the functions uniquely.
- Additional constraints are needed:
  - Space-time geometry reduces to Minkowski for $r \to \infty$.
  - There exists a smooth closed convex event horizon outside which the geometry is non-singular
- Under these conditions, the solution is **UNIQUE** (Theorems by Carter, Israel,...)

The Kerr metric

- Due to lack of spherical symmetry physicists struggled to find the solution until 1963 (Kerr). Mental note: no Birkhoff's theorem!.
- In Boyer-Lindquist form, the Kerr metric reads[8]

$$
\begin{aligned}
ds^2 &= \left(1-\frac{r_S r}{\Sigma}\right)dt^2 + \frac{2r_S ar\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 && (134)\\
&- \left(r^2+a^2+\frac{r_S ra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 && (135)
\end{aligned}
$$

with the definitions

$$
\begin{aligned}
r_S &= 2M && (136)\\
\Sigma &= r^2+a^2\cos^2\theta && (137)\\
\Delta &= r^2-r_S r+a^2 && (138)
\end{aligned}
$$

[8] R. Kerr, "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics" Physical Review Letters. 11 (1963) 237

The Kerr and Kerr-Newman metrics

- The Kerr metric describes the spacetime geometry outside of body of mass M rotating angular momentum $J = aM$ (a is the spin parameter). This follows from asymptotic limit:

$$\begin{aligned} ds^2 &= -\left[1 - \frac{2M}{r} + \mathcal{O}(r^{-2})\right] dt^2 + \left[\frac{4aM\sin^2\theta}{r} + \mathcal{O}(r^{-2})\right] dtd\phi \\ &+ \left[1 + \mathcal{O}(r^{-1})\right] (dr^2 + r^2 d\Omega^2) \,, \end{aligned} \tag{139}$$

  which is Lense-Thirring metric of a rotating spherical body on its weak-field limit.
- The Kerr metric can be extended to include the electric charge via the vector potential

$$A_\mu = (Qr/\rho^2, 0, 0, -aQr\sin^2\theta/\rho^2) \tag{140}$$

- Just add a contribution Q^2 in $\Delta = r^2 - r_S r - a^2$ of Kerr to obtain Kerr-Newman.
- Theorem (of uniqueness): The most general axisymmetric solution of the Einstein vacuum field equations is given by the Kerr-Newman one, characterized by mass M, charge Q, and angular momentum J.

Properties of the Kerr metric - I

- Two physically relevant surfaces. The first one, given by $g^{rr} = \Delta = 0$, reads

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} \tag{141}$$

- This is the analogue of both the event horizon and the Cauchy horizon of the spherically symmetric counterpart, provided that $M^2 \geq a^2$. For the $M = a^2$ we have extreme black holes, which set the limit of maximum rotation to $J = Ma$ (angular momentum is limited by mass).
- The second one corresponds to $g_{tt} = 0$:

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 \cos^2 \theta} \tag{142}$$

Due to the contribution of the $\cos^2 \theta$, this is an outer surface representing a flattened sphere that touches the event horizon at the poles of the rotation axis $\theta = \pi/2$.

Properties of the Kerr metric - II. Ergosphere

- The space between these two surfaces is known as the **ergosphere**. Since in this region g_{tt} is negative, any particles suffers a *f*rame dragging effect, that is, it unavoidably co-rotates with the black hole.
- To prove it, let us consider the emission of a photon at $r = \theta =$constant. The corresponding line element yields

$$0 = ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \tag{143}$$

which allows to solve for the corresponding angular velocity as

$$\Omega_{\pm} \equiv \frac{d\phi}{dt} = \frac{-g_{t\phi}}{g_{\phi\phi}} \mp \frac{\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} \tag{144}$$

- Therefore, for an observer with four-velocity $u^{\mu} = (u^t, 0, 0, u^{\phi})$ and $ds^2 > 0$, and bearing in mind that $g_{t\phi} < 0$, the values permitted for the angular velocity are given by

$$\Omega_- < \frac{d\phi}{dt} < \Omega_+ \tag{145}$$

- For every other observer outside of the ergoregion, in which $g_{tt} > 0$ it is verified $\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}/g_{\phi\phi} < g_{t\phi}/g_{\phi\phi}$ so that $\Omega_- < 0$ and therefore there is no restriction in the sign of the angular velocity.
- The above fact is a manifestation of the *frame-dragging effect*, namely, the state of rotating of a body affects the local notion of inertial reference system.

Properties of the Kerr metric - III. Ergosphere and ZAMOs

- The closest analogue to an "static" observer in the ergoregion to a rotating body is the one who is at rest with respect to the t =constant hypersurfaces, i.e., satisfying $u^\mu \propto \nabla^\mu t = (g^{tt}, 0, 0, g^{t\phi}$.
- These are the observers who have zero angular momentum (*ZAMO*: zero angular momentum observers))in their proper frame at large distances.
- Since $p^\mu \propto u^\mu$ one has $p_\phi = g_{t\phi} p^t + g_{\phi\phi} p^\phi \propto g_{t\phi} g^{tt} + g_{\phi\phi} g^{t\phi} = g_{\mu\phi} g^{t\mu} = \delta^t_\phi = 0$ and using the fact that $p^t = g^{tt} p_t$ and $p^\phi = g^{t\phi} p_t$ (since for these observers its momentum in the direction of the rotation axis vanishes, $p_\phi = 0$) we can compute

$$\frac{d\phi}{dt} = \frac{d\phi}{du}\frac{du}{dt} = \frac{u^\phi}{u^t} = \frac{p^\phi}{p^t} = \frac{g^{t\phi}}{g^{tt}} = -\frac{g_{t\phi}}{g_{\phi\phi}} \tag{146}$$

- For the Kerr black hole this yields the result

$$\Omega_{ZAMO} = \frac{a(r^2+a^2-\Delta)}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta} \tag{147}$$

- At the event horizon location, $r \to r_+$, this yield the result (since $\Delta = 0$ there)

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{148}$$

- Since this is the largest velocity every observer can take within the ergosphere, it is usually understood as the velocity the event horizon itself rotates.

Properties of the Kerr metric - III

- Note that the zeroes of both g_{tt} and g^{rr} correspond to *non-essential singularities*, that is, they are simply coordinate singularitie). From the curvature scalar $R^{\alpha}{}_{\beta\gamma\delta}R_{\alpha}{}^{\beta\gamma\delta} \sim 1/\rho^6$ one finds intrinsic singularities of the metric at

$$\Sigma = 0 \to (r = 0, \theta = \pi/2) \tag{149}$$

Since this corresponds to the edge of a disk it is called a **ring singularity**.
Exercise : Is it the ring singularity avoidable by any observer?.

Properties of the Kerr metric - IV

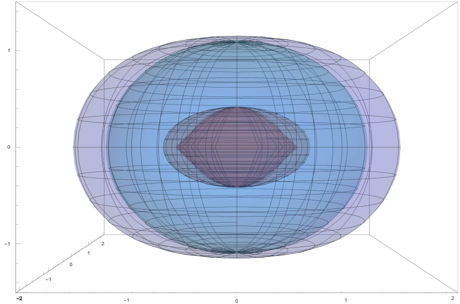


Figure: The Kerr-Newman black hole

Exercise : Can you write the Kerr metric in Eddington-Filkenstein coordinates?.

Exercise : Can particles run away from the ergosphere?.

Exercise : Can the ring singularity be avoided?.

Properties of the Kerr metric - III: Energy extraction

- **Penrose process** : suppose an asymptotic observer A with 4-velocity $u^\mu = \partial_t = (1,0,0,0)$ who drops into the black hole a particle with momentum p_A and energy $E_A = p_A t$, which subsequently decays into two particles as $E_A = E_B + E_C$.
- Within the ergoregion, where $g_{tt} < 0$, the vector $\xi = \partial_t$ is space-like, and therefore $p_t = p_\mu \xi^\nu$ is not necessarily positive. Therefore, it is possible for particle B to have $E_B < 0$, which implies that the asymptotic observer would measure an energy $E_C = E_A - E_B = E_A + |E_B| > E_A$ so that the emitted particle has a larger energy than the original one.
- This energy is extracted from the rotational energy of the black hole, which is therefore slowed down.
- **Superradiance**: The increase in the dispersed radiation by a body as compared to the incident one.
- In the simplest case one considers a scalar field $\psi = Re(\psi_0 e^{-i\omega t} e^{i\omega\phi})$ scattered against the black hole. One can then compute the temporal average of the scalar field flux through the normal n^μ to the event horizon as

$$\begin{aligned} < j_\mu n^\mu > &= < (\nabla_t \psi)^2 > + < (\nabla_t \psi)(\nabla_\phi \psi) > = < \mathrm{Re}(\psi_0)^2 \omega^2 \cos^2(m\phi - \omega t) \\ &+ \mathrm{Im}(\psi_0)^2 \omega^2 \sin^2(m\phi - \omega t) > - < \mathrm{Re}(\psi_0)^2 \omega m \cos^2(m\phi - \omega t) + \\ &+ (\mathrm{Im}\psi_0)^2 \omega m \sin^2(m\phi - \omega t) > \frac{1}{2}\omega(\omega - m\Omega_H)|\psi_0|^2 \qquad (150) \end{aligned}$$

- For frequencies $0 < \omega < m\Omega_H$ (where $\Omega_H = a/(r_+^2 + a^2)$ is the angular velocity on the horizon) this flux is negative, which means that the energy measured by an asymptotic observers is larger than the energy of the ingoing wave: this is the superradiance effect.

The Newman-Janis trick - I

- It is possible to generate axisymmetric solutions out of a spherically symmetric seed of the form
$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \qquad (151)$$
using a procedure introduced in[9]. It works as follows:
- Introduce advanced null coordinates $du = dt - f^{-1}dr$ so that the metric becomes
$$ds^2 = -fdu^2 - 2dudr + r^2 d\Omega^2 \qquad (152)$$
- To write the contravariant components of the metric in terms of the null tetrad $\{l^\mu, l^\nu, m^\mu, \bar{m}^\mu\}$ as $g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu$ via the identifications
$$l^\mu = \delta^\mu_r; n^\mu = \delta^\mu_u - \frac{f}{2}\delta^\mu_r; m^\mu = \frac{1}{\sqrt{2}r}\left(\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\phi\right) \qquad (153)$$
- Allow the coordinates u and r to take complex values. This complexification process is not unique. Common choices are $r \to \frac{1}{2}(r+\bar{r}) = \mathrm{Re}(r)$ and $r \to \frac{1}{2}(r^{-1}+\bar{r}^{-1})$.
- Perform a complex change of coordinates
$$u = u' + ia\cos\theta; r = r' - ia\cos\theta; \theta' = \theta, \phi' = \phi \qquad (154)$$

9 E. T. Newman and A. I. Janis, "Note on the Kerr Spinning-Particle Metric", Journal of Mathematical Physics 6.6 (1965) 915.

The Newman-Janis trick - II

- Write the tetrads in terms of the new coordinates

$$l'^{\mu}=\delta^{\mu}_{r};\, n'^{\mu}=\delta^{\mu}_{u}-\frac{\tilde{f}}{2}\delta^{\mu}_{r};\, m'^{\mu}=\frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(\delta^{\mu}_{\theta}+\frac{i}{\sin\theta}\delta^{\mu}_{\theta}-ia\sin\theta(\delta^{\mu}_{u}-\delta^{\mu}_{r})\right) \quad (155)$$

  where the original function $f(r)$ has been replaced by its complexified one $\tilde{f}(r,\bar{r})$, and a is a parameter interpreted as the angular momentum per unit mass.
- Construct the metric $g^{\mu\nu}$ from the new tetrad and lower the indices. In Kerr coordinates, this new metric reads

$$ds^2=-\tilde{f}(du-a\sin^2\theta d\phi)^2-2(du-a\sin^2\theta d\phi)(dr+a\sin^2\theta d\phi)+\rho^2 d\Omega^2 \quad (156)$$

  where $\rho^2=a^2+a^2\cos^2\theta$.
- Transforming to Boyer-Lindquist coordinates $du=dt'-g(r)dr$ and $d\phi=d\phi'-h(r)dr$ to remove the cross terms $g_{tr}=g_{r\phi'}=0$ amounts to the choice $g=\frac{r^2+a^2}{\Delta}$ and $h=a/\Delta$ with $\Delta=\tilde{f}\rho^2+a^2\sin^2\theta$ to finally achieve

$$ds^2=-\tilde{f}dt^2+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2+\frac{\Sigma^2}{\rho^2}\sin^2\theta d\phi^2+2a(\tilde{f}-1)\sin^2\theta dt d\phi \quad (157)$$

- This procedure works as long as Δ does not depend on θ, yet one still has to verify that the solution obtained this way does verify Einstein's equations.
- Examples that work: Kerr-Newman solution with $f=1-\frac{2M}{r}+\frac{Q^2}{r^2}$, non-linear electrodynamics...

LESSON V: SPACE-TIME SINGULARITIES

Space-time singularities

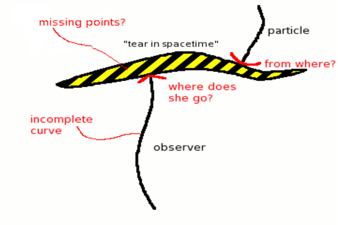
- Singularities signal a breakdown of GR in that its classical description cannot be expected to be valid at the extreme conditions near the center of black holes.
- On dimensional grounds one expects GR to break at scales characterized by Planck's length

$$l_P = \left(\frac{\hbar}{c^3}\right)^{1/2} \sim 10^{-33} cm \tag{158}$$

- If we see space-time singularities as indicative of a physically troublesome region, a natural guess is that something is going on ill with the geometry. Thus one would be tempted to define a space-time singularity as a "place" where curvature "blows up" or shows any other pathological behaviour.
- However, space-time singularities are fundamentally different from singularities on the fields living on a fixed space-time background! (e.g. Coulomb's divergence in classical electrodynamics). Indeed, a space-time is formally defined as a manifold M with a metric $g_{\mu\nu}$ living on it, so formally a space-time is not part of the manifold.
- We might be tempted to use curvature scalars (to avoid singularities that can be removed out of a coordinate choice) BUT
  - Space-times can be pathological despite all curvature scalars being finite.
  - Curvature divergences are not necessarily linked to any pathological behaviour!.

Geodesic incompleteness

- Where else can be look at?.
- A path of a particle in free-fall is given by a **time-like geodesic**. Light rays and the transmission of information are linked to null geodesics.
- It thus seems reasonable to demand any non-singular space-time to be, at least, null and time-like geodesically complete: in a physically consistent theory **nothing should cease to exist suddenly or "emerge" from nowhere!.**



- CAVEAT 1: We want to avoid the existence of space-times that are otherwise non-singular but have artificially removed points that can be mend by plugging back the removed patch. In technical terms this is call an **Inextendible space-times**: there is no way the space-time can be further extended.
- CAVEAT 2: There exist space-times which are time-like and null geodesically complete but contains inextendible time-like curves of bounded acceleration which have finite proper length (Geroch): that is, a rocketship with a sufficiently large but finite amount of fuel could end up its existence in finite proper time.

Raychaudhuri equation and focusing of geodesics - I

- Let us start with the Bianchi identity for a vector field

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) u^\alpha = R^\alpha{}_{\rho\mu\nu} u^\rho \tag{159}$$

- Contracting α with μ and then with u^ν one finds

$$u^\nu \nabla_\mu \nabla_\nu u^\mu - u^\nu \nabla_\nu \nabla_\mu u^\mu = R_{\rho\nu} u^\rho u^\nu \tag{160}$$

- Using Leibniz rule for the covariant derivative:

$$u^\nu \nabla_\nu \nabla_\mu u^\mu + \nabla_\mu u_\nu \nabla^\nu u^\mu - \nabla_\mu (u^\nu \nabla_\nu u^\mu) + R_{\rho\nu} u^\rho u^\nu = 0 \tag{161}$$

- If we assume u^μ to be a time-like geodesic vector field normal to a space-like hypersurface, then $u^\nu \nabla_\nu u^\mu = 0$ and the third term in the equation above vanishes
- Let us introduce now the spacial metric $h_{\mu\nu} = g_{\mu\nu} + u^\mu \nu^\nu$ and define the following objects:

$$\text{Expansion} \quad : \quad \theta \equiv h_{\mu\nu} \nabla^\nu u^\mu \tag{162}$$

$$\text{Shear} \quad : \quad \sigma_{\mu\nu} \equiv \nabla_{(\mu} u_{\nu)} - \frac{1}{3} \theta h_{\mu\nu} \tag{163}$$

$$\text{Twist} \quad : \quad \omega_{\mu\nu} \equiv \nabla_{[\nu} u_{\mu]} \tag{164}$$

Raychaudhuri equation and focusing of geodesics - II

- This way Raychaudhuri equation reads

$$u^\rho \nabla_\rho \theta = \frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\rho\nu}u^\rho u^\nu \tag{165}$$

- Parametrizing geodesics by their proper time, $u^\mu = \nabla^\mu \lambda$ makes the twist term to vanish.
- The term $\sigma_{\mu\nu}\sigma^{\mu\nu}$ is clearly non-negative because it is a"purely spatial" tensor.
- Then the sign of expansion will be determined by that of $R_{\rho\nu}u^\rho u^\nu$. If the latter is non-negative (congruence conditions), i.e.

$$R_{\rho\nu}u^\rho u^\nu > 0 \tag{166}$$

then it follows that

$$\frac{d\theta}{d\lambda} + \frac{1}{3}\theta^2 \leq 0 \rightarrow \frac{d\theta^{-1}}{d\lambda} \geq \frac{1}{3} \rightarrow \theta^{-1}(\lambda) \geq \theta_0^{-1} + \frac{\lambda}{3} \tag{167}$$

- The above equation implies that starting from the initial "time" θ_0 the time-like null geodesic will reach $\theta = -\infty$ within a proper time less than $3/|\theta_0|$. This is called a "focusing effect".
- Similarly, one can prove the focusing effect for null geodesics with the affine time interval $[0, 2/|\theta_0|$.

Energy conditions

- The focusing effect is a geometric condition independent of the particular theory. BUT, via GR equations
$$R_{\rho\nu}u^\rho u^\nu = T_{\mu\nu}u^\mu u^\nu \tag{168}$$
for u^ν null. Thus is $T_{\mu\nu}u^\mu u^\nu > 0$, the focusing effect holds. This is related to the **ENERGY CONDITIONS**.
- For a perfect fluid energy-momentum tensor (includes scalar and electromagnetic fields, among many others):
$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu} \tag{169}$$
where $u_\mu u^\mu = -1$ (time-like) with ρ and P energy density and pressure of the fluid. In a comoving system this energy-momentum tensor reads $T^\mu{}_\nu = \text{diag}(-\rho, p, p, p)$.
- There are four main kinds of energy conditions, which for a (very general) energy-momentum tensor $T^\mu{}_\nu = \text{diag}(-\rho, p_1, p_2, p_3)$ read
  - Null energy condition (NEC): $T_{\mu\nu}n^\mu n^\nu \geq 0 \to \rho + p_i \geq 0$ with n^μ a null vector. The energy density has to be positive if it is measured by an observer who goes through a null curve.
  - Weak energy condition (WEC): $T_{\mu\nu}u^\mu u^\nu > 0 \to \rho \geq 0, \rho + p_i \geq 0$ with u^μ a time-like vector: the energy density as measured by a local time-like observer is non-negative.
  - Strong energy condition (SEC): $T_{\mu\nu}u^\mu u^\nu \geq -T/2 \to \rho + \sum_i p_i \geq 0, \rho + P_i \geq 0$: gravity is attractive (violated in models of cosmic acceleration and inflation.
  - Dominant energy condition (DEC): $-T^\mu{}_\nu u^\nu$ is a future-oriented time-like or null quantity $\to \rho \geq 0, \rho \geq |p_i|$: speed of energy flow of matter is less than the speed of light.

Singularity theorems - I: Time-like geodesics

- A Cauchy surface Σ in a spacetime M is a closed achronal set satisfying that $D(\Sigma) = M$. A spacetime M is said to be globally hyperbolic if it contains a Cauchy surface.
- There are two main singularity theorems related to time-like geodesics.
- The **first theorem** assumes more but proves more too: If a (non-extensible) space-time $(M, g_{\mu\nu})$, satisfies the following conditions
  - $R_{\mu\nu} u^\mu u^\nu \geq 0$ for every time-like vector u^μ.
  - M contains a space-like surface S whose future expansion verifies that $\theta(p) \leq \theta_0 \leq 0 \forall p \in S$.

  then every future-pointing time-like curve starting from θ_0 has length at most $3/|\theta_0|$.
- Since global hyperbolicity seems to be too a strong condition which could be false in certain space-times, one can re-formulate the theorem to avoid this assumption, while playing the price that the trapped hypersurface must be compact, i.e., we are working with closed universes or with bounded regions of them, like black holes.
- In the **second theorem**, if a (non-extensible) strongly causal space-time $(M, g_{\mu\nu})$ satisfies the following conditions:
  - $R_{\mu\nu} u^\mu u^\nu \geq 0$ for every time-like vector u^μ.
  - M contains a compact, edgeless, achronal, smooth space-like hypersurface S with future convergence $\theta(p) \leq \theta_0 \leq 0$.

  then, there is at least one future inextendible future directed time-like geodesic starting in S whose length is no greater than $3/|\theta_0|$.

Singularity theorems - II: Null geodesics

- **First singularity theorem for null geodesics**: If a (non-extensible) space-time $(M, g_{\mu\nu})$, satisfies the following conditions
  - $R_{\mu\nu} u^\mu u^\nu \geq 0$ for every null vector u^μ.
  - M contains a non-compact, connected Cauchy hypersurface.
  - M contains a surface S whose future expansion verifies that $\theta(p) \leq \theta_0 \leq 0 \forall p \in S$.

  then $(M, g_{\mu\nu})$ contains at least one incomplete (future) null geodesic curve.
- **Second singularity theorem for null geodesics**:If a (non-extensible) space-time $(M, g_{\mu\nu})$, satisfies the following conditions
  - $R_{\mu\nu} u^\mu u^\nu \geq 0$ for every null vector u^μ.
  - Each (null or time-like) geodesic has a point such that $u_{[\rho} R_{\alpha]\beta\lambda[\mu} u_{\sigma]} u^\beta u^\lambda \neq 0$.
  - There are no closed causal curves.
  - At least one of the following properties hold:
    - M contains a compact achronal set without edge.
    - M contains a future-trapped surface.
    - There is a point $p \in M$ such that the expansion of the future-directed null geodesics emanating from p becomes negative along each geodesic in the congruence.

  then $(M, g_{\mu\nu})$ contains at least one incomplete time-like or null geodesic.

Important comments

- The non-extensibility requirement is essential: it implies that geodesic completeness cannot be restored by extending further the manifold to add further points to be occupied by the observer: for instance, Minkowski space-time with a single point removed it is geodesically incomplete, but not maximal, since it can be extended simply by restoring the point back to the manifold.
- This is a sufficient condition for singular space-times.
- Theorems **DO TELL US NOTHING** about the behaviour of curvature scalars. In other words, the divergence of some curvature scalars **DO NOT NECESSARILY IMPLY** the existence of a singularity, either in terms of geodesic completeness or other criteria. Actual counter-examples do exist.
- Indeed, the concept of geodesic completeness precludes any idea of “things” blowing up at some point!.
- BONUS TRACK: the development of a singularity is unavoidable in full gravitational collapse.

Upgrades of the singularity theorems

- A pattern can be identified in all singularity theorems: if a sufficiently differentiable manifold $\mathcal{M}$ satisfies:
  - A condition on curvature (to ensure focusing).
  - A condition on causality (to avoid uncomfortable situations like closed time-like curves and to guarantee the existence of geodesics with maximum proper time connecting events).
  - An initial/boundary condition to evolve data from.

  then $\mathcal{M}$ contains incomplete geodesics.
- Despite 60 years of research on this topic the theorems are still oblivious to the every other aspect of singularities (strength, character, removal) beyond the conditions under which occur.

Detailed theory of geodesics - I

- Geodesics are given by the equation

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}=0 \tag{170}$$

where $\Gamma^\mu_{\alpha\beta}$ are the components of the connection (here the Christoffel symbols).

- The geodesic equation can be derived from the action

$$\mathcal{L}=\frac{1}{2}\int d\lambda g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{171}$$

- Consider now a static, spherically symmetric line element

$$ds^2=-A(r)dt^2+B^{-1}(r)dr^2+r^2d\Omega^2 \tag{172}$$

therefore

$$\mathcal{L}=\frac{1}{2}\int d\lambda\left[-A(r)\dot{t}^2+B(r)^{-1}\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right] \tag{173}$$

- The momenta associated to the variables (t,r,θ,ϕ) are

$$p_t=-\frac{\partial L}{\partial\dot{t}};p_r=\frac{\partial L}{\partial\dot{r}}=\frac{\dot{r}}{B(r)};p_\theta=\frac{\partial L}{\partial\dot{\theta}}=r^2\dot{\theta};p_\phi=\frac{\partial L}{\partial\dot{\phi}}=r^2\sin^2\theta\dot{\phi} \tag{174}$$

Detailed theory of geodesics - II

- From the Hamiltonian $H = -p_t\dot{t} + p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ (for this case it actually coincides with the Lagrangian), the geodesic equations are written as

$$\dot{r}^\mu = \frac{\partial H}{\partial p_\mu} = g^{\mu\nu}p_\nu ; \dot{p}^\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2}(\partial_\mu g^{\alpha\beta})p_\alpha p_\beta \to \dot{p}_t = \dot{p}_\phi = 0 \tag{175}$$

Thus p_t and p_ϕ are constants of motion. These equations also imply that $dH/d\lambda = 0 \to H$ is another constant of motion. So, setting the motion at $\theta = \pi/2$ by convenience, we have:

$$p_t = \frac{dt}{d\lambda}A(r) = E; \quad p_\theta = \frac{d\phi}{d\lambda}r^2\sin^2\theta = L \tag{176}$$

$$2H = -\frac{p_t^2}{A(r)} + B(r)p_r^2 + \frac{p_\theta}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta} = -\frac{E^2}{A(r)} + \frac{\dot{r}^2}{B(r)} + \frac{L^2}{r^2} \tag{177}$$

- When $H \neq 0$ we can always re-scale $\lambda \to \lambda/\sqrt{2|H|}$ so only the sign of $H = k/2$ is physically relevant. Thus we can call $k = 0$ as null (photons), $k = -1$ as time-like (particles) and $k = +1$ as space-like. This way the geodesic equation reads:

$$\frac{A(r)}{B(r)}\left(\frac{dr}{d\lambda}\right)^2 = E^2 - A(r)\left(\frac{L^2}{r^2} - k\right)^2 \tag{178}$$

This is the geodesic equation for any static, spherically symmetric space-times. E and L are physically interpreted as the energy per unit mass and the angular momentum per unit mass.

Incomplete geodesics in the Schwarzschild space-time

- For a Schwarzschild black hole $A(r) = B(r) = 1 - r_S/r$, so that the geodesic equation reads

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - A(r)\left(\frac{L^2}{r^2} - k^2\right)^2 \tag{179}$$

This has the same structure as a particle with energy E moving in an effective potential

$$V_{eff} = \left(1 - \frac{r_S}{r}\right)\left(\frac{L^2}{r^2} - k\right) \tag{180}$$

Near the center of the black hole one has $V_{eff} = -\frac{r_S}{r}\left(\frac{L^2}{r^2} - k^2\right)^2$ which is an infinitely attractive potential towards the black hole center.

- Considering radial ($L = 0$) time-like ($k = -1$) geodesics, this equation is integrated as

$$\lambda(r) = \lambda_0 - \frac{2}{3}\sqrt{r^3/r_S} \tag{181}$$

where λ_0 is the value of the affine parameter at $r = 0$. These geodesics are incomplete in the future, because $r \geq 0$.

Exercise: Are there any more incomplete geodesic in the Schwarzschild geometry?.

Exercise [green]: Do the same with the Reissner-Nordström geometry, $a = 1 - r_S/r + Q^2/r^2$. Does it contains any incomplete geodesic? (hint: look for null radial geodesics).

Physical impact of curvature divergences - I. Congruences

- Curvature divergences can be used to estimate the physical effects of strong gravitational fields.
- Accepted criteria in the literature (Ellis, Schmidt):
  - STRONG singularity: all objects falling into it are crushed to zero volume.
  - WEAK singularity: objects have a change to remain bound together.
- IDEA: idealize a body as a set of points following geodesics of the background metric and study the evolution of the separation between nearby geodesics to determine its impact in the body.
- Mathematically one defines a CONGRUENCE OF GEODESICS labelled by two parameters: $x^\mu = x^\mu(\lambda, \xi)$, where λ: affine parameter, ξ: identifies the different geodesics in the congruence. For a given geodesic $u^\mu \equiv \frac{\partial x^\mu}{\partial \lambda}|_{\xi=const}$ is the tangent vector, and the separation between nearby geodesics (at fixed λ) is measured by the JACOBI VECTOR fields

$$Z^\mu \equiv \frac{\partial x^\mu}{\partial \xi} \tag{182}$$

which satisfies the **GEODESIC DEVIATION EQUATION**:

$$\frac{d^2 Z^\alpha}{d\lambda^2} + R^\alpha{}_{\beta\mu\nu} Z^\mu u^\nu = 0 \tag{183}$$

Impact of curvature divergences - II. Volume invariants

- Using an orthonormal tetrad parallel-transported along the congruence allows to define a basis $\{e_1, e_2, e_3\}$, such that the Jacobi fields can be projected into that basis as $Z^a e_a (a = 1, 2, 3)$.
- Solving the geodesic deviation equation yields six independent Jacobi fields: Z^a and $DZ^a/d\lambda$ starting at some point λ_i:
  - If $Z^a(\lambda_i)$ are not all zero then $Z^a(\lambda) = A^a{}_b(\lambda) Z^b(\lambda_i)$, where $A^a{}_b$ is some matrix (the identity at $\lambda = \lambda_i$).
  - If $Z^a(\lambda_i)$ are all zero then $Z^a(\lambda) = A^a{}_b(\lambda) \frac{dZ^b}{d\lambda}|_{\lambda=\lambda_i}$.
- With three linearly independent non-vanishg solutions $Z_i = Z_i^a e_a$ one can define a VOLUME ELEMENT as $V(\lambda) = det[Z_i^a, Z_i^b, Z_i^c]$ which can be written as

$$V(\lambda) = det[A(\lambda)] V(\lambda_i) \tag{184}$$

and then a STRONG SINGULARITY is met if

$$\lim_{\lambda \to 0} V(\lambda) = 0 \tag{185}$$

Exercise : prove that Schwarzschild's singularity is STRONG.

- By the principle of general covariance, observers with arbitrary motions should also have complete paths. *No discrimination is allowed between different observers - they all have the same right to live*.
- In presence of acceleration, the unitary tangent vector to the timelike curve $x^\mu(s)$ will not be parallel transported along it, but will change as

$$\frac{Du^\mu}{ds} \equiv u^\mu \nabla_\mu u^\nu = a^\nu \tag{186}$$

where a^μ is the acceleration vector and the covariant derivative is defined as usual as $\frac{DZ^\mu}{ds} = \frac{dZ^\mu}{ds} + \Gamma^\mu_{\rho\sigma} Z^\rho u^\sigma$.
- The above equation can be rewritten on a Frenet-Serret frame as

$$\frac{D\lambda_{(a)}}{ds} = \lambda_{(b)} A^{(b)}{}_{(a)} \tag{187}$$

where $A \equiv A^{(b)}{}_{(a)}$ is a 4×4 matrix of the form

$$A^{(b)}{}_{(a)} = \begin{pmatrix} 0 & k(s) & 0 & 0 \\ k(s) & 0 & \tau_1(s) & 0 \\ 0 & -\tau_1(s) & 0 & \tau_2(s) \\ 0 & 0 & -\tau_2(s) & 0 \end{pmatrix} \tag{188}$$

where the functions $\{k(s), \tau_1(s), \tau_2(s)\}$ are called the curvature, first torsion and second torsion of the curve $\gamma(s)$,

Accelerated observers - II

- If one assumes that there is only linear acceleration, i.e., there is no accelerated motion in the angular directions ($\tau_1 = \tau_2 = 0$), then the motion is described by

$$\left(\frac{dx}{d\lambda}\right)^2 + V_{eff}(x) = \left[E + \int_{x_0}^{x} \frac{k(\lambda)dx'}{\sqrt{1 + \frac{L^2}{r^2(x')}}}\right]^2 \tag{189}$$

where $V_{eff}(x) = C(x)(L^2/r^2(x) - k)$ is the usual effective potential of the geodesic case.

- One can use this expression to show that if we would like the accelerated curve to “escape” from the light-cone, it would need to achieve infinite energy, and consequently, the acceleration needed would also be infinite.
- Is there any physical scenario in which the trajectories of free-falling particles are complete, but some trajectories of accelerating observers are not?. Examples by Geroch (1968).

Regularity tests with scattering of waves

- As another test to verify that curvature divergences do not affect the well-posedness of physical laws in a given space-time, one can consider the propagation of scalar waves in such a background.
- Taking the scalar field equation $(\Box - m^2)\phi = 0$, we decompose it in modes of the form $\phi_{\omega,lm} = e^{-i\omega t} Y_{lm}(\theta,\varphi) f_{\omega,l}(x)/r(x)$, where $Y_{lm}(\theta,\varphi)$ represents spherical harmonics.
- Using the radial coordinate $y = \int dx/C$, this way the $f_{\omega,l}(x)$ are governed by a Schrödinger-like equation of the form

$$-f_{yy} + V_{eff} f = \omega^2 f \tag{190}$$

where

$$V_{eff} = \frac{r_{yy}}{r} + C(r)\left(m^2 + \frac{l(l+1)}{r^2}\right) \tag{191}$$

- Using the mode decomposition, we thus define an incoming wave packet from past null infinity (in the naked case, or from the inner horizon in the black hole case) and study its interaction with the wormhole. The behaviour will depend dramatically on the angular momentum of the incident mode.
- One then needs to expand the above potential in the potentially problematic region. Compute transmission coefficients for a given frequency. Are they well defined or not?.

LESSON VI: EXOTIC AND ALTERNATIVE COMPACT OBJECTS

Black hole mimickers

- Black hole **mimicker** encompass a plethora of objects that may look as a black hole but it is conceptually distinguished from it given the absence of an event horizon, i.e., the yare horizonless.
- A full taxonomy on these objects is complicated given the fact that the mechanisms under which they can go disguised as black holes and under which observational channels vary from model to model. Moreover, a given model may contain at the same time both black hole (Kerr-like) and horizonless solutions.
- The newly born field of multimessenger astronomy allow us to perform cross-tests with different carriers to unveil the nature of a given black hole (mimicker) candidate.

Non-linear electrodynamics - I

- Charge is usually neglected in astrophysical settings, since black holes tend to discharge due to several processes, for instance, via neutralization because of the highly ionized environment around black holes.
- However, the presence of a tiny charge is very relevant to discuss the innermost regions of black holes. Near those regions, the strong gravitational fields can induce pair-production of charged particles inducing corrections to Maxwell electrodynamics. How to compute them?.
- Classical models of the electromagnetic field have been considered for decades in order to remove the divergence of electron's self-energy.
- At low curvatures, backreaction of the gravitational field can be neglected and such corrections can be incorporated via *effective models*.
- Effective models of electrodynamics can be captured by **non-linear electrodynamics**

$$\mathcal{S}_m = \frac{1}{8\pi}\int d^4x\sqrt{-g}\varphi(X,Y) \tag{192}$$

where

$$X = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu};\, Y = -\frac{1}{2}F_{\mu\nu}F^{*\mu\nu} \tag{193}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor of the vector potential A_μ, and $F^{\star\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is its dual.

Exercise: if we introduce electric $E^i = -F^{0i}$ and magnetic $B^i = -\frac{1}{2}\varepsilon^{ijk}F_{jk}$ fields, can you write $F^{\mu\nu}$ and $F^{*\mu\nu}$ (hint: matrix representations are always useful for this).

Exercise: Can you tell me the form of the objects X, Y? (hint: E^i, B^i are components of vectors, but X, Y need to be scalars).

NEDs - II. Action and field equations

- The gravitational action is thus written as

$$\mathcal{S} = \mathcal{S}_G + S_{NED} = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \varphi(X,Y)\right] \qquad (194)$$

- Matter field equations (Exercise : prove it).

$$\nabla_\mu\left[\varphi_X F^{\mu\nu} + \varphi_Y F^{*\mu\nu}\right] = 0 \qquad (195)$$

+ Bianchi identity $\nabla_\mu F^{*\mu\nu} = 0$
- Energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = 2\left(\varphi_X F_{\mu\alpha}F^{\alpha}{}_{\nu} - \varphi_Y F_{\mu\alpha}F^{*}\alpha_{\nu}\right) - g_{\mu\nu}\varphi(X,Y) \qquad (196)$$

- We look again for electrostatic spherically symmetric solutions $F^{tr} \equiv E(r) \neq 0$. In such a case $X = E^2$ and $Y = 0$ and the energy-momentum tensor reads

$$T^0{}_0 = T^1{}_1 = 2\varphi_X E^2 - \varphi;\ T^2{}_2 = T^3{}_3 = -\varphi \qquad (197)$$

NEDs - III. Integration of the gravitational equations

- From the symmetry $T^0{}_0 = T^1{}_1$ one can reduce the static, spherically symmetric line element to (Exercise [easy]: prove it):

$$ds^2 = A(r)dt^2 - A(r)^{-1}dr^2 - r^2 d\Omega^2 \tag{198}$$

such that the electromagnetic field equations can be integrated as

$$r^2 \varphi_X E(r) = Q \tag{199}$$

Comment 1: Note that these equations are independent of the metric components. The same as in flat space!.
Comment 2: Because of Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$, this equation is compatible with Einstein's equations (i.e., one of them can be rewritten in terms of the others).

- The Einstein equations read explicitly (Exercise: prove it - time-consuming but not hard):

$$\frac{d}{dr}(rA(r) - r) = -8\pi r^2 T^0{}_0 = -8\pi r^2 (2\varphi_X E^2 - \varphi) \tag{200}$$

$$\frac{d^2}{dr^2}(rA(r)) = -16\pi r T^2{}_2 = 16\pi r \varphi \tag{201}$$

which can be readily integrated as

$$A(r,Q) = 1 - \frac{2M}{r} + \frac{8\pi}{r}\int_r^\infty R^2 T^0{}_0(R,Q)dR \tag{202}$$

M: asymptotic mass, and $\varepsilon_{ex} = 4\pi \int_r^\infty R^2 T^0{}_0(R,Q)dR$ is the contribution total energy in flat sphere outside a sphere of radius r.

NEDs - IV. Examples and properties

- Maxwell electrodynamics:

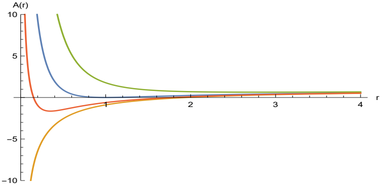
$$\varphi(X) = \alpha X; E(r) = \frac{Q}{\alpha r^2}; r^2 T^0{}_0 = \frac{Q^2}{\alpha r^2}; \varepsilon_{ex}(r,Q) = \frac{4\pi}{\alpha r}; A(r,Q) = 1 - \frac{2M}{r} + \frac{8\pi Q^2}{\alpha r} \tag{203}$$

- Born-Infeld electrodynamics:

$$\varphi(X) = \frac{2}{\mu^2}(1 - \sqrt{1 - \mu^2 X - \frac{\mu^4}{4} Y^2}) \tag{204}$$

$$E(r,Q) = \frac{Q}{\sqrt{r^4 + \mu^2 Q^2}}; T^0{}_0(r,Q) = 2\frac{\sqrt{r^4 + \mu^2 Q^2} - r^2}{\mu^2 r^2} \tag{205}$$

$$\varepsilon_{ex}(r,Q) = \frac{8\pi r}{3\mu^2}[r^2 - \sqrt{r^4 + \mu^2 Q^2} + \frac{2\mu^2 Q^2}{r^2} {}_2F_1(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{\mu^2 Q^2}{r^4})]\text{(FINITE ENERGY!)}$$



<u>Exercise</u> Determine the conditions for different horizons [hint: check the behaviour at the center, and bear in mind the finiteness of the energy].

NEDs - V. Regular black holes: de Sitter cores

- Consider an anisotropic fluid $T_\mu{}^\nu = \text{diag}(-\rho, p_r, p_\perp, p_\perp)$.
- Under the assumptions of i) regularity of density $\rho(r)$, ii) finiteness of the mass parameter $m(r)$ characterizing the gravitational configurations and iii) dominant energy condition, the most general static, spherically symmetric metric holding regularity of all curvature scalars is given by $ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2$ with

$$f(r) = 1 - \frac{2m(r)}{r}; m(r) = 4\pi \int d^4 r r^2 \rho(r) dr \tag{206}$$

and has a de Sitter asymptotic at $r \to 0$, that is

$$f(r) = 1 - \frac{\Lambda r^2}{3} \tag{207}$$

where the energy-momentum tensor is that of a cosmological constant term, $T_{\mu\nu} = \Lambda g_{\mu\nu}$.
- **No-go theorem** (K. A. Bronnikov, Phys. Rev. Lett. 85 (2000) 4641): For a NED to achieve a regular de Sitter core at $r \to 0$ one must have $\varphi \to 0$ and $\varphi_X \to 1$ as $X \to 0$ (i.e. a Maxwellain behaviour there), since the regularity of the energy-momentum tensor requires $|X\varphi_X| < \infty$. However, the first integral (squared) implies $X\varphi_X^2 = q^2/r^4 \to \infty$ at $r = 0$, which entails that $\varphi \to \infty$ as $X \to 0$, which is a strongly non-Maxwellian behaviour.
- Therefore, a regular electrically charged structure is not compatible with the Maxwell weak field limit at the center, and regular black holes in electrically charged NEDs are not possible.

Wormholes - I. Main elements

- Wormholes are hypothetical tunnels connecting space-time regions that are causally disconnected.
- Let us assume a static, spherically symmetric space-time possessing two asymptotically flat regions, conveniently recast as (remember that only two functions are independent):

$$ds^2 = e^{2\phi(r)}dt^2 - dl^2 - r^2(l)d\Omega^2 \tag{208}$$

- Comments:
  - The coordinate l (proper time) covers the entire range $(-\infty, +\infty)$.
  - If there are no horizons (so-called *traversable* wormholes) then $\phi(l)$ must be everywhere finite.
  - Two asymptotically flat regions requires $\lim_{l\to\pm\infty}(r(l)/l) \to 1$, which guarantees $\lim_{l\to\pm\infty}\phi(l) = \phi_\pm$ finite.
  - The radius of the wormhole throat is defined by $r_0 = \min(r(l))$.

Wormholes - II. Coordinates and throat

- In Schwarzschild coordinates, Eq.(208) can re-cast as

$$ds^2 = -e^{2\phi_\pm(l)}dt^2 + \frac{dr^2}{1-\frac{b_\pm(r)}{r}} + r^2 d\Omega^2 \tag{209}$$

- Comments:
  - Two coordinate patches are now required, each covering $[r_0, \infty]$ and joined at r_0.
  - The relation between coordinates can be written as

$$l(r) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{1-\frac{b_\pm(r')}{r'}}} \tag{210}$$

  - The mass of the object as seen from asymptotic infinity is $b_\pm = 2GM_\pm$.
  - At the throat $dr/dl = 0$ (r is a minimum at $l = 0$). Thus as one moves away from the throat, $d^2r/dl^2 > 0$, which implies the **flare-out condition**: $b'(r) < b(r)/r$

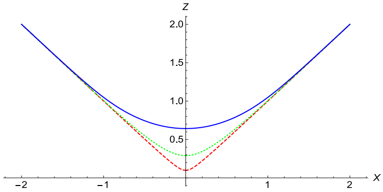


Figure: Typical shape of a wormhole structure

Wormholes - III. Violation of energy conditions

- The non-zero components of the energy-momentum tensor read

$$\rho = -T^0{}_0 \frac{b'}{8\pi r^2} \tag{211}$$

$$\tau = -T^1{}_1 = \frac{1}{8\pi}\left[\frac{b}{r^3} - 2\left(1-\frac{b}{r}\right)\frac{\phi'}{r}\right] \tag{212}$$

$$p = T^2{}_2 = T^3{}_3 = \frac{1}{8\pi}\left[(1-\frac{b}{r})\left[\phi'' + \phi'(\phi' + \frac{1}{r})\right] - \frac{1}{2r^2}(b'r - b)(\phi' + \frac{1}{r})\right] \tag{213}$$

- At the wormhole throat

$$\rho(r_0) = \frac{b'(r_0)}{8\pi r_0^2}; \tau(r_0) = \frac{1}{8\pi r_0^2}; P(r_0) = \frac{1-b'(r_0)}{16\pi r_0}(\phi' + \frac{1}{r_0}) \tag{214}$$

- On the other hand

$$8\pi(\rho - \tau) = -\frac{e^{2\phi}}{r}\left[e^{-2\phi}(1-b/r)\right]' \tag{215}$$

and we know that $e^{-2\phi}(1-b/r)|_{r_0} = 0$ and that $\forall r > r_0$ one has $[e^{-2\phi}(1-b/r)]' = 0$ so

$$\rho(r_0) - \tau(r_0) \leq 0 \tag{216}$$

which is a violation of the null energy condition!.
<u>Exercise</u>: Probe that wormholes actually violate all pointwise energy conditions (hint: just look for appropriate combinations of the expressions above].

- **Conclusion**: wormholes are supported by exotic matter sources...in GR!.

Junction conditions - I

- Many models of gravitating bodies of both theoretical and observational interest involve the consideration of two different patches of space-time, glued at some hypersurface separating the interior from the exterior.
- Let us consider twoo smooth four-dimensional manifolds $(M^{\pm}, g_{\mu\nu})$, and let us denote by $V^{\pm}$ two bounded regions living in $M^{\pm}$ with boundaries $\Sigma^{\pm}$. These regions are matched at a time-like hypersurface Σ with the natural identification of their boundaries as $\Sigma^{+} = \Sigma^{-}$
- The space-time metric $g_{\mu\nu}$ is assumed to be well defined through the entire manifold and, in particular, to be continuous (but not necessarily differentiable) at Σ
- Since there may be discontinuities in several geometric quantities across Σ, the suitable tool to deal with this scenario is that of tensorial distributions, namely, tensor fields with compact support on the manifold.
- The distributional form of the stress-energy tensor is given by

$$\underline{T}_{\mu\nu} = T^{+}_{\mu\nu}\underline{\theta} + T^{-}_{\mu\nu}(\underline{1} - \underline{\theta}) + \tau_{\mu\nu}\underline{\delta}^{\Sigma} \tag{217}$$

where underbars indicate distributions; $T^{\pm}_{\mu\nu}$ are the stress-energy tensors in $V^{\pm}$; $\underline{\theta}$ is the scalar distribution defined by the (locally integrable) Heaviside function, the latter taking the value of 1 in V^{+}, 0 in V^{-}, and any intermediate reference value in Σ; $\underline{\delta}^{\Sigma}$ is a scalar Delta-type distribution with support on Σ acting upon any test function X as $<\underline{\delta}, X> \equiv \int_{\Sigma} X$; and $\tau_{\mu\nu}$ accounts for the singular part of the stress-energy tensor on Σ.
- Similarly, the distributional form of the trace of the stress-energy tensor reads

$$\underline{T} = T^{+}\underline{\theta} + T^{-}(\underline{1} - \underline{\theta}) + \tau\underline{\delta}^{\Sigma} \tag{218}$$

where $\tau \equiv \tau^{\mu}{}_{\mu}$ is the trace of the singular part of the stress-energy tensor.

Junction conditions - II

- Since $R = -\kappa^2 T$, its distributional version yields the junction condition $[T] = 0$, where brackets represent a discontinuity ("jumps") across Σ in the quantity contained there. This condition indicates the need for the continuity of the trace of the stress-energy tensor across Σ, which was already expected on grounds of continuity and standard differentiability of the tensorial equations.
- The distributional form of the Einstein tensor takes the form

$$\underline{G}_{\mu\nu} = G^{+}_{\mu\nu}\underline{\theta} + G^{-}_{\mu\nu}(\underline{1} - \underline{\theta}) + \mathcal{G}_{\mu\nu}\underline{\delta}^{\Sigma} \tag{219}$$

  where $\mathcal{G}_{\mu\nu}$ represents the singular part of the Einstein tensor on Σ.
- Now, let us introduce n_μ as the unit vector normal to Σ defined via $\nabla_\mu \underline{\theta} = n_\mu \underline{\delta}^{\Sigma}$, so that the projector on Σ is defined as

$$h_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu \tag{220}$$

  (commonly known as the first fundamental form), which must be continuous across Σ.
- $\mathcal{G}_{\mu\nu}$ can be expressed as

$$\mathcal{G}_{\mu\nu} = -[K_{\mu\nu}] + h_{\mu\nu}[K^{\rho}_{\rho}] \tag{221}$$

  where

$$K^{\pm}_{\mu\nu} \equiv h^{\rho}{}_{\beta} h^{\sigma}{}_{\mu} \nabla^{\pm}_{\rho} n_{\sigma} \tag{222}$$

  is the second fundamental form on $V^{\pm}$, respectively.

Junction conditions - III

- The singular part of the Einstein equations $\mathcal{G}_{\mu\nu} = \kappa^2 \tau_{\mu\nu}$ thus yields the junction condition

$$-[K_{\mu\nu}] + h_{\mu\nu}[K^{\rho}_{\rho}] = \kappa^2 \tau_{\mu\nu} \tag{223}$$

- Its trace reads

$$2[K^{\rho}_{\rho}] = \kappa^2 \tau \tag{224}$$

  and, therefore, the brane tension in GR is non-vanishing in general.
- The Bianchi identities hold in the distributional sense, that is $\nabla_{\mu} \underline{G}^{\mu}{}_{\nu} = 0$, and can be explicitly written as two sets of equations:

$$\begin{aligned} (K^{+}_{\rho\sigma} + K^{-}_{\rho\sigma})\mathcal{G}^{\rho\sigma} &= 2n^{\rho} n^{\sigma}[R_{\rho\sigma}] - [R] &(225)\\ D^{\rho}\mathcal{G}_{\rho\nu} &= -n^{\rho} h^{\sigma}_{\nu}[R_{\rho\sigma}] \,, &(226) \end{aligned}$$

  where $D_{\rho} \equiv h_{\rho}{}^{\alpha}\nabla_{\alpha}$ denotes the covariant derivative on Σ.
- These equations can be properly worked out to yield the two junction conditions

$$\begin{aligned} D^{\rho}\tau_{\rho\nu} &= -n^{\rho} h^{\sigma}{}_{\nu}[T_{\rho\sigma}] &(227)\\ (K^{+}_{\rho\sigma} + K^{-}_{\rho\sigma})\tau^{\rho\sigma} &= 2n^{\rho} n^{\sigma}[T_{\rho\sigma}] &(228) \end{aligned}$$

- These four conditions close the formalism, and relate the behaviour of the geometrical quantities with the necessary matter content on the shell Σ (when present).

- Thin-shell wormholes arise from surgically matched patches of space-time at the hypersurface Σ (the throat). The formalism of junction conditions is needed.
- In the hypersurface Σ we have three independent basis vectors $e_i \equiv \partial/\partial\xi^i$ with components $e_i^\mu = \partial x^\mu/\partial\xi^i$, being ξ^i the coordinates on Σ. The induced metric on Σ is then expressed as $h_{ij} = g_{\mu\nu} e_i^\mu e_j^\nu$, and we note that $n_\mu e_i^\mu = 0$. The second fundamental form is thus defined by $K_{ij} = e_i^\mu e_j^\nu \nabla_\mu n_\nu$, which is symmetric.
- Now, differentiating $n_\mu e_i^\mu = 0$ with respect to ξ^j, one can write the useful formula

$$K_{ij}^{\pm} = -n_\mu \left(\frac{\partial^2 x^\mu}{\partial\xi^i\partial\xi^j} + \Gamma^{\mu\pm}_{\alpha\beta} \frac{\partial x^\alpha}{\partial\xi^i} \frac{\partial x^\beta}{\partial\xi^j} \right) \tag{229}$$

  for the computation of the second fundamental form.
- Let us now introduce two spherically symmetric space-times whose line elements on $\mathcal{M}_\pm$ are

$$ds_\pm^2 = -A_\pm(r_\pm)dt^2 + \frac{1}{B_\pm(r_\pm)} dr_\pm^2 + r_\pm^2 d\Omega^2 \tag{230}$$

  described by the functions $A_\pm(r_\pm), B_\pm(r_\pm)$, respectively, and where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the unit volume in the two-spheres.
- The induced line element on the matching hypersurface Σ can be written as

$$ds_\Sigma^2 = -d\tau^2 + R^2(\tau)d\Omega^2 \tag{231}$$

  which is parameterized in terms of the proper time of an observer comoving with Σ. Here R is the radius of the shell and $4\pi R^2(\tau)$ measures its area.

Wormholes - IV. Thin shells - B

- This spherical three-dimensional hypersurface has coordinates $x^\mu(\tau,\theta,\varphi)=(t(\tau),R(\tau),\theta,\varphi)$, and since the tangent vectors to it are $e^\mu_\theta=(0,0,1,0)$ and $e^\mu_\varphi=(0,0,0,1)$, setting the velocity vector as $U^\mu \equiv dx^\mu/d\tau=(t_\tau,R_\tau,0,0)$ (where $t_\tau \equiv dt/d\tau, R_\tau \equiv dR/d\tau$), it follows that the normal vector (assumed to be oriented from $\mathcal{M}_-$ to $\mathcal{M}_+$) must be of the form $n^\mu=\pm(n^t,n^r,0,0)$.
- Now, using the fact that $n^\mu n_\nu=+1$ (space-like character) and $n_\mu U^\mu=0$ (orthogonality condition) one finds that the components of the normal vector, n^t and n^r, are given by

$$n^t=\frac{R_\tau}{\sqrt{AB}};n^r=\sqrt{B+R_\tau^2} \tag{232}$$

where the metric functions A and B are the evaluation of the metric functions) on $\mathcal{M}_\pm$ at the shell radius $r=R(\tau)$. Due to the continuity of the space-time metric across Σ they must match there.
- It is now immediate to compute the non-vanishing components of the second fundamental form $K^i{}_j=\text{diag}(K^\tau{}_\tau,K^\theta{}_\theta,K^\theta{}_\theta)$ as

$$K^{\tau\,\pm}_{\tau} = \pm\frac{B^2A_R+(BA_R-AB_R)R_\tau^2+2ABR_{\tau\tau}}{2AB\sqrt{B+R_\tau^2}}, \tag{233}$$

$$K^{\theta\,\pm}_{\theta} = \pm\frac{\sqrt{B+R_\tau^2}}{R}. \tag{234}$$

where $A_R \equiv dA/dR$ and so on.

Wormholes - IV. Thin shells - C

- The matter stress-energy tensor on Σ can also be written as a diagonal matrix

$$S^{\mu}{}_{\nu} = \text{diag}(S^{\tau}{}_{\tau}, S^{\theta}{}_{\theta}, S^{\theta}{}_{\theta}) \tag{235}$$

with components $S^{\tau}_{\tau} = -\sigma$ and $S^{\theta}_{\theta} = \mathcal{P}$, where σ is the surface energy density and $\mathcal{P}$ is the tangential surface pressure.

- The junction conditions allow to relate these components to the jumps in the extrinsic curvature as

$$\sigma = -\frac{1}{4\pi}[K^{\theta}{}_{\theta}] \; ; \; P = -\frac{1}{8\pi}([K^{\tau}{}_{\tau}] + [K^{\theta}{}_{\theta}]) \tag{236}$$

- Given a spherically symmetric line element these equations allow to compute the energy density and pressure of the fluid supporting the shell, or the other way round.
- The conservation equation yields another equation

$$\dot{\sigma} = -2(\sigma + P)\dot{R} + \Upsilon \dot{R} \to \sigma' = -\frac{2}{R}(\rho + P) + \Upsilon \tag{237}$$

where the shape of Υ depends on the choice for the metric functions. For a Schwarzschild thin-shell wormhole this equations yields $\dot{\sigma} = -2(\sigma + P)\dot{R}/R$.

- The (linear) stability of the wormhole is analyzed by rewriting the equation of motion as

$$\frac{1}{2}\dot{R} + V(R) = 0 \tag{238}$$

and expanding in series the potential around an assumed static solution R_0.

$$V(R) = V(R_0) + V'(R_0)(R - R_0) + \frac{1}{2}V''(R_0)(R - R_0)^2 + \mathcal{O}(R - R_0)^3 \tag{239}$$

Since around a static solution $\dot{R}_0 = \ddot{R}_0$ then $V(R_0) = V'(R_0) = 0$ and stability is achieved if $V''(R_0) > 0$.

Curvature-free charged space-times

- A common trend is to build *ad hoc* metric components which are free of curvature singularities everywhere (typically via a dS core)
- Bardeen - perhaps the most well known one and more widely used:

$$A(r) = 1 - \frac{2Mr^2}{(r^2+Q^2)^{3/2}} \tag{240}$$

- Hayward is an alternative example given by

$$A(r) = 1 - \frac{2Mr^2}{(r^3+2Ml^2} \tag{241}$$

supported by a matter field satisfying

$$\rho = -p = \frac{8}{\kappa^2}\frac{3M^2l^2}{(r^3+2Ml^2)^2}; p_\perp = \frac{8}{\kappa^2}\frac{3M^2l^2(r^3-Ml^2)}{(r^3+2Ml^2)^3} \tag{242}$$

- Ayón-Beato-Garcia solution is yet another example supported by an NED and given by

$$A(r) = 1 - \frac{2Mr^2}{(r^2+Q^2)^{3/2}} + \frac{Q^2r^2}{(r^2+Q^2)^2} \tag{243}$$

- (Charged) black-bounces

$$A(x) = 1 - \frac{2M}{r(x)} + \frac{Q^2}{r^2(x)} \tag{244}$$

where the function $r(x)$ implements a bouncing, wormhole-like behaviour at $x=0$. Simplest case: $r(x) = \sqrt{x^2+a^2}$.

Compact objects supported by scalar fields

- Finding solutions to GR equations + scalar fields is quite difficult (the symmetry ${T^t}_t = {T^r}_r$ is no longer there, which overrules the tricks employed for vacuum space-times).
- But there is *one* exact analytical solution of GR + free real scalar field (Wyman):

$$ds^2_{GR} = -e^{\nu} dt^2 + \frac{e^{\nu}}{W^4} dy^2 + \frac{1}{W^2} (d\theta^2 + \sin\theta^2 d\varphi^2) \;, \tag{245}$$

  where ν and W are functions of the radial coordinate y. These are coordinates in which $\phi_{yy} = 0$ (thus $\phi(y) = y$).
- Demanding asymptotic flatness, the metric functions take the form

$$e^{\nu} = e^{\beta y} \tag{246}$$

$$W = \gamma^{-1} e^{\beta y/2} \sinh(\gamma y) \;, \tag{247}$$

  where $\beta = -2M$, and $\gamma \equiv \sqrt{\beta^2 + 2\kappa^2}/2$. The asymptotic limit corresponds to $y \to 0$, where the scalar field vanishes, while the center of the (spherical) solution is reached at $y \to \infty$.
- Central curvature

$$\lim_{y \to \infty} R \approx \frac{\kappa^2 e^{\left(2\sqrt{\beta^2+2\kappa^2}+\beta\right)y}}{(\beta^2 + 2\kappa^2)^2} \;. \tag{248}$$

- Incomplete geodesics. Interpretation?.
- Phenomena triggered by scalar fields: superradiance, scalar clouds, boson stars, black hole bombs,..., numerical analysis is needed!.

Boson stars

- A (scalar) boson star is supported (in the simplest case) by a Klein-Gordon field with action
$$\mathcal{S} = \int \sqrt{-g}\left(\frac{R}{\kappa^2} - \nabla_a\bar{\Phi}\nabla^a\Phi - \mu^2\bar{\Phi}\Phi\right) d^4x, \tag{249}$$
- It leads to a system of Einstein + Klein-Gordon, leading to a set of three coupled equations whose resolution requires proper numerical methods and suitable boundary conditions at the star's center.
- Similarly, a (vector) Proca star
$$\mathcal{S} = \int \sqrt{-g}\left(\frac{R}{16\pi} - \frac{1}{4}F_{ab}\bar{F}^{ab} - \frac{1}{2}\mu^2 A_a\bar{A}^a\right) d^4x, \tag{250}$$
- It leads to four coupled equations with similar comments on their resolution.
- Typically these objects are not compact enough to hold a critical curve (although exceptions are known), which nonetheless may act as observational discriminators in shadow observations under certain circumstances, but can equally imitate black hole shadows good enough to pass unseen[10].

[10] C. A. R. Herdeiro, A. M. Pombo, E. Radu, P. Cunha, V.P. and N. Sanchis-Gual, JCAP **04** (2021) 051.

Gravastars

- Gravastars (Gravitational condensate stars) are ultra-compact objects whose radius is arbitrarily close to the Schwarzschild radius and without a central singularity. The interior is replaced by a dS core, surrounded by a thin-shell of matter with EOS, and matched to an external Schwarzschild solution:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega^2 && (251)\\ & \text{Region I} : 0 \leq r \leq r_1, P = -\rho && (252)\\ & \text{Region II} : r_1 \leq r \leq r_2, P = +\rho && (253)\\ & \text{Region III} : r_1 \leq r, P = -\rho && (254) \end{aligned}$$

- Therefore, the formalism of thin-shells must be called upon (twice).
- In region I, the dS core satisfies $f(r) = Ch(r) = C(1 - H_0^2 r^2)$ supported by a density $\rho = 3H_0^2/8\pi$.
- In region II, the shell satisfies three coupled differential equations, which can be exactly solved in the infinitely thin-shell limit.

Hairy black holes

- The **uniqueness theorems** on black hole physics tell us that *The only possible stationary, axisymmetric, asymptotically flat, solution of GR field equations with spherical topological horizons is given by the Kerr one*. If you add a Maxwell field, then the theorems extend that solution to the Kerr-Newman one.
- Closely related to it (or perhaps simply a corollary) is the **No-hair conjecture**, which states that *There are not regular, stable perturbations of a KN BH which cannot be expressed in terms of a change to mass, charge, or angular momentum change to the original BH*. In other words, every single black hole of the Universe is entirely described by the trio $\{M, Q, J\}$.
- These results, however, do not prevent us for considering alternative black holes coupled to other matter sources, which *may* (note the italic) introduce additional *charges* or *hairs* necessary to describe them $\Rightarrow$ **hairy black holes**.
- The (perhaps) two most common matter sources to produce hairy black holes are scalar and non-abelian gauge fields.

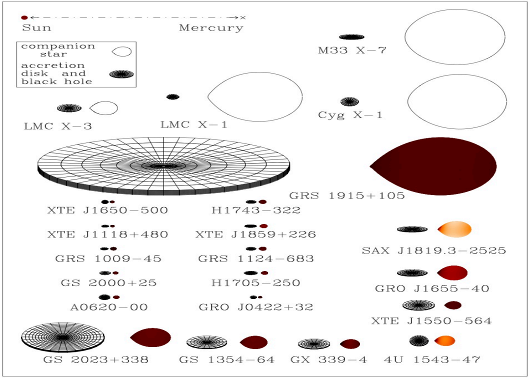
On the universality of Kerr's hypothesis

- 1

LESSON VII: OBSERVATIONAL SEARCHES WITH LIGHT AND GWS

Searches with X-ray spectroscopy

- A fuel-exhausted stellar object with mass below the Chandrasekhar's limit $M_{Ch} \approx 1.4 M_{\odot}$ unavoidably collapse under its event horizon yielding a black hole.
- Stellar-mass black holes ($\lesssim 100 M_{\odot}$) can be detected basically via X-ray emissions out of their accretion disks (continuum fitting method and X-ray reflection spectroscopy):
  • C. Bambi, "Astrophysical Black Holes: A Review, arXiv: 1906.03871 [astro-ph.HE].



- Historically this was the first evidence (1971) for the existence of black holes in binary star systems in our galaxy - Cygnux X-1.

Gravitational waves from black holes - I

- GWs are perturbations in the fabric of space-time. To study them we use linearized theory, assuming an observer who is far away from a given static matter distribution, in such a way that we can consider a perturbation upon the Minkowski space-time as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{255}$$

  where the perturbation is assumed to satisfy $|h_{\mu\nu}| \ll 1$.
- It is more convenient to work in the so-called trace-reverse metric perturbations

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h_\alpha h^\alpha \tag{256}$$

- There is an implicit gauge invariance related to the local coordinate transformations $x^\mu \to x^\mu + \xi^\mu$ under which the trace-reversed perturbation transforms as

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial_\lambda \xi^\lambda \tag{257}$$

  Using this freedom one can choose the Lorentz gauge $\partial_\mu \tilde{h}^{\mu\nu} = 0$, in which one writes

$$\partial'^\mu \tilde{h}'_{\mu\nu}(x') = \partial^\mu h_{\mu\nu}(x) - \Box \xi_\nu \tag{258}$$

- In order for the Lorentz gauge to be satisfied, one needs to choose $\Box \xi_\nu = \partial^\mu \tilde{h}_{\mu\nu}$ so that the linearized Einstein equations read

$$G_{\mu\nu} = -\frac{1}{2}\Box \tilde{h}_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{259}$$

Gravitational waves from black holes - II

- In vacuum, the wave equation above reads simply

$$\Box\tilde{h}^{\mu\nu} \equiv \partial_\mu\partial^\mu\tilde{h}^{\mu\nu} = \left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\tilde{h}^{\mu\nu} = 0 \qquad (260)$$

  and its solution is given by a superposition of harmonic solutions as

$$\tilde{h}_{\mu\nu} = \int d^3k(A_{\mu\nu}(\vec{k})e^{ik_\alpha x^\alpha} + A^\star_{\mu\nu}(\vec{k})e^{-ik_\alpha x^\alpha}) \qquad (261)$$

  where $A^{\mu\nu}, A^{*\mu\nu}$ are the constant (symmetric) polarization tensors, encoding the information on the amplitude and polarization of the GW, while k_α is the constant wave vector, determining the propagation direction of the GW and its frequency.
- Hilbert's gauge implies that $A^{\mu\nu}k_\nu = 0$, i.e., $A^{\mu\nu}$ and k_μ are orthogonal, and reads as four conditions on $A^{\mu\nu}$ reducing its number of independent components from 10 to 6.
- Replacing this solution in the wave equation yields the additional constraint $k_\mu k^\mu = -k_0^2 + (k_1^2 + k_2^2 + k_3^2) = 0$, which means that the wave propagates on the light cone, i.e., its velocity is $v = c$ and its frequency $\omega = k^0$.
- There is a further gauge freedom related to the four components of ξ^μ: this further reduces the number of components of $A^{\mu\nu}$ to just 2!.
- One choice is the transverse–traceless or TT gauge, in which i) only the spatial components of $\tilde{h}_{\mu\nu}$ are non-vanishing, $\tilde{h}_{0i} = 0$, which means that the GW is transverse to its direction of propagation and ii) the sum of its diagonal terms is vanishing, $h_{\mu\mu} = h_{00} + h_{11} + h_{22} + h_{33} = 0$. In this gauge we denote $h^{TT}_{\mu\nu} = \tilde{h}_{\mu\nu} = h_{\mu\nu}$.

Gravitational waves from black holes - III

- Therefore, we can write the two independent components of the polarization tensor as (assuming a GW propagating in the z-direction) as

$$A^{\mu\nu} = h_+ \varepsilon_+^{\mu\nu} + h_\times \varepsilon_\times^{\mu\nu} \tag{262}$$

where $h_+, h_\times$ are the (dimensionless= amplitudes of each mode, while the unit polarization tensors $\varepsilon_+^{\mu\nu}, \varepsilon_\times^{\mu\nu}$ are given by

$$\varepsilon_+^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \varepsilon_\times^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{263}$$

- The effect of a passing GW can be analyzed from the geodesic deviation equation

$$\frac{d^2\xi^k}{dt^2} = -R^k{}_{0j0}^{TT} \xi^j \tag{264}$$

where we have assumed to geodesic trajectories along their proper times $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu(\tau)$, while $R^k{}_{0j0}^{TT} = -\frac{1}{2}\frac{\partial^2}{\partial t^2} h_{jk}^{TT} \approx \frac{\partial^2 \Phi}{\partial x^j \partial x^k}$ is the Riemann tensor in the linearized theory in the TT gauge, and with the last approximation corresponding to the Newtonian regime.

- The tidal force acting upon a particle of mass m is thus given by $f^k \approx -mR^k{}_{0j0}^{TT} \xi^j$.

Gravitational waves from black holes - IV

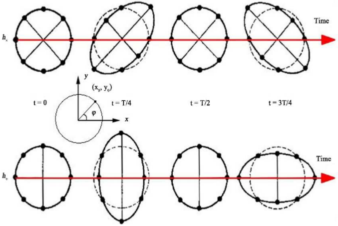
- Let us consider a GW with polarization $+$ only, travelling in the z-direction:

$$h^{\mu\nu} = h_{+}\varepsilon^{\mu\nu}\cos[\omega(t-z)] \tag{265}$$

- Let two free-falling particles with an original separation ξ_0^x along the x-direction, and another two placed along the y-direction, hit by the GW. After the passing of the GW, the respective distances will evolve as

$$\frac{\xi^x}{\xi_0^x} = 1 - \frac{1}{2}h_{+}\cos[\omega(t-z)]; \; \frac{\xi^y}{\xi_0^y} = 1 + \frac{1}{2}h_{+}\cos[\omega(t-z)] \tag{266}$$

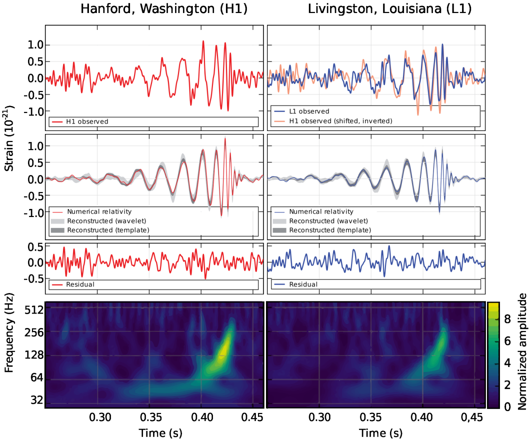
- This implies that the coordinate distances along the two axes oscillates out of phase: when the distance between the the two particles along the x-axis is maximum, then along the y-axis is minimum and vice-versa, and their are switched after half a period.



- Should we choose the polarization $\times$, then the oscillations would be out of phase at an angle of 45 degrees with respect to the $+$ polarization.

The inspiral-merger-ringdown phases

- The process of a binary merger consists of three phases:



- Inspiral: rely on post-newtonian approach.
- Merger: numerical resolution of Einstein's equations is needed.
- Ringdown: settled into a Kerr solution after any hair is emitted out. Detection of signatures of new physics?.

Quasi-normal modes - I

- Quasi-normal modes (QNMs) are complex numbers emerging out of (electromagnetic or gravitational) perturbations over a compact body[11]. They are typically computed using a WKB approximation associated to scalar perturbations.
- We perturb the black hole with some probe minimally coupled to a massive scalar field Φ:

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\right)\Phi=\mu^2\Phi \tag{267}$$

- Next one performs a mode decomposition of the scalar field as

$$\Phi(r,t,\theta,\phi)=\int d\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}(\omega)\Phi_{lm\omega}(r,t,\theta,\phi);\Phi_{lm\omega}=e^{-i\omega t}\frac{\psi_{l\omega}(r)}{r}Y_{lm}(\theta,\phi) \tag{268}$$

where $\psi_{l\omega}(r)$ is the radial part and $Y_{lm}(\theta,\phi)$ are the usual spherical harmonics.
- This way, the scalar field equation is transformed in a Schrödinger-like equation

$$\frac{d^2\psi}{dr_\star^2}+(\omega-V_l(r_\star))\psi_{l\omega}(r_\star)=0 \tag{269}$$

where $r_\star$ is the *tortoise* coordinate, defined by

$$dr/dr_\star=\sqrt{AB} \tag{270}$$

and the effective potential is

$$V_l(r)=A(r)\left(\frac{A'(r)}{r}+\frac{l(l+1)}{r^2}+\mu^2\right) \tag{271}$$

[11] K. D. Kokkotas and B. G. Schmidt, Living Rev. Rel. **2** (1999), 2.

Quasi-normal modes - II

- Boundary conditions are imposed to account for asymptotic infinity, and the presence of a horizon (in a BH space-time):

$$\psi(r) \to \begin{cases} Be^{+i\omega r_\star} & \text{if} \quad r \to \infty (r_\star \to +\infty) \\ Ae^{-i\omega r_\star} & \text{if} \quad r \to r_h (r_\star \to -\infty) \end{cases} \tag{272}$$

- These two boundary conditions result into a numerical shooting-problem that yields a discrete spectrum of complex (due to the non-Hermitian - dissipative - boundary conditions) numbers:

$$\omega = \omega_R + i\omega_I \tag{273}$$

- The associated families of wavefunction solutions, $\psi_{ln}(r)$, described by an overtone number n, with higher overtones decaying exponentially faster, are interpreted as short-live resonances with ω_R accounting for their frequency and ω_I for their damping.
- *Bonus track*: Similarly one can study the absorption of black holes via a simple change in the boundary conditions. Assuming solutions representing a scalar wave incoming from the past null infinity these are

$$\psi(r) \to \begin{cases} e^{-i\omega r_\star} + e^{i\omega r_\star} R_{\omega l} & \text{if} \quad r \to \infty (r_\star \to +\infty) \\ T_{\omega l} e^{-i\omega r_\star} & \text{if} \quad r \to r_h (r_\star \to -\infty) \end{cases} \tag{274}$$

where transmission and reflection coefficients are given by $|R_{\omega l}|^2$ and $|T_{\omega l}|^2$, respectively.

Quasi-normal modes - III

- Exact solutions to the QNM problem are rare and one needs to resort to numerical methods. However, analytical expressions can be found in the eikonal limit, $l \to \infty$.
- In such a case, the (massless) Klein-Gordon equation can be written under the form

$$\frac{d^2}{dr_\star^2}\psi + Q_0\psi = 0 \quad ; \quad Q_0 \approx \omega^2 - A\frac{l^2}{r^2} \tag{275}$$

- Standard WKB methods manage to solve this equation to provide the relation

$$\frac{Q_0(r_0)}{\sqrt{2Q_0^{(2)}}} = i(n+1/2) \tag{276}$$

where $Q_0^{(2)} \equiv d^2Q_0/dr_\star^2$ and the point r_0 is such that $dQ_0/dr_\star = 0$. For this value one finds that $2A(r_0) = r_0A'(r_0)$, $\Rightarrow$ it coincides with the value of the null circular orbit, $r = r_c$.
- Using this result, WKB methods also allow to find the frequency

$$\omega_{QNM} = l\sqrt{\frac{A(r_c)}{r_c^2}} - \frac{(n+1/2)}{\sqrt{2}}\sqrt{-\frac{r_c^2}{A(r_c)}\left(\frac{d^2}{dr_\star^2}\frac{A}{r^2}\right)} \tag{277}$$

which can actually be rewritten under the most simple form using null geodesic quantities on circular curves $(E/L)^2 = A(r_c)/r_c^2$ as ($\Omega_c = \frac{\dot{\varphi}}{\dot{t}} = \frac{A(r_c)^{1/2}}{r_c}$ is the coordinate angular velocity)

$$\omega_{QNM} = \Omega_c l - i(n+1/2)|\gamma_L| \tag{278}$$

- Therefore, in the eikonal limit, the (real and the imaginary parts of the) of the QNMs (for every SSS space-time) turn out to be multiples of those associated to the frequency and instability time of null circular orbits!.

Echoes in GWS from exotic compact objects- I

- The post-merger ringdown waveform of exotic ultracompact objects would be initially identical to that of a black hole, but putative corrections at the horizon scale will appear as periodic releases of secondary GW bursts after the main burst.
- Suppose an ultracompact object whose surface's radius is arbitrarily close to its Schwarzschild horizon, $r_0 = 2M + l$, where $l \ll M$.
- Consider this time the scattering of a massless scalar wave $\Box\Phi = 0$ (no assumption on the time-dependence) with equation of motion (in the tortoise coordinate)

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_\star^2} - V_l(r)\right]\Psi_{lm}(r,t) = 0 \tag{279}$$

  where the potential $V_l(r)$ meets its Schwarzschild values for $r > r_0$, but is model-dependent for $r \leq r_0$.
- This equation can be numerical solved with initial conditions $\Psi_{lm}(r,0) = 0$, $\frac{\partial \Psi_m}{\partial t}(0,r) = e^{(-r_\star - r_g)^2}/\sigma$, where r_g and σ are model-dependent.
- Near the would-be Schwarzschild horizon, the presence of a hard surface makes the potential to develop a minimum and, therefore, an innermost stable photonsphere, which can trap low-frequency modes.
- Such trapped modes can be leaked through the potential barrier under the form of periodic echoes. The typical time scale for this to happen is roughly the time needed for light to take a round trip between the two maxima of the potential barrier as

$$\Delta t \sim 2\int_{r_{min}}^{3M} \sqrt{AB} dr \tag{280}$$

Echoes in GWS from exotic compact objects - II

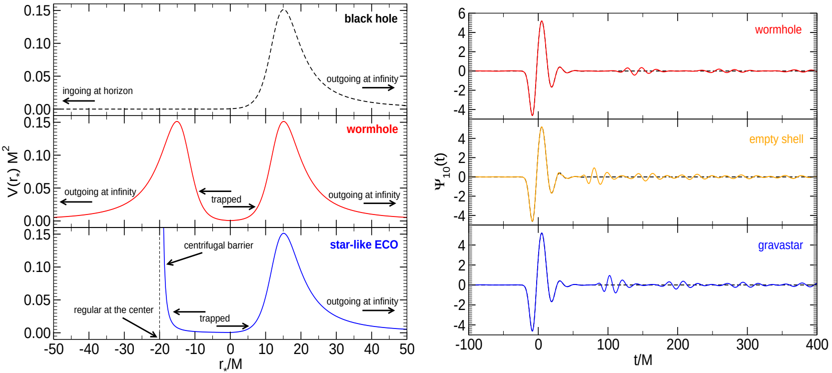


Figure: The effective potential (left) and the wave profile (right) for different exotic compact objects as compared to the black hole ones. Extracted from Cardoso et al., arXiv:1608.08637 [gr-qc].

The replacement of the would-be horizon by additional peaks in the effective potential, induces an effective partially-reflective surface allowing for a cascade of additional models of stable period but decreasing amplitude. The *echoes*. Detectability?.

The current EHT observations

- Supermassive black holes: located at the center of every large and middle-size galaxy and probably also in small galaxies.
- Detected via black hole shadows, like in the galaxy M87/SgrA* via the Event Horizon Telescope.

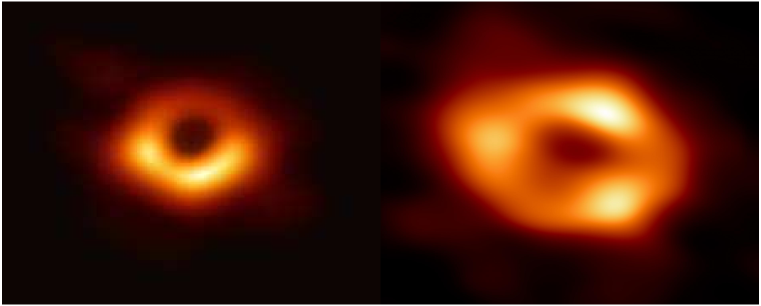


Figure: The image created by the acreting plasma in orbit around the supermassive central object of M87 galaxy (left) and on SgrA* (right).

- Consistent with the Kerr solution, but open space for deviations from it.
- Mysteries: they exist even in very distance galaxies (~ 1 Gyr). And where are the intermediate black holes?.

Shadows from accretion disks - I. Critical curve

- The motion of photons around any spherically symmetric compact body with line element $ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$ is governed by the equation

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - V(r) \tag{281}$$

  where λ is the affine parameter, $b \equiv L/E$ is the impact parameter defined as the ratio of the photon's angular momentum and its energy, and the effective potential is given by

$$V(r) = \frac{A(r)}{r^2} \tag{282}$$

- Zeroes on the right-hand side of (281) correspond to *turning points*, i.e., a photon travelling from asymptotic infinity approaches to a minimum distance r_0 of the black hole before being scattered back to asymptotic infinity.
- The minimum value of the impact factor for which this may occur is given by

$$b_c = \frac{r_{ps}^2}{A(r_{ps})} \tag{283}$$

  and the corresponding value of the radial function r_{ps} is known as the critical curve or the photon sphere.
- Mathematically, the critical curve corresponds to the maximum of the potential.
- Physically, the critical curve corresponds to photons winding around the black hole an infinite number of times. Yet, it is an unstable curve under any small perturbation.
- Exercise: prove that for a Schwarzschild black hole, $b_c = 3\sqrt{3}M$ and $r_m = 3M$.

Shadows from accretion disks - II. Ray-tracing

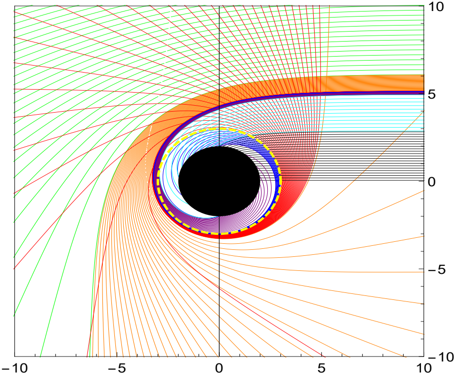
- In order to find all trajectories approaching the critical curve, it is more convenient to rewrite the geodesic equation in terms of the variation of the azimuthal angle ϕ with respect to the radial coordinate as

$$\frac{d\phi}{dr} = \mp \frac{b}{r^2\sqrt{1-\frac{b^2 A(r)}{r^2}}} \,, \tag{284}$$

- To find the optical appearance of a black hole one is not interested on all the light rays emitted from the source and scattered in all directions, but just on those that have managed to arrive to the observer's screen.
- This is done by implementing a *ray-tracing procedure*: light rays arriving to the screen of the observer at asymptotic infinity are traced back to the point of the sky they originated from bearing in mind its deflection by the gravitational field of the black hole .
- Mathematically one integrates the geodesic equation (284) backwards.
- Physically, this corresponds to a black hole illuminated from behind by a planar source which emits isotropically and with uniform brightness.
- In their path from the screen to its emission source a given light ray with impact parameter b will turn a given angle around the black hole, even winding one or more times around them depending on how close b is to b_c.
- This can be captured by the (normalized) change in the azimuthal angle, that is, $n(b) \equiv \frac{\phi}{2\pi}$, which depends on how close the impact parameter is to the critical one. In this setup, light rays in straight motion (i.e. not being deflected at all by the black hole) have $n = 1/2$, and those with $b = b_c$ have $n \to \infty$.

Ray-tracing for Schwarzschild black hole

- Integrate the geodesic equation for a bunch of light rays approaching the critical curve from above and below. Implemented via a numerical integrator, for instance, within Mathematica.



- In the ray-tracing for the Schwarzschild black hole, the observer is located on the far right of the screen, the region inside the event horizon is represented by a black circle, and the photon sphere radius by the filled black circumference. The different coloured bunch of curves correspond to $1/2 < n < 3/4$ (green and cyan), $3/4 < n < 5/4$ (orange and purple) and $n > 5/4$ (red and blue). Those in black correspond to $n < 1/4$ (for $b < b_c$).

- While the background geometry determines the critical curve, the actual optical appearance of a black hole is strongly dependent on the modelling of the accretion disk providing its main source of illumination:
  - Optical properties: whether the disk is transparent to its radiation (optically thin) or not (optically thick).
  - Geometrical properties: whether the disk is spherical, infinitely thin, or any shape in between.
  - Emission properties: governed by the radiative transfer (Boltzmann) equation (neglecting scattering)[12]

$$\frac{d}{d\lambda}\left(\frac{dI_\nu}{d\nu^3}\right)=\left(\frac{j_\nu}{\nu^2}\right)-(\nu\alpha_\nu)\left(\frac{I_\nu}{\nu^3}\right)\,, \tag{285}$$

    where I_ν is the intensity for a given frequency ν, j_ν is the emissivity, α_ν the absorptivity, and quantities inside parenthesis are frame-independent.
- Solving this problem thus requires precise information on those properties to be implemented in Genera-Relativistic Magneto-HydroDynamical (GRMHD) simulations[13].

[12] George B. Rybicki, Alan P. Lightman, "Radiative Processes in Astrophysics" (New York: Wiley-VCH, 2004).

[13] R. Gold, et al. Astrophys. J. 897 (2020) 148.

Shadows from accretion disks - IV. Optically thin disk

- If the disk is optically thin, on each intersection with the equatorial plane every light ray will pick up additional brightness in a way that largely depends on the particular assumed emission modelling of the disk.
- In the Schwarzschild case, the bending angle can be computed analytically to go logarithmically with n, entailing an exponential suppression of every subsequent loop around the black hole. For the sake of the image, therefore, only three contributions are relevant
    - Direct emission: corresponding to light rays intersecting the equatorial plane (on its front) just once, and defined by $1/2 < n \leq 3/4$. This is the dominant contribution to the optical appearance of the object, both in terms of luminosity and width of the associated ring of radiation (more on this later). This corresponds to $b/M \notin (5.02, 6.17)$.
    - Lensed emission: corresponding to light rays intersecting the equatorial plane twice (on its front and its back, respectively), and defined by $3/4 < n \leq 5/4$, being the sub-dominant contribution to the luminosity. It corresponds to $b/M \in (5.02, 5.19) \cup b/M \in (5.23, 6.17)$
    - Photon ring emission: corresponding to light rays intersecting the equatorial plane at least thrice, and defined by $n > 5/4$. It covers the range $b/M \in (5.19, 5.23)$.

Shadows from accretion disks - V. Spherical disk

- Consider a spherically symmetric disk. The central brightness depression (aka the shadow) in the observed image coincides precisely with those light rays that terminate on the event horizon. The boundary of the shadow is thus bounded by the critical curve.
- Approaching the shadow edge, given the logarithmic divergence of null geodesics paths', the image brightness also diverges logarithmically at the critical curve, resulting in a bright "photon ring" encircling the black hole shadow.
- To get the specific intensity received on the observer's screen one integrates the specific intensity along a given photon's path γ_i as

$$I_{ob}(\nu_0) = \int_{\gamma_i} g^3 j(\nu_e) dl_p \tag{286}$$

where $j(\nu_e)$ is the emitted emissivity per-unit volume, $g = \nu_0/\nu_e$ is the redshift factor accounting for the ratio between the observed and the emitted frequency, and dl_p is the infinitesimal proper length given by

$$dl_p = \sqrt{\frac{1}{A(r)} + r^2 \left(\frac{d\varphi}{dr}\right)} dr \tag{287}$$

- Assuming a monochromatic emission with ν_f and radial profile $1/r^2$, i.e., $j(\nu_e) = \delta(\nu_e - \nu_f)/r^2$, and using $v_0 = A^{1/2} v_e$, the observed specific intensity reads

$$I_{ob}(\nu_0) = \int_{\gamma_i} \frac{A^{3/2}(r)}{r^2} \sqrt{\frac{1}{A(r)} + r^2 \left(\frac{d\varphi}{dr}\right)} dr \tag{288}$$

- Consider an infalling spherical accretion where the radiating material is moving towards the black hole along the radial direction. The redshift factor is now velocity-dependent as

$$g_i = \frac{\mathcal{K}_\rho u_0^\rho}{\mathcal{K}_\sigma u_e^\sigma} \tag{289}$$

where $\mathcal{K}^\mu = \dot{x}^\mu$ is the photon's 4-velocity and $u_0^\mu = (0,0,0,1)$ is the static obs' 4-velocity.

- The infalling's gas 4-velocity u_e^μ components are

$$u_e^t = A^{-1}(r); u_e^r = -\sqrt{\frac{1-A(r)}{A(r)B(r)}}, u_e^\theta = u_e^\phi = 0 \tag{290}$$

- From the null condition $\mathcal{K}_\mu \mathcal{K}^\mu = 0$ and the fact that $\mathcal{K}_t$ must be a constant, one gets

$$\mathcal{K}_t = \frac{1}{b}; \frac{\mathcal{K}_r}{\mathcal{K}_t} = \pm\sqrt{B(r)\left(\frac{1}{A(r)} - \frac{b^2}{r^2}\right)} \tag{291}$$

where $+/-$ for a photon approaching/departing the black hole, so the redshift factor is

$$g_i = \left(u_e^t + \left(\frac{\mathcal{K}_r}{\mathcal{K}_e} u_e^r\right)\right)^{-1} \tag{292}$$

- Since the proper distance is now $dl_p = \mathcal{K}_\mu u_e^\mu d\lambda = \frac{\mathcal{K}_t}{g_i|\mathcal{K}_r|} dr$, the observed intensity reads

$$I_{ob} \propto \int_{\gamma_i} \frac{g_i^3 \mathcal{K}_t}{r^2|\mathcal{K}_r|} dr \tag{293}$$

Some comments on the theoretical modelling

- The precise shape of the image is prone to many approximations and simplifications in the theoretical modelling:
  - Geometrical vs optically thick/thin?: EHT uses optically thin but geometrically thick. Other models (e.g. Novikov-Thorne uses different hypothesis).
  - Real BHs do rotate, but even at maximum allowed speed (which is physically realistic at the moment of formation/merger, since BHs de-rotate themselves pretty fast) deviations from the circularity in the space of the shadow is $\lesssim 7\%$.
  - Inclination of the image must be taken into account: M87 at ~ 17 degrees and $SgrA^*$ at ~ 30 degrees???. Minor influence in the shape of the image unless one goes to extreme inclinations ~ 80 degrees.
  - Truncated shapes of the emission profile of the disk are quite unnatural, since the disk extends all the way down to the event horizon $\Rightarrow$ unfortunate, since this fact most spoil the chances of observing the photon ring structure with the EHT capabilities $\Rightarrow$ the ngEHT should be able to cope with this.
  - Limited resolution of the images so far means that we only hope to see a blurred image $\Rightarrow$ apply Gaussian filter $1/12$ the field of view to simulate nominal resolutions.
  - If the object is not a BH, so that a horizon is not present (horizonless ultra-compact objects) $\Rightarrow$ new signatures could be present, similarly as in the echoes of GW emission. (Almost) perfectly reflective surfaces are claimed by EHT to be heavily constrained[14]. Meanwhile, wormholes are (mostly) unaffected.

[14] The counterarguments of *R. Carballo-Rubio, F. Di Filippo, S. Liberati and M. Visser, [arXiv:2205.13555 [astro-ph.HE]* claim that these arguments do not take into account the bouncing of radiation back to the object's surface and subsequent absorption, which largely softens such constraints even for very small absorption coefficients, $\kappa \sim 10^{-5}$.

Constraints from SgrA* shadow's size

- EHT Collaboration claims[15] that under suitable circumstances (optically thin at observed wavelength, geometrically thick, disk up to the event horizon, mass-distance relation well known) and after proper calibration (the mass-to-distance ratio, while different sources of error can be controlled to some extend), the measurement of the direct emission ring can act as a reliable proxy for the size of the shadow itself.

$$r_{sh} = \frac{r_m}{\sqrt{A(r_m)}} \to r_{sh} = 3\sqrt{3}M \approx 5.196M(\text{for Schwarzschild}) \tag{294}$$

- Consequently, every modification from the Schwarzschild/Kerr paradigm with a different shadow size is suitable to be constrained by this observation alone (and to be ruled out as viable alternatives?). Mostly independent of the disk's modelling!.
- Using two different surveys for the mass-to-distance ratio of Sgr A*, the EHT set bounds on the fractional deviation $\delta \equiv (d_{sh}/d_{sh,sch}) - 1$ between the inferred shadow radius and that of a Schwarzschild black hole of angular size $\theta_{sh} = 3\sqrt{3}\theta$:
  - Keck: $\delta = -0.04^{+0.09}_{-0.10} \to 4.5M \lesssim r_{sh} \lesssim 5.5M$.
  - VLTI: $\delta = -0.08^{+0.09}_{-0.09} \to 4.3M \lesssim r_{sh} \lesssim 5.3M$
- Moreover, assuming both sets to be uncorrelated, one can average over both results to get[16]

$$\delta = -0.0060 \pm 0.0063 \to 4.55M \lesssim r_{sh} \lesssim 5.21M(1\sigma); 4.22M \lesssim r_{sh} \lesssim 5.54M(2\sigma) \tag{295}$$

[15] K. Akiyama et. al. [Event Horizon Telescope], Astro. J. Lett. **930** (2022) L17.

[16] S. Vagnozzi, R. Roy, Y. D. Tsai and L. Visinelli, [arXiv:2205.07787 [gr-qc]].

Constraining the size of Reissner-Nordström black holes

- For a Reissner-Nordström black hole a photon sphere is present whenever $0 < Q^2 < (9/8)M^2$ (which includes in particular overcharged black holes, i.e., naked singularities)

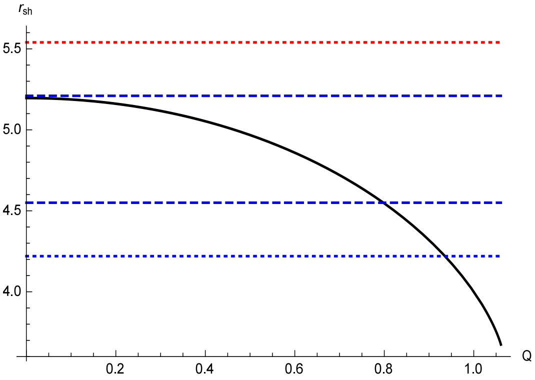


Figure: The shadow radius r_{sh} of the Reissner-Nordström black hole as a function of the charge

- At 1σ the electric charge is constrained within the range $0 \leq Q \lesssim 0.8$, and at 2σ within the range $0 \leq Q \lesssim 0.9$.

Shadows from accretion disks - VII. Geometrically thin disk

- In a geometrically thin disk the emission can only be emitted from the equatorial plane so that the specific intensity I_ν^{em} only depends on the radial coordinate.
- Analytical models can be implemented upon a two simplifications: i) neglecting absorption $\alpha_\nu = 0$, ii) monochromatic emission, $j_\nu \sim \nu^2$, so that the transfer equation implies that I_ν/ν^3 is conserved along a photon's trajectory, iii) isotropic emission, i.e., $I_\nu^{em} = I(x)$.
- Gravitational redshift acts upon the frequency of the photon in the rest frame of the gas in the disk, ν_e ,with associated intensity I_{ν_e} to, via Liouville's theorem, yield a photon frequency measured by the distant observer ν_o with intensity $I_{\nu_0}^{ob} = (\nu_e/\nu_0)^3 I_{\nu_e}^{em}$, which for a spherically symmetric geometry implies that $I_{\nu_0}^{ob} = A^{3/2}(x) I_{\nu_e}$. Integrating over the full spectra of frequencies, $I^{ob} = \int d\nu_e I_{\nu_e}^{ob}$, one finds the result[17] $I^{ob} = A^2(x) I(x)$.
- Additional intersections with the disk contribute to the total luminosity on the observer's screen, so that we can write (keeping only up to the third intersection by the arguments above):

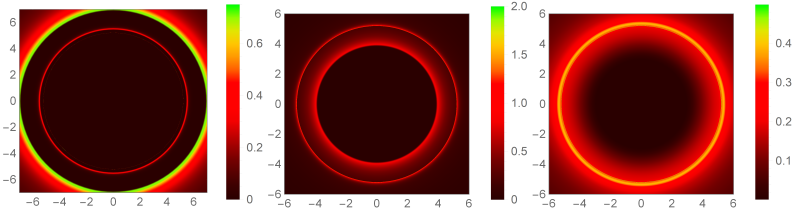
$$I^{ob} = \sum_m A^2(x) I(r) \Big|_{x=x_m(b)} \ , \qquad (296)$$

  where the so-called *transfer function* $x_m(b)$ encodes the location of the m-th intersection of the light ray with impact parameter b with the disk.
- The net effect is the development of an infinite sequence of strongly-lensed self-similar rings, exponentially decreased in brightness, and approaching a central brightness depression whose size is not determined by the critical curve but by the location of the inner edge of the disk.
- Note that the profile for $I(r)$ is still a free-function!.

[17] S. E. Gralla, D. E. Holz and R. M. Wald, Phys. Rev. D 100 (2019) 024018.

Light rings and shadow of the Schwarzschild black hole

- The optical appearance of the object is very different depending on the profile for $I(r)$, modelled in terms of the (truncated) location of the inner edge of the disk + decay with r.
- We run three simulations on face-on orientation with $r_{ie} = r_{isco} = 6M$ (left), $r_{ie} = r_m = 3M$ (middle) and $r_{ie} = r_h = 2M$ (right). Admittedly too strong an assumption from a physical point of view, yet necessary for computations.



- The new light rings are barely appreciable because of their dimness and the fact that they are typically superimposed with the direct emission.
- The size of the shadow is determined by the location of the inner edge of the disk. The minimum size corresponds to every model in which $r_{ie} = r_h = 2M$: the *inner shadow*[18].
- The sharpness of the new light rings makes them to die-off slowly in the Fourier domain: they will dominate the interferometric signal for very-high frequencies (baseline lengths of $\sim 10^{10} - 10^{11}$ wavelengths): ideal playground for very-long baseline interferometry[19].

[18] A. Chael, M. D. Johnson and A. Lupsasca, Astrophys. J. **918** (2021) no.1, 6.

Lyapunov exponents

- Lyapunov exponents are a measure of the rate at which nearby geodesics diverge (or converge) in the phase space. Let us first rewrite the geodesic equation under the form

$$\left(\frac{dr}{du}\right)^2 = \mathcal{V}(r) \tag{297}$$

where $\mathcal{V}(r) = E^2 - A(r)L^2/r^2$. In this notation, circular unstable orbits, $r = r_{ps}$ are determined by $\mathcal{V}(r_{ps}) = \mathcal{V}'(r_{ps}) = 0$.

- Linearizing around the photon sphere, $r = r_m + \delta r$, one finds

$$\frac{d\delta r}{du} = \sqrt{\mathcal{V}(r_m + \delta r)} \approx \sqrt{\frac{1}{2}V''(r_m)}\delta r = \frac{L}{(3M)^2}\delta r \tag{298}$$

where we have first imposed the photon sphere condition and next used the Schwarzschild values.

- In coordinate time, using the conserved variables, one can rewrite this expression as

$$\frac{d\delta r}{dt} \approx \sqrt{\frac{A(r_m)^2\mathcal{V}''(r_m)}{2E^2}}\delta r = \frac{\delta r}{3\sqrt{3}M} \tag{299}$$

which can be easily integrated to give[20]:

$$\delta r(r) \approx e^{\gamma_L t}\delta r_0 \quad ; \quad \gamma_L = \sqrt{\frac{\mathcal{V}''(r_m)}{2\dot{t}^2}} = \frac{1}{3\sqrt{3}M} \tag{300}$$

- This equation implies that slightly perturbed near-critical orbits diverge exponentially in coordinate time, where positive values of the coefficient γ_L indicate a strong sensibility to initial conditions (with an instability scale $T \sim 1/\gamma_L$).

[20] A more rigorous derivation of this result can be found in arXiv:0812.1806 [hep-th].

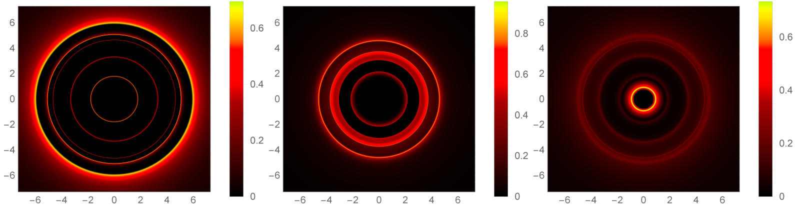
Multi-ring structure as observational discriminators - I

- If the potential qualitative departs from the Schwarzschild one in the inner region (while keeping the asymptotic structure compatible with weak-field tests), new features arise.
- Not only the original light rings can be infused with additional luminosities, but new light rings not foreseen in the Schwarzschild/Kerr solutions may arise.
- Toy-example: Black Bounce-Type II:

$$A(x) = B^{-1}(x) = 1 - \frac{2Mx^2}{r^3(x)}; r^2(x) = x^2 + a^2 \tag{301}$$

with a a new parameter. GR-frame or modified gravity?. Take an agnostic approach.

- For $\frac{4\sqrt{3}}{9} < \frac{a}{M} < \frac{2\sqrt{5}}{5}$ it describes a family of traversable wormholes with two photon spheres (on each side).



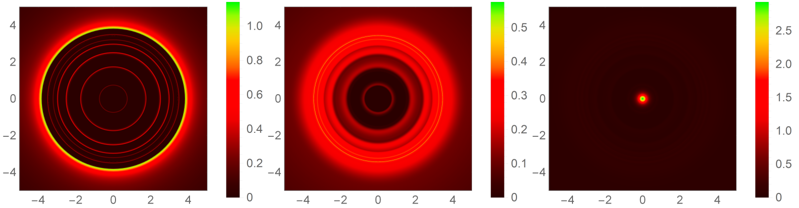
- In a model in which $r_{ie} = r_{isco}$, $r_{ie} = r_{ps}$, and $r_{ie} = r_{th}$ the optical appearance changes a lot!.

Multi-ring structure as observational discriminators - II

- Toy-example: Simpson-Visser *"eye of the storm"*

$$A(r) = 1 - \frac{-2Me^{-l/r}}{r} \tag{302}$$

- Solutions with $-0.7358 \lesssim l/M < 0.8$ have two photon spheres, no event horizon, and no curvature singularity either.



- The presence of additional light rings not foreseen in the Schwarzschild-Kerr solutions seems to have a generic property of those models having a reflective surface at the center (i.e. an effective potential taking finite values there).
- The presence of minima in the potential (anti-photon spheres) may however bring in instabilities[21], though the time-scale of them seem to be strongly model-dependent.

[21] V. Cardoso, L. C. B. Crispino, C. F. B. Macedo, H. Okawa, and P. Pani, Phys. Rev. D 90, 044069 (2014).

The Novikov-Thorne accretion disk model

Figure: Under construction

Null geodesics in a Kerr space-time

- The Kerr geometry admits a complete set of first integrals of motion [22]: the energy E, the angular momentum L, and the Carter constant K.
- Using the representation $l^a = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ for the null vector $g_{ab}l^a l^b = 0$ tangent to the geodesic, $l^a \nabla_a l^b = 0$, and the time-translation $\xi_t^a = \partial_t^a$ and axial $\xi_\theta^a = \partial_\phi^a$ Killing vectors in this geometry, these quantities are (here $\Phi \equiv \frac{2Mr}{\Sigma}$)

$$E \equiv g_{ab}l^a \xi_t^b = (1-\Phi)\dot{t} + a\Phi \sin^2\theta\dot{\phi} \quad (303)$$

$$L \equiv -g_{ab}l^a \xi_\phi^b = -a\Phi \sin^2\theta \dot{t} + (r^2 + a^2 + \Phi \sin^2\theta)\sin^2\theta\dot{\phi} \quad (304)$$

$$K \equiv 2\Sigma l^a l^b \tilde{l}_a \tilde{n}_b = \Delta\left(\dot{t} - \frac{\Sigma\dot{r}}{\Delta} - a\sin^2\theta\dot{\phi}\right)\left(\dot{t} + \frac{\Sigma\dot{r}}{\Delta} - a\sin^2\theta\dot{\phi}\right) \quad (305)$$

- Enforcing the null condition above, and rewriting the above system a la Chandrasekhar, one finds the system

$$\Sigma\dot{t} = \frac{1}{\Delta}\left[E\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right) - 2aMrL\right] \quad (306)$$

$$\Sigma^2\dot{r}^2 = \mathcal{R}(r) \quad (307)$$

$$\Sigma^2\dot{\theta}^2 = \Theta(\theta) \quad (308)$$

$$\Sigma\dot{\phi} = \frac{1}{\Delta}\left[2EaMr + (\Sigma - 2Mr)\frac{L}{\sin^2\theta}\right] \quad (309)$$

with the definitions $\mathcal{R}(r) = (E(r^2+a^2) - aL)^2 - K\Delta$, $\Theta(\theta) = L - (\frac{L}{\sin^2\theta} - aE\sin\theta)^2$.

[22] Chandrasekhar S., 1983, The mathematical theory of black holes. Oxford University Press, 646 p., Oxford

Photon shell of the Kerr black hole

- For a Kerr black hole, unstable bound null geodesics do not form a sphere, existing instead at a photon shell determined by a radius (in the Boyer-Lindquist coordinates) $r_c^- \leq r_c \leq r_c^+$ given by

$$r_{c,\pm} = 2M \left[1 + \cos\left(\frac{2}{3} \arccos(\pm a/M) \right) \right] \tag{310}$$

  where $+/-$ for retrogade/direct orbits.
- The bound orbits at $b = b_{c,\pm}$ are confined to the equatorial plane $\theta = \pi/2$, while those in the range $r_c^- < r_c < r_c^+$ oscillate between the two polar angles $\theta_\pm$ given by

$$\theta_\pm = \arccos(\mp\sqrt{u_+}) \tag{311}$$

$$u_\pm = \frac{r}{a^2(r-M)^2}\left[-r^3 + 3M^2 r - 2a^2 M \pm 2\sqrt{M\Delta(2r^3 - 3Mr^2 + a^2 M)} \right] \tag{312}$$

- In other words, the photon shell is made of

$$\boxed{r_c^- \leq r_c \leq r_c^+,\ \theta_- \leq \theta \leq \theta_+,\ 0 \leq \phi < 2\pi,\ -\infty < t < +\infty}$$

  and degenerates to the photon sphere of the Schwarzschild solution when $a \to 0$.

Basis of magnetohydrodynamics simulations

- Since both M87 and SrgA* accretion rate is significantly below the Eddington's limit, while the plasma is typically hot, strongly magnetized and turbulent, the most "successful" models are magnetically arrested ones incorporating plasma flow near the black hole via general relativistic magnetohydrodynamics (GRMHD) simulations. The main bottleneck is the suitable modelling of the relativistic radiative transport equations.
- The Boltzmann equation for unpolarized photons (scattering can be safely neglected) is

$$\frac{d}{d\lambda}\left(\frac{I_\nu}{\nu^3}\right) = \left(\frac{j_\nu}{\nu^3}\right) - (\nu\alpha_\nu)\left(\frac{I_\nu}{\nu^3}\right) \tag{313}$$

  where parenthesis denote frame-invariant quantities, ν is the photon's frequency, I_ν the intensity, j_ν the emissivity, and α_ν the absorptivity.
- Analytical models for the emissivity and absorptivity are typically set as[23]

$$j_\nu = Cn\left(\frac{\nu}{\nu_p}\right)^{\alpha}; \alpha_\nu = ACn\left(\frac{\nu}{\nu_p}\right)^{-(2.5+\alpha)} \tag{314}$$

  where A, C, α some constants, $\nu_p = 230$GHz is the pivotal frequency, while

$$n = n_0 \exp[-\tfrac{1}{2}\left[\left(\tfrac{r}{10}\right)^2 + z^2\right] \tag{315}$$

  is the particle number, with $z = h\cos\theta$ and h the disk's height.
- Standardized Imaging Tests are then run using different assumptions for these constants.

WARNING: Heavy numerical relativity methods are required.

[23] R. Gold, *et al.* Astrophys. J. **897** (2020) no.2, 148

Primordial black holes

- The idea that gravitationally collapsed objects of very low mass (primordial black holes - PBHs) could have been formed by large amplitude density perturbations in the very early universe was proposed by S. Hawking more than 40 years ago[24].
- Comparison of the cosmological distance at a time t after the Big Bang with the density associated to a BH of mass M suggest such PBHs to have a mass of order

$$M \sim \frac{c^3 t}{G} \sim 10^{15} \left(\frac{t}{10^{-23} s} \right) g \tag{316}$$

- This spans a huge range of masses: from 10^5 at a Planck's time $t \sim 10^{-43}$s to $10^5 M_\odot$ at $t \sim 1$s, while the existence of quantum instabilities inherent to the existence of the event horizon which render such PBHs quantum mechanically unstable, implies that only PBHs with masses above 10^{15}g could have survived until today.
- Mechanisms for formation: collapse from large inhomogeneities? result of phase transitions? quantum effects during inflation?.
- Constraints on their evaporation come from different sources[25]: BB nucleosynthesis, CMB, extragalactic gamma and cosmic rays, PBH explosions,. . .

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[24] S. Hawking, Gravitationally collapsed objects of very low mass, Mon. Not. Roy. Astron. Soc. 152 (1971) 75.

[25] B. Carr, K. Kohri, Y. Sendouda and J. Yokoyama, Rept. Prog. Phys. **84** (2021) no.11, 116902.

LESSON VIII. BLACK HOLES BEYOND GR

Recommended literature:

- J. Beltran Jimenez, L. Heisenberg, G. J. Olmo D. Rubiera-Garcia, Phys. Rept. **727** (2018)1.
- A. De Felice, S. Tsujikawa, Living Rev. Rel. **13** (2010)3.
- S. Capozziello, M. De Laurentis, Phys. Rept. **509** (2011)167.
- T. Clifton, P. G. Ferreira, A. Padilla, C. Skordis, Phys. Rept. **513** (2012)1.
- S. Nojiri, S. D. Odintsov, V. K. Oikonomou, Phys. Rept. **692** (2017)1.
- L. Heisenberg, Phys. Rept. **796** (2019)1.

Why going beyond GR?

- GR a successful theory: solar system experiments, GW + gamma rays bursts, ΛCDM, recent galactic observations, compatibility with shadows . . . Why going beyond GR?.
- Phenomenological side: need of adding extra fields to match GR (ΛCDM) with observational data:
  - Dark matter - no evidence at particle accelerators/cosmic rays/so on.
  - Dark energy - violation of energy conditions. Evidence?
  - Inflation - degeneracy of models?,
- Theoretical side:
  - Compatibility between GR and quantum mechanics. Quantum gravity at Planck's scale?.
  - Unavoidability of spacetime singularities deep inside black holes and in the early Universe. Related to quantum gravity?.
  - Further compact objects may exist with similar/different GW signatures.
- Need to rethink fundamental principles, and understand their phenomenological consequences.

The era of strong-field tests of GR

- Modified gravity after GW150914 and GW170817+GW170817A: compatible with GR expectations for coalescence of a black holes and neutron stars.
- LISA-VIRGO + satellite collaborations reported two main results of interest for modified gravity:
  - Gravitational waves propagate at the same speed as electromagnetic radiation. Experimental constraint: one part in $\sim 10^{-16}$.
  - Slaughter on modified theories of gravity[26] (mainly those motivated by cosmological considerations)
    - Viable: GR, quintaessence/K-essence, Brans-Dicke/$f(R)$, Kinetic Gravity Braiding, Derivative Conformal, Disformal Tuning, some DHOST, ...
    - Non-viable: quartic/quintic Galileons, Fab Four, de Sitter Hordenski, Gauss-Bonnet, quartic/quintic GLPV, some DHOST, ...
  - Purely tensor polarizations strongly favoured over purely scalar/vector polarizations.
- Tests on the Kerr (and Kerr-like) solution from black hole shadows (+ Iron - Kα line, etc).
- Tests of GR from compact and sub-stellar objects: neutron stars, white, red and brown dwarfs, and main-sequence stars. Limiting mass and stellar evolution.
- All sum up: the birth of multimessenger astronomy: astronomy with different carriers (light, neutrinos, and GWs).

[26]. M. Ezquiaga and M. Zumalacárregui, Phys. Rev. Lett. **119** (2017) no.25, 251304.

Parametrizing deviations from the Kerr black hole - I

- To test deviations from the Kerr hypothesis at the horizon scale, it is useful to develop model-independent frameworks parametrizing generic axisymmetric black hole geometries through a finite number of adjustable quantities, which can be later constrained with observational data.
- In a suitable coordinate system a generalization of the Kerr metric can be written as

$$ds^2 = -\frac{-N^2 - W^2 \sin^2\theta}{K^2} dt^2 - 2Wr\sin^2\theta dt d\theta + K^2 r^2 \sin^2\theta d\phi^2 + \Sigma\left(\frac{B^2}{N^2}dr^2 + r^2 d\theta^2\right) \tag{317}$$

where the functions $N(r,\theta)$, $W(r,\theta)$, $K(r,\theta)$, $\Sigma(r,\theta)$ and $B(r,\theta)$ must be chosen in such a way that these coefficients can be constrained from the PPN limit[27], e.g., in the spherically symmetric limit

$$g_{tt} \approx 1 - \frac{2M}{r} + 2(\beta-\gamma)\frac{M^2}{r^2}; g_{rr} \approx 1 + \frac{2\gamma M}{r} \tag{318}$$

- In GR: $\beta = \gamma = 1$, while Lunar Laser Ranging experiment set the bound $|\beta - 1| \leq 2.3 \times 10^{-4}$.
- Near the event horizon, the metric functions are parametrized in series of the variable $x = 1 - 2M/r$, and observational data - e.g. size of the shadow - would put constraints upon the different terms in the expansion.

[27] D. Psaltis *et al.* [Event Horizon Telescope], Phys. Rev. Lett. **125** (2020) no.14, 141104.

Parametrizing deviations from the Kerr black hole - II

- Examples of parametrizations (in units of M)
- Johannsen-Psaltis[28]:

$$g_{tt}^{JP} = -\left(1-\frac{2}{r}\right)\left(1+\sum_{i=2}^{\infty}\frac{\alpha_i}{r_i}\right) \tag{319}$$

- Rezzola-Zidenko[29]:

$$g_{tt}^{RZ} = -x[1-\varepsilon(1-x)+(a_0-\varepsilon)(1-x^2)+\tilde{A}(x)(1-x)^3] \tag{320}$$

where

$$\tilde{A}(x) = \frac{a_1}{1+\frac{a_2 x}{1+\frac{a_3 z}{\dots}}} \tag{321}$$

- These metric coefficients can be constrained by several means, the simplest of them once again the size of the shadow.
- EHT observations of Sgr A* report that

$$-1.1 \lesssim \alpha_2 \lesssim 0.5, -3.1 \lesssim \alpha_3 \lesssim 1.5, -7.8 \lesssim \alpha_4 \lesssim 4.6 \quad \text{for JP metric} \tag{322}$$

$$-0.2 \lesssim a_0 \lesssim 0.7, -0.3 \lesssim a_1 \lesssim 1.0 \quad \text{for RZ metric} \tag{323}$$

- It can be combined with other tests: S2 orbit, pulsar timing, universality of free-fall, GWs,...

[28] T. Johannsen, Phys. Rev. D **87** (2013) no.12, 124017.

[29] R. Konoplya, L. Rezzolla and A. Zhidenko, Phys. Rev. D **93** (2016) no.6, 064015.

- $f(R)$ is perhaps the simplest extension of GR. It just requires to promote R to a more general function.
- Field equations

$$f_R R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box] f(R) = \kappa^2 T_{\mu\nu} \tag{324}$$

- Tracing over $g^{\mu\nu}$ one finds

$$R f_R - 2f + 3\Box f(R) = \kappa^2 T \tag{325}$$

  This implies that $f(R)$ can be interpreted as a dynamical scalar dof (aka scalar field $\phi \equiv f_R$).
- The fourth-order equation nature of the eom + new propagating dof greatly complicates the obtention + interpretation of solutions.
- Solutions can be obtained within the *constant-field approximation*: $f_R(R_0) - 2f(R_0) = 0$. **Physical viability and interpretation?**.
- Solutions with non-constant curvature are also known, in both analytical and numerical form. Too much literature to summarized on these slides.
- A problem related to the definition of the mass, since the contribution from the scalar field is never zero. Are vacuum solutions those of GR?. Does Birkhoff's theorem hold?.

Black holes in Gauss-Bonnet gravity

- A theory with second-order field equations that admit exact black hole solutions based on the Gauss-Bonnet invariant.
- In four space-time dimensions Gauss-Bonnet is a topological invariant and yields no modified dynamics. Promote the theory to D dimensions. Action:

$$\mathcal{S}_{GB} = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}[(R-2\Lambda)+\alpha_{GB}(R^2-4R_{\mu\nu}R^{\mu\nu}+R^{\alpha}{}_{\beta\mu\nu}R_{\alpha}{}^{\beta\mu\nu})] \qquad (326)$$

- Field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu}+\alpha_{GB}G^{GB}_{\mu\nu}=\kappa^2 T_{\mu\nu} \qquad (327)$$

$$G_{\mu\nu} = 2[RR_{\mu\nu}-2R_{\mu\alpha}R^{\alpha}{}_{\nu}-2R^{\alpha\beta}R_{\mu\alpha\nu\beta}+R_{\mu}{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}]-\frac{1}{2}g_{\mu\nu}L_{GB} \qquad (328)$$

- Surprisingly, this theory admits (non-linear) electrostatic, spherically symmetric solutions under closed form (though via a somewhat lengthly calculation!)

$$g_{\alpha}^{-}(r)=1+\frac{r^2}{l_{\alpha}^2}\left(1-\sqrt{1+\frac{2l_{\alpha}^2}{r^{D-1}}\left(m-\frac{2\kappa^2}{D-2}\omega_{D-2}\int_r^{\infty}R^{D-2}T_0^0(R,q)dR-\frac{r^{D-1}}{l_{\Lambda}^2}\right)}\right), \qquad (329)$$

where we have defined $l_{\alpha}^2=2\widetilde{\alpha}$.

- Main features: several branches of solutions, not all defined everywhere. Up to two horizons. Singularities everywhere. Physical impact?.

Modified gravity with scalar fields: the Horndeski family

- The most general gravitational action with a single scalar field having second-order equations of motion is called the Horndeski family, which is given by[30]

$$\mathcal{S}=\sum_{i=2}^{5}\int d^4x\sqrt{-g}\mathcal{L}_i(X,\phi) \tag{330}$$

where the $\mathcal{L}_i$ terms represent the following contributions

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi,X) && (331)\\
\mathcal{L}_3 &= G_3(\phi,X)\Box\phi && (332)\\
\mathcal{L}_4 &= G_4(\phi,X)R-2G_{4,X}(\phi,X)[(\Box\phi)^2-\phi_{\mu\nu}^2] && (333)\\
\mathcal{L}_5 &= G_5(\phi,X)G_{\mu\nu}\phi^{\mu\nu}+\frac{G_{5,X}}{3}[(\Box\phi)^3+2\phi_{\mu\nu}^3-3\phi_{\mu\nu}^2\Box\phi] && (334)
\end{aligned}$$

where the functions $G_i(X,\phi)$, which depend on both the scalar field ϕ and its kinetic term $X=\nabla_\mu\phi\nabla^\mu\phi$, characterize the particular member in the family (note that GR corresponds to $\{G_4=0,G_2=G_3=G_5\}$), and the notations $\phi_{\mu\ldots\nu}\equiv\nabla_\mu\ldots\nabla_\nu\phi$, and $\Box\phi\equiv g^{\mu\nu}\phi_{\mu\nu}$.
- This family can be further extended to a more general class of healthy theories having higher-order equations but such that the application of a number of hidden constraints prevent the propagation of the Ostrogradsky ghosts. It includes as members the beyond Horndeski class [31] and the even more general class called degenerate higher-order scalar-tensor (DHOST) theories.

[30] D. Langlois and K. Noui, JCAP **02** (2016), 034.

[31] M. Zumalacárregui and J. García-Bellido, Phys. Rev. D **89** (2014), 064046

Playing with the Horndeski family

- For instance, the beyond Horndeski model adds two extra terms

$$\mathcal{L}_4^{bH} = F_4(X)\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\sigma}^{\alpha\beta\gamma}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma} \tag{335}$$

$$\mathcal{L}_5^{bH} = F_5(X)\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}\phi_{\sigma\delta} \tag{336}$$

- Chamaleon/Vainshtein mechanisms must be called upon in order to "screen" the additional scalar fields far from the matter sources, so as not to get into conflict with weak field/solar system experiments[32].
- Analytical solutions can be found in specific branches of the theory[33]. An example is the "Quartic Horndeski" square-root model with

$$G_2 = \eta X; G_4 = \frac{M_P^2}{16\pi} + \beta\sqrt{-X}; G_3 = G_5 = F_4 = F_5 \tag{337}$$

with η, β free dimensionless parameters, and another is the Quartic beyond-Horndeski model with

$$G_2 = \eta X; G_4 = \frac{M_P^2}{16\pi}; F_4 = \gamma(-X)^{-3/2}; G_3 = G_5 = F_5 = 0 \tag{338}$$

- Go on with your favourite choice of free functions to produce as many papers on new solutions as desired.

[32] K. Koyama, G. Niz and G. Tasinato, Phys. Rev. D **88** (2013), 021502.

[33] E. Babichev, C. Charmousis and A. Lehébel, JCAP **04** (2017), 027.

Modified gravity with vector fields: the Proca family

- Why stopping with addition of scalar fields? vectors fields are also at our disposal.
- The most general derivative self-interactions for a massive vector field with second order equations of motion and three propagating degrees of freedom on flat space-time (up to disformal transformations) corresponds to the generalized Proca theory, which is given by the Lagrangian density[34]

$$\mathcal{L}_{gp} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{n=2}^{5} \alpha_n \mathcal{L}_n \tag{339}$$

where the self-interaction terms to the electromagnetic kinetic term are given by

$$\mathcal{L}_2 = f_2(X, F, Y) \tag{340}$$

$$\mathcal{L}_3 = f_3(X)\partial_\mu A^\mu \tag{341}$$

$$\mathcal{L}_4 = f_4(X)[(\partial \cdot A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho] \tag{342}$$

$$\begin{aligned}\mathcal{L}_5 = {} & f_5(X)[(\partial \cdot A)^3 - 3(\partial \cdot A)\partial_\rho A_\sigma \partial^\sigma A^\rho + 2\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma] \\ & + \tilde{f}_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^{\beta}_{\mu}\partial_\alpha A_\beta \end{aligned} \tag{343}$$

$$\mathcal{L}_6 = f_6(X)\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}\partial_\alpha A_\mu \partial_\beta A_\nu \tag{344}$$

where $X = -A_\mu A^\mu/2$ and $Y = A^\mu A_\nu F_\mu{}^\alpha F_{\nu\alpha}$ and overtildes denote duals.

- Further generalizations of this action are possible, even mixing vector with scalar and tensor fields, and even considering non-abelian and multi-field models.
- Finding exact/numerical black hole solutions for specific choices is quickly degenerating.

[34] L. Heisenberg, Phys. Rept. **796** (2019) 1-113.

Further alternatives

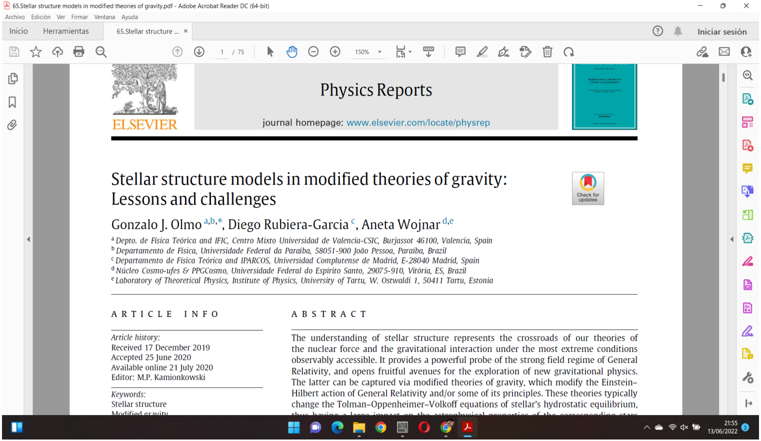
- While these are (arguably) the most popular current extensions, other proposals are available:
  - Break/add physical core principles imbued in GR: Lorenz-breaking theories, doubly special-relativity...
  - Break/change the way matter and gravity couple: non-minimal gravity-matter couplings, action-dependent theories...
  - Do not stop at adding/re-defining spin-0 and spin-1 additional fields but consider also spin-2: massive gravity, mimetic gravity...
  - Increment number of space-time dimensions: brane-world models.
  - Mix theories between themselves: Einstein-Maxwell-Dilation-GB-Proca-Whatever.
  - And of course fundamental approaches to quantization of gravity are still lurking, even if their role is nowadays quite diminished: string theory, loop quantum gravity...
- Not tired yet?. Propose your favourite model of gravity!.

Main difficulties faced

- Every such alternative proposal to GR faces a number of common troubles for the sake of black hole physics.
- Theoretical front:
  - Field equations tend to be significantly harder to solve than their GR counterparts. Analytical solutions may even be impossible except under excessively constrained conditions.
  - Shortcut, GR-based, methods to solve equations do not necessarily work beyond of it (e.g. Janis-Newman trick).
  - The body of knowledge developed within GR (cosmic censorship, gravitational collapse, thermodynamic laws, etc) does not necessarily hold beyond of it.
- Numerical front:
  - Most available numerical recipes are tightly attached to the structure of GR Einstein's field equations, thus rendering the task of adapting them to every modification of GR very costly from a resources and human power point of view.
  - Indeed, getting to a well-posed set of equations for numerical equations to be run is not guaranteed a priori.
- Observational front:
  - Theories with extra fields, or changing the way (different sectors of the) matter propagates get very easily in trouble with weak-field limit observations.
  - Conversely, those modifying GR-dynamics only in high curvature/density regimes, may be impossible to be distinguished at horizon/photon sphere-scale observations.

Stellar and sub-stellar objects

- Besides black holes and exotic horizonless objects, of course both stellar and sub-stellar objects are at our disposal to test modified gravity effects.
- Relativistic: neutron stars and (to some extend) white dwarfs:
- Non-relativistic stars: red and brown dwarfs, main-sequence stars and (to some extend) white dwarfs.



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Stellar structure models in modified theories of gravity: Lessons and challenges

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ABSTRACT

The understanding of stellar structure represents the crossroads of our theories of the nuclear force and the gravitational interaction under the most extreme conditions observably accessible. It provides a powerful probe of the strong field regime of General Relativity, and opens fruitful avenues for the exploration of new gravitational physics. The latter can be captured via modified theories of gravity, which modify the Einstein–Hilbert action of General Relativity and/or some of its principles. These theories typically change the Tolman–Oppenheimer–Volkoff equations of stellar's hydrostatic equilibrium,

- A great playground for modified theories of gravity!.

LESSON IX. BLACK HOLES IN THE METRIC-AFFINE APPROACH

A quick digression: the three formulations of GR

- In general, an affine connection can be split into three independent pieces:

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {}^{\lambda}{}_{\mu\nu} \right\} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} \tag{345}$$

  - Levi-Civita connection of the metric $g_{\mu\nu}$ (associated to curvature) : $\left\{ {}^{\lambda}{}_{\mu\nu} \right\}$.
  - Contortion (associated to torsion $T^{\lambda}{}_{\mu\nu} \equiv 2\Gamma^{\lambda}{}_{[\mu\nu]}$)

$$K^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} T^{\lambda}{}_{\mu\nu} + T_{(\mu}{}^{\lambda}{}_{\nu)} \tag{346}$$

  - Disformation (associated to nonmetricity $Q_{\rho\mu\nu} \equiv \nabla_{\rho} g_{\mu\nu}$)

$$L^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\beta} \left(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu} \right) \tag{347}$$

- Three equivalent (modulo boundary terms technicalities) formulations of GR [35]
  - Riemannian-based GR: $R \neq 0, T = 0, Q = 0$.
  - Teleparallel equivalent of GR: $R = 0, T \neq 0, Q = 0$.
  - Symmetric (or coincident) teleparallel GR: $R = 0, T = 0, Q \neq 0$.

[35] See Jimenez et al. arXiv:1710.03116 [gr-qc].

A simple extension of GR: Einstein-Cartan

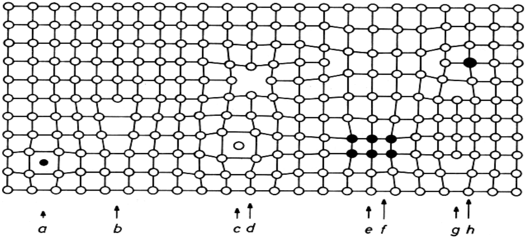
- Perhaps the simplest geometry of this kind is that of adding torsion (and it is dubbed as Einstein-Cartan), since it can be embedded within GR itself, with formally the same action.
- Torsion must be considered whenever one considers the presence of fermions in the theory (since they see the connection).
- Defining torsion as $T^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\lambda}_{[\mu\nu]}$, the affine (Cartan) connection is split into two pieces $\Gamma^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu}$ made up of the Riemannian (torsionless) connection and the contortion tensor $K_{\alpha\mu\nu} = T_{\alpha\mu\nu} + 2T_{(\mu\nu)\alpha}$.
- The matter Lagrangian contains now a contribution in the torsion piece, $\mathcal{L}_m(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}, \psi_m)$ in such a way that the gravitational field equations are also GR-like, $\tilde{G}_{\mu\nu} = T^{\text{eff}}_{\mu\nu}$ where the effective energy-momentum tensor $T^{\text{eff}}_{\mu\nu} = T_{\mu\nu} + U_{\mu\nu}$ contains now a torsion-induced piece which can be computed on a case-by-case basis.
- Torsion is NOT a dynamical field and DOES NOT propagate in vacuum, but instead satisfies the so-called Cartan's equations

$$T^{\alpha}{}_{\beta\gamma} + \delta^{\alpha}_{\beta} T_{\gamma} - \delta^{\alpha}_{\gamma} = \kappa^2 s^{\alpha}{}_{\beta\gamma} \tag{348}$$

  where $T_{\beta} \equiv T^{\gamma}{}_{\beta\gamma}$ is the torsion vector and $s^{\alpha\mu\nu} \equiv \frac{\mathcal{L}_m}{\delta K_{\alpha\mu\nu}}$ is the spin tensor with dimensions of energy/area.
- Therefore, torsion effects should appear at very high-energy (spin) densities, which manifest via new matter-type terms to the Einstein-like equations, but also on the dynamics of fermions/bosons coupled to gravity.
- Black hole solutions can be *"easily"* found by setting a set of matter fields, and computing the $U_{\mu\nu}$ tensor to get to the new Einstein equations.

Crystalline structures with defects - I

- A microstructure with a macroscopic continuum limit is found in condensed matter systems such as Bravais crystals or graphene.
- Crystalline structures may have defects of different kinds



a) Interstitial impurity atom, b) Edge dislocation, c) Self interstitial atom d) Vacancy, e) Precipitate of impurity atoms, f) Vacancy type dislocation loop, g) Interstitial type dislocation loop, h) Substitutional impurity atom

- In real crystals, the density of defects is generally non-zero.
- Defects have dynamics: upon the action of forces or heat, defects can move and interact with the same or other kinds of defects.

Crystalline structures with defects -II

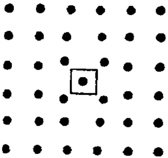
- The continuum limit of these structures is naturally described in terms of a metric-affine space
  - At each point we find 2 or 3 lattice vectors defining the microstructure.
  - Moving along those vectors we jump from atom to atom.
  - Distances can be measured by step counting along crystallographic directions to provide an intuitive idea of metric:

$$ds^2 = g_{ij} dx^i dx^j$$

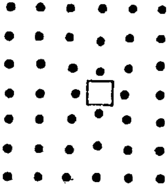
    with $g_{ij} = \delta_{ij}$ and $\Gamma^a_{bc} = 0$ in suitable coordinates.
  - An affine connection $\Gamma^\lambda_{\mu\nu}$ is used to transport vectors and define geodesics.

Crystalline structures with defects - III

- The step-counting procedure breaks down when considering point defects.



Self-interstitial



Vacancy

- The continuum limit must be done with care:
  - Determine the density of defects, whose average separation scale can be much larger than the interatomic separation.
  - Knowledge of that density allows to determine the deformations of lengths, areas, and volumes w.r.t. an idealized reference structure without defects: $g_{\mu\nu}^{Phys} = D_\mu^\alpha q_{\alpha\nu}^{Aux}$, where D_μ^α depends on the density of defects.
- The idealized geometry has a well defined parallel transport, $\nabla_\alpha^\Gamma q_{\mu\nu} = 0$, but there is non-metricity on the physical geometry $Q_{\alpha\mu\nu} \equiv \nabla_\alpha^\Gamma q_{\mu\nu} \neq 0$,
- Cartan torsion, $S_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \Gamma_{\gamma\beta}^\alpha$ is the continuum version of crystal dislocation.
- Independent $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\lambda$ are necessary to account for microscopic defects.

Non-Riemannian geometries

- The usual Riemannian description of GR is not the single consistent geometry to describe gravitational phenomena
- Riemannian geometry is not enough to deal with continuous systems with defects on their microstructure.
- Could this lack of versatility of Riemannian geometry be the reason for the existence of space-time singularities?.
- What happens if gravitation is formulated in non-Riemannian spaces?. Why rule out this possibility *a priori*?.
- Whether the space-time geometry is Riemannian or otherwise is to be determined upon experimentation, not by tradition/convention.

Ricci-based gravities in metric-affine approach

- A large family of theories built out as scalars based on the Ricci tensor: ${M^\mu}_\nu = g^{\mu\alpha} R_{\alpha\nu}$ are called **Ricci-based theories of gravity**, and can be systematically analyzed.
- Consider the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}L_G[g_{\mu\nu}, R_{\mu\nu}(\Gamma)] + S_M[g_{\mu\nu}, \psi_m] \tag{349}$$

  Examples: GR, $f(R)$, $f(R, R_{\mu\nu}R^{\mu\nu})$, Born-Infeld inspired theories of gravity...
- Caveat 1: Torsion is not included here, as for minimally coupled bosonic fields it can be removed out of a gauge choice.
- Caveat 2: The antisymmetric part of the Ricci tensor is removed to avoid ghost-like instabilities Beltrán, Delhom, arXiv:1901.08988 [gr-qc] .
- If connection is metric-compatible, $\nabla_\mu^\Gamma(\sqrt{-g}g^{\alpha\beta}) = 0$ (Metric approach) then EOM:

$$\delta g^{\mu\nu} \Rightarrow \left(\frac{\delta L_G}{\delta g^{\mu\nu}} - \frac{L_G}{2}g_{\mu\nu}\right) + \nabla_\lambda\left[g_{\gamma\nu}\frac{\delta L_G}{\delta \Gamma^\mu_{\lambda\gamma}} - g_{\beta\mu}g_{\gamma\nu}g^{\alpha\lambda}\frac{\delta L_G}{\delta\Gamma^\alpha_{\beta\gamma}}\right] = \kappa^2 T_{\mu\nu} \tag{350}$$

- Difficulties: higher-order EOM?, ghosts?, $c_g \neq c$?, compatibility with solar system experiments? viable as effective models of quantized GR?...
- If $g_{\mu\nu}$ and $\Gamma^\alpha_{\beta\gamma}$ are independent (Palatini/metric-affine approach) then EOM:

$$\delta S = \int d^4x\left[\sqrt{-g}\left(\frac{\delta L_G}{\delta g^{\mu\nu}} - \frac{L}{2}g_{\mu\nu}\right)\delta g^{\mu\nu} + \sqrt{-g}\frac{\delta L_G}{\delta\Gamma^\alpha_{\beta\gamma}}\delta\Gamma^\alpha_{\beta\gamma}\right] + \delta S_M \tag{351}$$

$$\delta g^{\mu\nu} \Rightarrow \frac{\delta L_G}{\delta g^{\mu\nu}} - \frac{L_G}{2}g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{352}$$

Einstein-frame representation of RBGs field equations

- RBG theories **always** admit an Einstein-frame representation

$$G^{\mu}{}_{\nu}(q) = \frac{\kappa^2}{|\hat{\Omega}|^{1/2}}\left[T^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\left(L_G + \frac{T}{2}\right)\right] \quad (354)$$

- Definitions:
  - $\nabla^{\Gamma}_{\mu}(\sqrt{-q}q^{\alpha\beta}) = 0$ (i.e., Γ is Levi-Civita of q).
  - $G^{\mu}{}_{\nu}(q) \equiv q^{\mu\alpha}R_{\alpha\nu}(q) - \frac{1}{2}\delta^{\mu}{}_{\nu}R(q)$ is the Einstein tensor of $q_{\mu\nu}$
  - q and g are related via a **deformation matrix** as

$$q_{\mu\nu} = g_{\mu\alpha}\Omega^{\alpha}{}_{\nu} \quad (355)$$

  - For every $\mathcal{L}_G$, the deformation matrix $\Omega^{\alpha}{}_{\nu}$ can always be written on-shell as a function of the energy-momentum tensor of the matter fields.
  - In GR, $\mathcal{L}_G = R$, then $G_{\mu\nu}(q) = \kappa^2 T_{\mu\nu}$ and $q_{\mu\nu} = g_{\mu\nu}$.

- Nice features of RBGs:
  - Second-order field equations.
  - Vacuum solutions are those of GR.
  - No ghost-like instabilities.
  - $c_g = c$ and two tensorial polarizations.
  - Compatibility with solar system experiments and with GW observations so far.

Spherically symmetric space-times

- In general, solving the RBG field equations requires removing all dependences on $g_{\mu\nu}$ in terms of $q_{\mu\nu}$ on their right-hand side using the fundamental relation $q_{\mu\nu} = g_{\mu\alpha}\Omega^{\alpha}{}_{\nu}$. Only possible for scenarios with a large amount of symmetry.
- For matter-energy sources whose energy-momentum tensor is of the form as $T^{\mu}{}_{\nu} = \text{diag}(-\rho, -\rho, K(\rho), K(\rho)$, where $K(\rho)$ is a certain function characterizing the matter fields, the RBG line element can be cast as

$$ds^2 = -\frac{A(x)}{\Omega_1(x)}dt^2 + \frac{dx^2}{A(x)\Omega_2(x)} + r^2(x)d\Omega^2$$

where the functions $\Omega_{1,2}$ characterize the particular combination of RBG + matter field description, and typically contain the mass and charge of the solution as well as additional parameters coming from the RBG Lagrangian density. As for the radial function it satisfies

$$r^2(x) = \frac{x^2}{\Omega_2(x)}$$

where the relation between $r^2 > 0$ (which measures the area of the two-spheres) and the coordinate $x \in (-\infty, +\infty)$ does not need to be monotonic.

An example: electrostatic solution of Born-Infeld gravity - I

- Consider Born-Infeld gravity coupled to an electrostatic (Maxwell) field

$$\mathcal{S}_{EiBI} = \frac{1}{\kappa^2\varepsilon}\int d^4x\left[\sqrt{-\lambda|g_{\mu\nu}+\varepsilon R_{\mu\nu}|}-\sqrt{-g}\right]+\frac{1}{8\pi}\int d^4x\sqrt{-g}X$$

  where ε is new parameter with dimensions of length squared. At curvature scales $|R_{\mu\nu}| \ll 1/\varepsilon$, one has $S_{EiBI} \approx (R-2\Lambda)/(2\kappa^2)+\varepsilon(R^2-R_{\mu\nu}R^{\mu\nu}/2)+\mathcal{O}(\varepsilon^2)$ with $\Lambda=(\lambda-1)/\varepsilon$.

- Electrostatic spherically symmetric solutions of this theory can be written in ingoing Eddington-Finkelstein coordinates as

$$ds^2 = -A(x)dv^2+\frac{2}{\sigma_+}dvdx+r^2(x)d\Omega^2 \ , \tag{356}$$

  where

$$A(x) = \frac{1}{\sigma_+}\left[1-\frac{r_S}{r}\frac{(1+\delta_1 G(r))}{\sigma_-^{1/2}}\right] \tag{357}$$

$$\delta_1 = \frac{1}{2r_S}\sqrt{r_q^3/l_\varepsilon};\sigma_\pm = 1\pm r_c^4/r^4(x); r^2(x) = \frac{x^2+\sqrt{x^4+4r_c^4}}{2} \ , \tag{358}$$

  with $r_c=\sqrt{l_\varepsilon r_q}$, $l_\varepsilon=-2\varepsilon$, $r_q^2=2G_Nq^2$, $M_0=r_S/2$, while the function $G(z)$, with $z=r/r_c$, can be written as an infinite power series expansion of the form

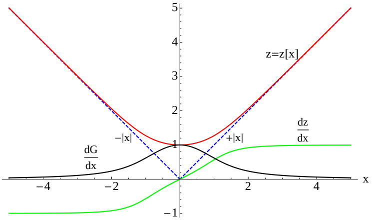
$$G(z) = -\frac{1}{\delta_c}+\frac{1}{2}\sqrt{z^4-1}\left[f_{3/4}(z)+f_{7/4}(z)\right] \ , \tag{359}$$

Electrostatic solution of Born-Infeld gravity - II

- For $z \gg 1$, $G(z) \approx -1/z$ yields the RN solution of GR:

$$A(x) \approx 1 - \frac{r_S}{r} + \frac{r_q^2}{2r^2} \ . \tag{360}$$

- Behaviour of radial function $r^2(x)$ inform us about the existence of a wormhole structure.



- No violation of energy conditions needed!
- Modified structure of horizons
  - If $\delta_1 > \delta_c$: RN-like solutions (two, one -extreme-, or none horizons).
  - If $\delta_1 < \delta_c$: Schwarzschild-like solutions (a single horizon).
  - If $\delta_1 = \delta_c$: A single horizon or none (below a number of charges $N_q < N_c \simeq 16.55$).
- Location is almost coincident with GR predictions save by small r_s, r_q (microscopic BHs).
- Curvature scalars blow up at the wormhole throat $x = 0$, but not for $\delta_1 = \delta_c$. Interpretation?.

Geodesics in metric-affine gravity

- A geodesic curve $\gamma^\mu = x^\mu(\lambda)$ with tangent vector $u^\mu = \frac{dx^\mu}{d\lambda}$ and affine parameter λ satisfies (e.g. Chandrasekhar's book):

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{361}$$

- Comments:
  - The metric defines a natural connection (Christoffel) and defines a set of geodesics.
  - The independent connection can be used to define a different set of geodesics.
  - Assuming the EEP, matter is not coupled directly to the independent connection, so geodesics are those associated to the metric (but GWs couple to $q_{\mu\nu}$![36]).

[36] J. Beltran Jimenez, L. Heisenberg, G. J. Olmo and D. Rubiera-Garcia, JCAP **10** (2017), 029.

Mechanisms for restoration of geodesic completeness - I

- For any static, spherically symmetric space-time $ds^2 = C(x)dt^2 - B(x)^{-1}dx^2 + r^2(x)d\Omega^2$, the geodesic equation can be written as

$$\frac{C}{B}\left(\frac{dx}{d\lambda}\right)^2 = E^2 - C\left(\frac{L^2}{r^2(x)} - k\right)$$

where λ is the affine parameter (the proper time for a time-like observer), E, L are the energy and angular momentum per unit mass, and $k = 0, -1$ for null and time-like observers.

- For the line element above this equation turns into

$$\frac{1}{\Omega_1^2}\left(\frac{dx}{d\lambda}\right)^2 = E^2 - \frac{A(x)}{\Omega_1}\left(\frac{L^2}{r^2(x)} - k\right)$$

(note that this equation can be rewritten as the one of a particle moving in a one-dimensional effective potential $V_{eff} = \frac{A(x)}{\Omega_1}\left(\frac{L^2}{r^2(x)} - k\right)$).

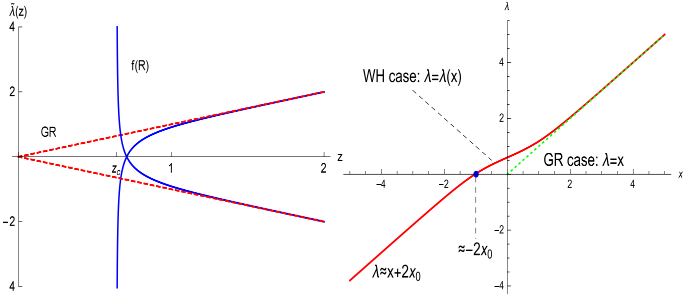
- Null radial geodesics are particularly informative, since they ignore the effective potential:

$$\lambda - \lambda_0 = \pm \int \frac{1}{E\Omega_1(x)} dx$$

and everything revolves around the behaviour of $\Omega_1(x)$ and $r^2(x)$.

Mechanisms for restoration of geodesic completeness - II

- Two mechanisms to restore geodesic completeness
  1. The central region is pushed to the future (or past) boundary of the space-time in such a way that every (null and time-like) geodesic takes an infinite time to reach to it.
  2. Some *bounce* arises in $r^2(x)$ near that region where the point-like singularity should be, $x = x_c$, allowing geodesics to defocus and continue their path to another region of space-time,.



- In the first case, no information can get to (or come out of) the bouncing region $z = z_c$.
- In the second case, both photons and extended observers can get to $x = 0$ in finite affine time. Effects of curvature?. In general it is divergent at the bounce!.

The effect of curvature upon a congruence

- Physical (extended observers) can be modelled by a *congruence*: each particle of the body is assumed to free-fall upon its geodesic, but tidal forces among each other disrupt their individual trajectories.
- Technically, a basis of (any) three Jacobi-vectors Z^a ($a = 1,2,3$) defines a volume as (Tipler, Krolak, Nolan):

  $$V(\lambda) = \det[Z^a_{(1)}, Z^b_{(2)}, Z^c_{(3)}]$$

  If $\lim_{\lambda \to 0} V(\lambda) = 0$, a crushing-type singularity occurs.
- In the Schwarzschild case, there is an infinite stretching in the radial direction and an infinite contraction in the angular ones: *spaghettization* occurs with entails a loss of causal contact among the different parts of the body.
- In our modified black holes, $V(\lambda) \to \infty$, which has not a proper interpretation!.
- Alternative approach: play with causality to send light rays from any two parts of the body on a round trip[37]. Time of flight?.
- In this approach, and using comoving coordinates, we verified that each part of the congruence are in causal contact with any other part even as the problematic (divergent-curvature) region is crossed. This means that the physical interactions sustaining the body may be effectively transmitted.

[37] G. J. Olmo, DRG, A. Sanchez-Puente, Class. Quant. Grav. **33**, 115007 (2016).

Projective invariance

- Rooted on the equivalence principle, test particle paths in GR are determined by the geodesic equation, which in an arbitrary parametrization λ reads

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{x^\beta}{d\lambda}=f(\lambda)\frac{dx^\mu}{d\lambda} \tag{362}$$

- An affine parametrization u is that in which $f(\lambda)=u_{\lambda\lambda}/u_\lambda$ with $u_\lambda=du/d\lambda$, which makes the *fictitious force* appearing in the right-hand side to go away.
- The geodesic equation (particles' paths), on each parametrization, is invariant under the set of *projective transformations*

$$\tilde{\Gamma}^\mu_{\alpha\beta}=\Gamma^\mu_{\alpha\beta}+\xi_\alpha\delta^\mu_\beta \tag{363}$$

  where the 1-form field $\xi\equiv\xi_\alpha dx^\alpha$ satisfies $\xi_\alpha\frac{dx^\alpha}{d\lambda}=-f(\lambda)$.
- However, neither the Riemann tensor nor the Ricci one are invariant, transforming as

$$R^\alpha{}_{\beta\mu\nu}(\tilde{\Gamma})=R^\alpha{}_{\beta\mu\nu}(\Gamma)+\delta^\alpha_\mu F_{\beta\nu}; R_{\mu\nu}(\tilde{\Gamma})=R_{\mu\nu}(\Gamma)+F_{\mu\nu} \tag{364}$$

  where the antisymmetric tensor $F_{\mu\nu}=\partial_\mu\xi_\nu-\partial_\nu\xi_\mu$.
- In the Einstein-Hilbert action of GR, the fact that only the symmetric part of the Ricci tensor enters into it makes it to be projectively-invariant as well.

Projective invariance and curvature scalars

- Having non-projectively invariant terms in the action is an invitation for the presence of ghost-like instabilities in the theory, unless suitable torsion is called upon to eat such ghosts up[38].
- The Kretschmann scalar is NOT projectively invariant:

$$K(\tilde{\Gamma}) = K(\Gamma) + 4F_{\mu\nu}F^{\mu\nu} \tag{365}$$

- This equation implies that, if the Kretschmann scalar diverges for a connection Γ, it is always possible to find a new gauge ξ_μ in which it is finite. Does this mean that the curvature divergence of the Schwarzschild black hole can be gauged away?.
- There is yet another curvature invariant

$$P \equiv R^{\alpha\beta}_{\mu\nu} R^{\mu\nu}_{\alpha\beta} = \frac{48M^2}{r^6} \tag{366}$$

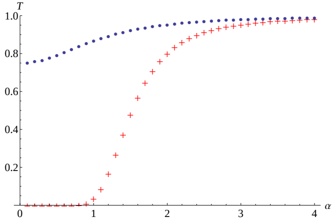
  which is projectively-invariant, so Schwarzschild curvature divergence cannot be gauged away by the projective invariance trick.
- It is however yet another warning in identifying space-time singularities with ill-behaviours of some selected set of curvature scalars.

[38] J. Beltrán Jiménez and A. Delhom, Eur. Phys. J. C **79** (2019) no.8, 656.

Scattering of waves off the wormhole

- Since real particles are not idealized geodesics but have some finite extension, another test is to discuss the problem of scattering of waves off the problematic region.
- Take a free real scalar field $\Box\phi = 0$ and $\phi_{\omega,lm} = e^{-i\omega t} Y_{lm}(\theta,\varphi) f_{\omega,l}(x)/r(x)$, and study its behaviour around the curvature-divergent region [39].

$$f_{y'y'} + (\alpha^2 \pm \frac{1}{\sqrt{|y'|}})f = 0$$



- Transmission and emission coefficients can be computed in all cases. No evidence of pathologies!.

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39 G. J. Olmo, DRG, A. Sanchez-Puente, Eur. Phys. J. C **76**, 143 (2016).

Paths of accelerated observers

- By the principle of general covariance, observers with arbitrary motions should also have complete paths. *No discrimination is allowed between different observers - they all have the same right to live*.
- Motion for accelerated observers with linear acceleration $k(\lambda)$ is described by [40]

$$\left(\frac{dx}{d\lambda}\right)^2 + C(x)\left(1+\frac{L^2}{r^2(x)}\right) = \left[E + \int_{x_0}^{x} \frac{k(\lambda)dx'}{\sqrt{1+\frac{L^2}{r^2(x')}}}\right]^2$$

- For quadratic gravity, $\mathcal{L}_G = R + a(R^2 + bR_{\mu\nu}R^{\mu\nu})$, the trajectory of an accelerated particle of charge q and mass m satisfies

$$\frac{1}{\sigma_+^2}\left(\frac{dx}{d\lambda}\right)^2 + C(x)\left(1+\frac{L^2}{r^2(x)}\right) = \left(E + I_L^{quadratic}(r)\right)^2$$

where $I_L^{quadratic}(r) = -\frac{Qq}{m}\frac{\text{EllipticF}\left[\sin^{-1}\left(\sqrt{\frac{L^2+r_c^2}{L^2+r^2}}\right),\frac{L^2-r_c^2}{L^2+r_c^2}\right]}{\sqrt{r_c^2+L^2}}$.

- Accelerated paths (finite local acceleration) can be indefinitely extended in all cases!.

[40] G. J. Olmo, DRG, A. Sanchez-Puente, Class. Quant. Grav. **35**, 055010 (2018).

(No) correlations between pathology markers

- Take a simple RBG model: $f(R) = R - \sigma R^2$ with perfect fluid $T^{\mu}{}_{\nu} = diag(-\rho, -\rho, \rho + \beta\rho^2, \rho + \beta\rho^2)$ (fulfilment of energy conditions, reduction to RN limit, equivalence to NEDs).
- Exact analytical solutions can be obtained for each combination of the signs of (σ, β): four cases[41]

45.What is a singular black hole beyond general relativity.pdf - Adobe Acrobat Reader DC

WHAT IS A SINGULAR BLACK HOLE BEYOND GENERAL ... PHYSICAL REVIEW D **95**, 064043 (2017)

TABLE I. Summary of the features of the four families of configurations studied in Sec. V [the case $\{\sigma > 0, \beta < 0\}$ with $\gamma = 1$ hides some peculiarities (see Sec. V A) so it is not contained in this table]. The metric components, energy density and curvature scalars refer to the behavior at the wormhole throat (when it exists) or otherwise at the innermost region of the solutions. Incomplete geodesics refer to the existence of (at least) a single incomplete null or timelike geodesic curve. The breakdown of the correlations among these three concepts is clear.

	Wormhole	Metric components	Energy density	Curvature scalars	Geodesics
Case I ($\sigma > 0, \beta < 0$)	YES if $\gamma > 1$ NO if $\gamma < 1$	Divergent	Finite	Divergent	Complete if $\gamma > 1$ Incomplete if $\gamma < 1$
Case II ($\sigma > 0, \beta > 0$)	NO	Finite	Divergent	Finite	Incomplete
Case III ($\sigma < 0, \beta < 0$)	NO	Divergent if $\delta_1 \neq \delta_c^{(\gamma)}$ Finite if $\delta_1 = \delta_c^{(\gamma)}$	Finite	Divergent if $\delta_1 \neq \delta_c^{(\gamma)}$ Finite if $\delta_1 = \delta_c^{(\gamma)}$	Incomplete? if $\delta_1 > \delta_c^{(\gamma)}$ Complete if $\delta_1 = \delta_c^{(\gamma)}$ (de Sitter core) Incomplete if $\delta_1 < \delta_c^{(\gamma)}$
Case IV ($\sigma < 0, \beta > 0$)	YES	Divergent	Finite	Divergent	Complete

- No correlation between geodesic (in) completeness and (divergence of) curvature scalars!.

[41] C. Bejarano, G. J. Olmo, DRG, Phys. Rev. D **95** (2017) 6, 064043.

Some thoughts

- Space-times singularities continue to defy our understanding and physical intuition of gravitational phenomena under the most extreme conditions.
- The singularity theorems have aged well, virtually being untouched since their formulation fifty years ago.
- (At least) four different criteria can be used to look for pathologies:
    - Geodesic completeness restoration can be achieved either via a bounce or by pushing the conflictive region to infinite affine distance.
    - Curvature divergences and intuitively infinite tidal forces do not necessarily disrupt physical observers in an utterly destructive way.
    - Upgrading idealized trajectories of point-like particles to (toy-model) capture its fundamental extended (quantum) behaviour poses no problem.
    - Accelerated observers have the same right to live as geodesic ones, and they actually do.
- No implication of total regularity (because we DO NOT understand what space-time singularities are YET): at best we can keep looking for further types of pathologies in specific space-times, while we await for a better description of the gravitational field in the high-energy regime.
- Metric-affine theories of gravity represent a promising framework to study such a regime.

Advanced method: the mapping procedure

- Let us recall the structure of the RBG field equations in $q_{\mu\nu}$ frame

$$G^{\mu}{}_{\nu}(q) = \frac{\kappa^2}{|\hat{\Omega}|^{1/2}} \left[T^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\left(L_G + \frac{T}{2}\right)\right]$$

- Fundamental difficulty:
  - Note that $q_{\alpha\beta} = g_{\alpha\rho}\Omega^{\rho}{}_{\beta}$ and $\Omega^{\rho}{}_{\beta}$ is a nonlinear function of $T^{\mu}{}_{\nu}$, which itself depends on $g^{\mu\nu}$.
  - There are certain configurations with high symmetry (cosmology, BHs, ...) in which specific models can be treated (as we just showed).
  - Dynamical scenarios with less symmetry are plagued by technical difficulties.
  - The application of numerical methods must be strongly model dependent and computationally expensive because of the need to invert the relation between the metrics at each step.
- Idea: rewrite the EOM in pure Einstein field equations form: $G_{\mu\nu}(q) = \kappa^2 \tilde{T}_{\mu\nu}(q)$.
- To do it so we rewrite the RBG action as (introducing suitable auxiliary fields)

$$\mathcal{S}(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}, \psi_m) = \frac{1}{2\kappa^2}\int d^4x\sqrt{-q}q^{\mu\nu}R_{\mu\nu}(\Gamma) + \tilde{S}_m(g_{\mu\nu}, \psi_m) \tag{367}$$

which is nothing but GR for the metric $q^{\mu\nu}$ coupled to a new matter action $\tilde{S}_m$ for the set of matter fields ψ_m. Its energy-momentum tensor is written in the usual way as

$$\tilde{T}_{\mu\nu}(q) \equiv -\frac{2}{\sqrt{-q}}\frac{\delta\tilde{S}_m}{\delta q^{\mu\nu}} = \frac{1}{|\hat{\Omega}|^{1/2}}\left[T^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\left(L_G + \frac{T}{2}\right)\right] \tag{368}$$

Casting the mapping

- Assuming two anisotropic fluid in the RBG and GR frames

$$T^{\mu}_{\nu}(g) = (\rho + p_{\perp})\, u^{\mu} u_{\nu} + p_{\perp} \delta^{\mu}_{\nu} + (p_r - p_{\perp})\, \chi^{\mu}\chi_{\nu} \quad (369)$$

$$\tilde{T}^{\mu}_{\nu}(q) = \left(\rho^q + p^q_{\perp}\right) v^{\mu} v_{\nu} + p^q_{\perp} \delta^{\mu}_{\nu} + \left(p^q_r - p^q_{\perp}\right) \xi^{\mu}\xi_{\nu} \quad (370)$$

this mapping implies that[42]

$$p^q_{\perp} = \frac{1}{|\hat{\Omega}|^{1/2}} \left[\frac{\rho - p_r}{2} - \mathcal{L}_G\right] \quad (371)$$

$$\rho^q + p^q_{\perp} = \frac{\rho + p_{\perp}}{|\hat{\Omega}|^{1/2}} \quad (372)$$

$$p^q_r - p^q_{\perp} = \frac{p_r - p_{\perp}}{|\hat{\Omega}|^{1/2}} \quad (373)$$

- Cooking recipe:
  - Select a known RBG coupled to some matter source $(\rho, p_r, p_{\perp})$ and compute $|\hat{\Omega}|$.
  - Use the mapping equations to find $(\rho^q, p^q_r, p^q_{\perp})$ and reconstruct the matter Lagrangian on the GR side.
  - Use any known solution for that GR matter source $q_{\mu\nu}$ to generate the one in RBG $g_{\mu\nu}$ via $q_{\alpha\beta} = g_{\alpha\rho}\Omega^{\rho}{}_{\beta}$.
- Consistence checked for electromagnetic[43]and scalar[44] fields.

[42] V. I. Afonso, G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D **97** (2018) no.2, 021503.

[43] V. I. Afonso, G. J. Olmo, E. Orazi and D. Rubiera-Garcia, Eur. Phys. J. C **78** (2018) no.10, 866

[44] V. I. Afonso, G. J. Olmo, E. Orazi and D. Rubiera-Garcia, Phys. Rev. D **99** (2019) no.4, 044040.

“Alchemy-ing” the mapping

- The mapping has an *alchemy* side to it, since it actually tends to transfer the functional structure between the matter and gravity sectors. Considering two field invariants (in the RBG and GR frames, respectively) $X = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $Z = -\frac{1}{2}Z_{\mu\nu}Z^{\mu\nu}$ one finds the mapping:

$$\mathrm{EiBI} + X \Leftrightarrow \mathrm{GR} + \frac{4\pi}{\kappa^2\varepsilon}\left(\sqrt{1+\frac{\varepsilon\kappa^2 Z}{2\pi}} - 1\right) \tag{374}$$

$$\mathrm{EiBI} + \frac{4\pi}{\kappa^2\varepsilon}\left(1 - \sqrt{1-\frac{\kappa^2\varepsilon X}{2\pi}}\right) \Leftrightarrow \mathrm{GR} + Z \tag{375}$$

- The matter sector is actually described by a Born-Infeld-type electrodynamics, provided that one makes the identification $\beta^2 = 2\pi/(\varepsilon\kappa^2)$ and takes ε to be negative in the first case, and positive in the second.
- Analytical solutions for GR coupled to both Maxwell and Born-Infeld are known both in the static case (Reissner-Norström) and in the rotating one (Kerr-Newman). Solutions in the RBG side (EiBI) can be thus generated out of the mapping.
- The largest advantage of this procedure is that it only involves algebraic transformations: no actual differential equation-solving is required!.
- Similar alchemy can be performed when $f(R)$ Lagrangians or scalar fields are considered.

Rotating black holes out of the Kerr-Newman solution

- The mapping may look a bit too unaesthetic, but finding new solutions out of a given (GR-based) *seed* using it is outragiously simple as compared to direct-resolution procedures.
- The matter on the KN solution is characterized by a fluid (in coordinates of the q-frame) $\rho^q = \frac{Q^2}{8\pi x^4} = -p_r^q = p_\perp^q$. The mapping thus act upon this seed to generate the new solution in the EiBI + BI side as[45]:

$$\begin{aligned} ds^2 &= -\left(1 - f + \epsilon\kappa^2\rho^q \frac{(\Delta + a^2\sin^2\theta)}{\Sigma}\right) dt^2 - 2a\left(f - \epsilon\kappa^2\rho^q \frac{(\Delta + x^2 + a^2)}{\Sigma}\right)\sin^2\theta dt d\phi \\ &+ \frac{(1+\epsilon\kappa^2\rho^q)\Sigma}{\Delta} dx^2 + (1 - \epsilon\kappa^2\rho^q)\Sigma d\theta^2 \\ &+ \left[\left(x^2 + a^2 + fa^2\sin^2\theta\right) - \epsilon\kappa^2\rho^q \frac{(x^2+a^2)^2 + a^2\Delta\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 , \end{aligned}$$

where f, Δ, Σ, a are the canonical objects of the Kerr black hole, while the ϵ-EiBI corrections are fed by the energy density of the electromagnetic field ρ^q.
- Astrophysical phenomenology: GWs, echoes, lensing[46], shadows...

[45] M. Guerrero, G. Mora-Pérez, G. J. Olmo, E. Orazi and D. Rubiera-Garcia, JCAP **07** (2020), 058.

[46] M. Sabir Ali, S. Kaushal, arXiv:2106.08464 [gr-qc].

Rotating Kerr-Newman black holes in EiBI gravity - I

- An upgrade has also been recently achieved to map the *entire space of NEDs between GR and EiBI*, including the most physically interesting case of EiBI + Maxwell.
- The seed solution is an infinite family of solutions given by a Kerr-like line element

$$
\begin{aligned}
ds_q^2 &= -\left(1-\frac{2\eta r}{\Sigma}\right)dt^2+\frac{\Sigma}{\Delta}dr^2-2a\sin^2\theta\frac{2\eta r}{\Sigma}dtd\phi \\
&+ \Sigma d\theta^2+\frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2-a^2\Delta\sin^2\theta)\right]d\phi^2 \quad , \qquad (376)
\end{aligned}
$$

with the usual definitions:

$$
\begin{aligned}
\Sigma &= r^2+a^2\cos^2\theta, &(377)\\
2\eta &= r(1-f), &(378)\\
\Delta &= r^2f+a^2=r^2-2\eta r+a^2 \ , &(379)
\end{aligned}
$$

while for NEDs we have:

$$
f(r)=1-\frac{2M}{r}-\frac{1}{r}\int_r^\infty R^2T_0^0(R,Q)dR \ , \qquad (380)
$$

Rotating Kerr-Newman black holes in EiBI gravity - II

- The mapping provides an answer to its countepart on the RBG side after selecting the target theories for the gravitational and matter sectors
- For EiBI coupled to Maxwell theory the mapping tells us its counterpart on the GR side to be GR + BI-type electrodynamics. In such a case one has

$$f(r) = 1 - \frac{2M}{r} + \frac{2\beta^2}{3}\left[r^2 - \sqrt{r^4 + Q^2/\beta^2} + \frac{2Q^2}{\beta^2 r^2}\, {}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{\beta^2 r^4}\right]\right] \quad (381)$$

$$\rho^q = \frac{2\beta r^2}{\Sigma^2}\left(\sqrt{\beta^2 r^4 + Q^2} - \beta r^2\right); K = -\rho^q + \frac{2\beta}{\Sigma}\left(2\beta r^2 - \frac{2\beta^2 r^4 + Q^2}{\sqrt{\beta^2 r^4 + Q^2}}\right) \quad (382)$$

  where ${}_2F_1[a,b,c;z]$ is a hypergeometric function.
- The mapping-generated, space-time metric is formally given by $g_{\mu\nu} = q_{\mu\nu} + \varepsilon h_{\mu\nu}$, with $q_{\mu\nu}$ the seed rotating GR + BI metric, and the correction term $h_{\mu\nu}$ given by

$$h_{tt} = -\tfrac{1}{\Sigma}\left(\rho^q a^2 \sin^2\theta + K(\rho^q)\Delta\right) \quad (383)$$

$$h_{rr} = \tfrac{\Sigma}{\Delta} K(\rho^q) \quad (384)$$

$$h_{\theta\theta} = -\Sigma\,\rho^q \quad (385)$$

$$h_{t\phi} = \tfrac{a\sin^2\theta}{\Sigma}\left(\rho^q(r^2+a^2) + K(\rho^q)\Delta\right) \quad (386)$$

$$h_{\phi\phi} = -\tfrac{\sin^2\theta}{\Sigma}\left(\rho^q(r^2+a^2)^2 + a^2\sin^2\theta K(\rho^q)\Delta\right) \quad (387)$$

- Infinitely many new exact rotating solutions can be generated out of this procedure by selecting other seed GR-based metrics and working out the correspondences of theories between frames.

Bonus track: the gauge approach to gravity

Figure: Under construction

Proposals for thesis on black hole physics

- Rotating Kerr-like black holes .
- Hawking's semiclassical integral of quantum vacuum polarization effects.
- Information loss paradox.
- Black holes embedded in Anti-de Sitter spaces.
- Phase transitions/black hole evaporation.
- Photon sphere, strong gravitational lensing, and observables.
- Black hole shadows.
- Perturbations and stability.
- Gravitational waves and/or echoes.
- Maximal extensions.
- Gravitational collapse and/or cosmic censorship conjecture.
- Dynamical solutions.
- Energy extraction: Penrose process and superradiance.
- Accretion processes and mass inflation instability.
- Geodesic behaviour in rotating black holes.
- Microscopic black holes and remnants.
- Horizonless compact objects.
- Wormholes.
- Non-linear electrodynamics/non-abelian black holes.
- Stellar structure limits and limiting masses.
- Observational tests of GR/tests of Kerr solution.
- Connections to quantum gravity/beyond GR black holes.

THANK YOU VERY MUCH FOR YOUR ATTENTION!

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Diego Rubiera-Garcia, *“Elementary and advanced black hole physics: a modern practitioner’s guide”*, Lecture Notes, unpublished, 2022.