3-D coherent dispersion effect with space charge

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Outline

1. Combined effect of space charge and dispersion
2. Space-charge-modified dispersion function
3. Coherent dispersion mode
4. Beam instabiltiies with dispersion
5. Splitting effect in 3-D bunched beam

Combined effect of space charge and dispersion

- Two fundamental effects of accelerators

Space charge

(high-intensity linac)

Dispersion

(synchrotron, transport arc section)

combined effect

Space charge + dispersion

(High-intensity circular machines)

- Beam dynamics in high-intensity circular accelerator are subject to ***combined effect*** of space charge and dispersion

Combined effect of space charge and dispersion

- Dispersion effect (only) → increase horizontal beam size

- Space charge (only)
  - ✓ Increase beam size
  - ✓ modify the lattice functions
  - ✓ envelope modes/ instabilities

- Dispersion effect + space charge
  - ✓ Increase horizontal beam size
  - ✓ modify the lattice functions
  - ✓ coherent **dispersion** (envelope) mode
  - ✓ induce coherent beam instabilities

Tools: combined rms envelope approach

- Starting potint Hamiltonian for single particle with s.c. and disp.

$$\mathrm{H} = \frac{1}{2}\left(\mathrm{p}_{\mathrm{x}}^{2} + \mathrm{p}_{\mathrm{x}}^{2}\right) + \frac{\kappa_{x0}(s)}{2} x^{2} + \frac{\kappa_{y0}(s)}{2} \mathrm{y}^{2} + \frac{m^{2}c^{4}}{E_{0}^{2}} \delta^{2} - \frac{x}{\rho(s)} \delta + V_{\mathrm{sc}}(x, y, s)$$

- Using Venturnin-Reiser and Lee-Okmoto theory

$$x = \bar{x} + \delta D_{x} \qquad x' = \bar{x}' + \delta D_{x}' \qquad y = \bar{y} \qquad y' = \bar{y}'$$

Betatron coordinate with s.c. ($\bar{x}$)

Off-momtenum coordinate with s.c. (δD_x)

Two fundamental articles:

M. Venturini and M. Reiser, Phys. Rev. Lett. 81, 96 (1998)

S. Y. Lee and H. Okamoto, Phys. Rev. Lett. 80, 5133 (1998)

Tools: combined rms envelope approach

- Following V-R and L-O theory, beam coherent motion { betatron montion / dispersion motion }

- How to seperate/identify the two parts in the total beam size?

  - By defining the space charge-modified dispersion

$$\frac{\mathrm{d}^2 D_x}{\mathrm{d}s^2} + \left[\kappa_{x0}(s) - \frac{\mathrm{K_{sc}}}{2X(X+Y)} \right] D_x = \frac{1}{\rho(s)}$$

Space charge depression

X,Y : total beam size

Space-charge modified dispersion function

$$\frac{d^2 D_x}{ds^2} + \left[\kappa_{x0}(s) - \frac{K_{sc}}{2X(X+Y)}\right] D_x = \frac{1}{\rho(s)}$$

- Ansatz 1 $\langle \bar{x}\delta \rangle = 0$
  - Second order moment of the total beam size $X^2 = (\bar{x} + \delta D_x)^2 = \bar{x}^2 + 2D_x\langle \bar{x}\delta \rangle + (\delta D_x)^2 = \bar{x}^2 + (\delta D_x)^2 = \sigma_x^2 + \sigma_p^2 D_x^2$

- Ansatz 2
  - rms equivalence of Frank Sacherer holds with dispersion

$$\left\langle x \frac{\partial V_{sc}}{\partial x} \right\rangle = -\frac{K_{sc}}{2} \frac{X}{X+Y}$$

Space-charge modified dispersion function

- With the definition of Dx, the parts are independent with each other.

$$X^2 = \left(\bar{x} + \delta D_x\right)^2 = \bar{x}^2 + \left(\delta D_x\right)^2 = \sigma_x^2 + \sigma_p^2 D_x^2$$

- Two *independent* parts
  - Betatron beam size σ_x
  - Dispersion beam size $\sigma_p D_x$

Envelope rms equation with dispersion

- By using the coordinate transformation

$$\varepsilon_{dx} = \sqrt{\langle \bar{x}^2 \rangle \langle \bar{x}'^2 \rangle - \langle \bar{x}' \bar{x} \rangle^2}$$

- The rms envleope equaion combined with dispersion

$$\frac{d^2\sigma_x}{ds^2} + \left[\kappa_{x0}(s) - \frac{K_{sc}}{2X(X+Y)}\right]\sigma_x - \frac{\varepsilon_{dx}^2}{\sigma_x} = 0$$

$$\frac{d^2\sigma_y}{ds^2} + \left[\kappa_{y0}(s) - \frac{K_{sc}}{2Y(X+Y)}\right]\sigma_y - \frac{\varepsilon_{dy}^2}{\sigma_y} = 0$$

$$\frac{d^2 D_x}{ds^2} + \left[\kappa_{x0}(s) - \frac{K_{sc}}{2X(X+Y)}\right]D_x = \frac{1}{\rho(s)}$$

Pioneer works on the genelized envelope equation set

J. A. Holmes, V. V. Danilov, J. D. Galambos, D. Jeon, and D. K. Olsen, Phys. Rev. ST Accel. Beams 2, 114202 (1999).

CFD model

➢ **C**onstant **F**ocusing channel with **D**ispersion (smooth approximation model)

- e.g. bending solenoid focusing structure, the matched beam sizes are given by

$$\underbrace{\left[\kappa_{x0}(s)-\frac{\mathrm{K}_{sc}}{2X(X+Y)}\right]}_{\kappa_x(s)}D_{x0}=\frac{1}{\rho(s)} \quad \Rightarrow \quad \sigma_{x0}=\left(\frac{\varepsilon_{dx}^2}{\kappa_x(s)}\right)^{\frac{1}{4}}$$

$$\underbrace{\left[\kappa_{y0}(s)-\frac{\mathrm{K}_{sc}}{2Y(X+Y)}\right]}_{\kappa_y(s)}\sigma_{y0}-\frac{\varepsilon_{dy}^2}{\sigma_{y0}}=0 \quad \Rightarrow \quad \sigma_{y0}=\left(\frac{\varepsilon_{dy}^2}{\kappa_y(s)}\right)^{\frac{1}{4}}$$

Characterisitcs of space-charge modified dispersion

➢ Two characterisitcs of the s.c.-modified dispersio can be shown by CFD

(1) For matched beams

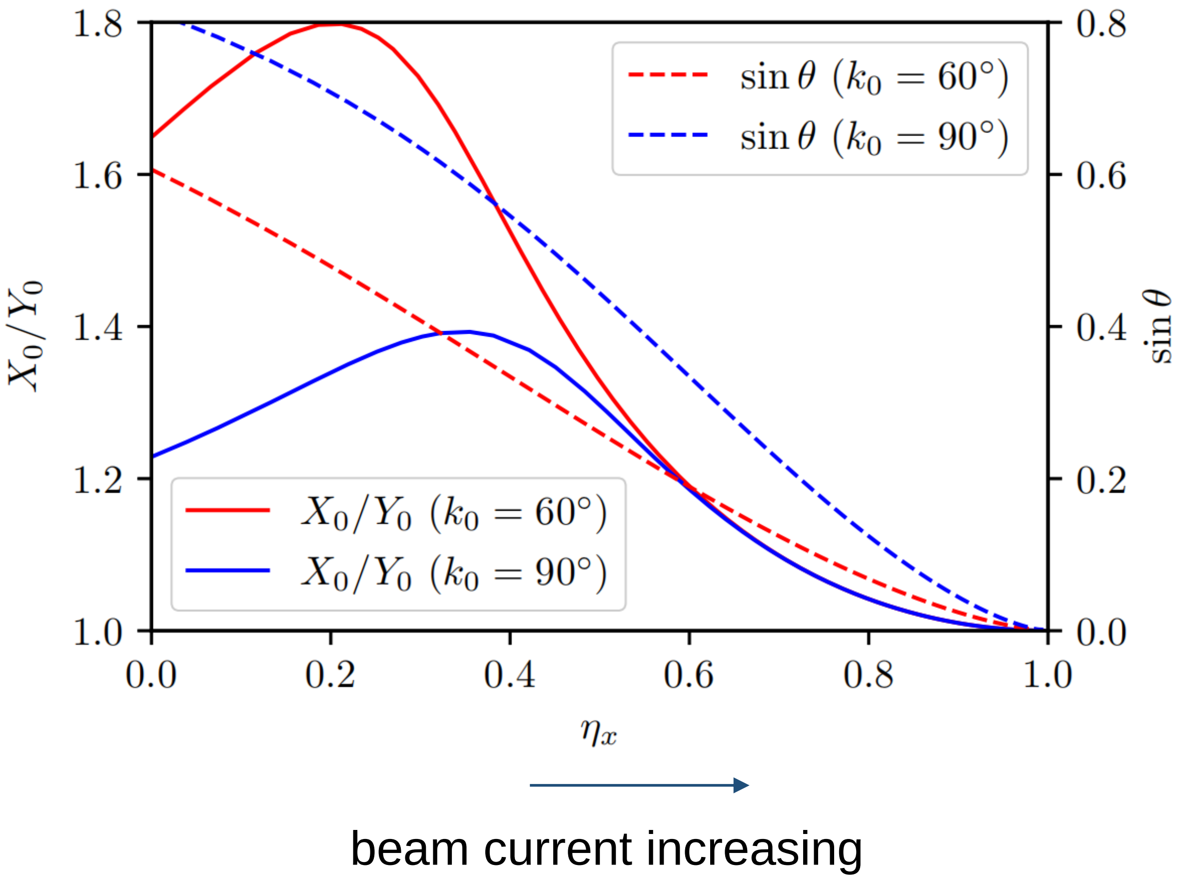
- As beam current increasing, the beam experiences two stages

1. X/Y increasing:
dispersion dominated stage

2. X/Y decreasing:
space-charge dominated stage

- Tune depression

$$\eta_{x,y} = \frac{k_{0,x,y} - k_{x,y}}{k_{0,x,y}}$$



Red: phase advance 60 deg

Blue: phase advance 90 deg

Characterisitcs of space-charge modified dispersion

(2) For mismatch oscillations

- Coherent oscillaton of mismatched beams can be analyzed via perturvation on the combined envelope equation

$$\xi'' + a_0\xi + a_1\eta + a_2 d_x = 0$$

$$\eta'' + a_1\xi + a_3\eta + a_4 d_x = 0$$

$$d_x'' + \frac{a_2}{\sigma_p^2}\xi + \frac{a_4}{\sigma_p^2}\eta + a_5 d_x = 0$$

$$\sigma_x = \sigma_{x,m} + \xi$$
$$\sigma_y = \sigma_{y,m} + \eta$$
$$D_x = D_{x,m} + d_x$$

- Here coefficients a_1 to a_5 are functions of matched beams and emittance

Characterisitcs of Space-charge modified dispersion

(2) For mismatch oscillations

- Envelope modes ϕ_1, ϕ_2

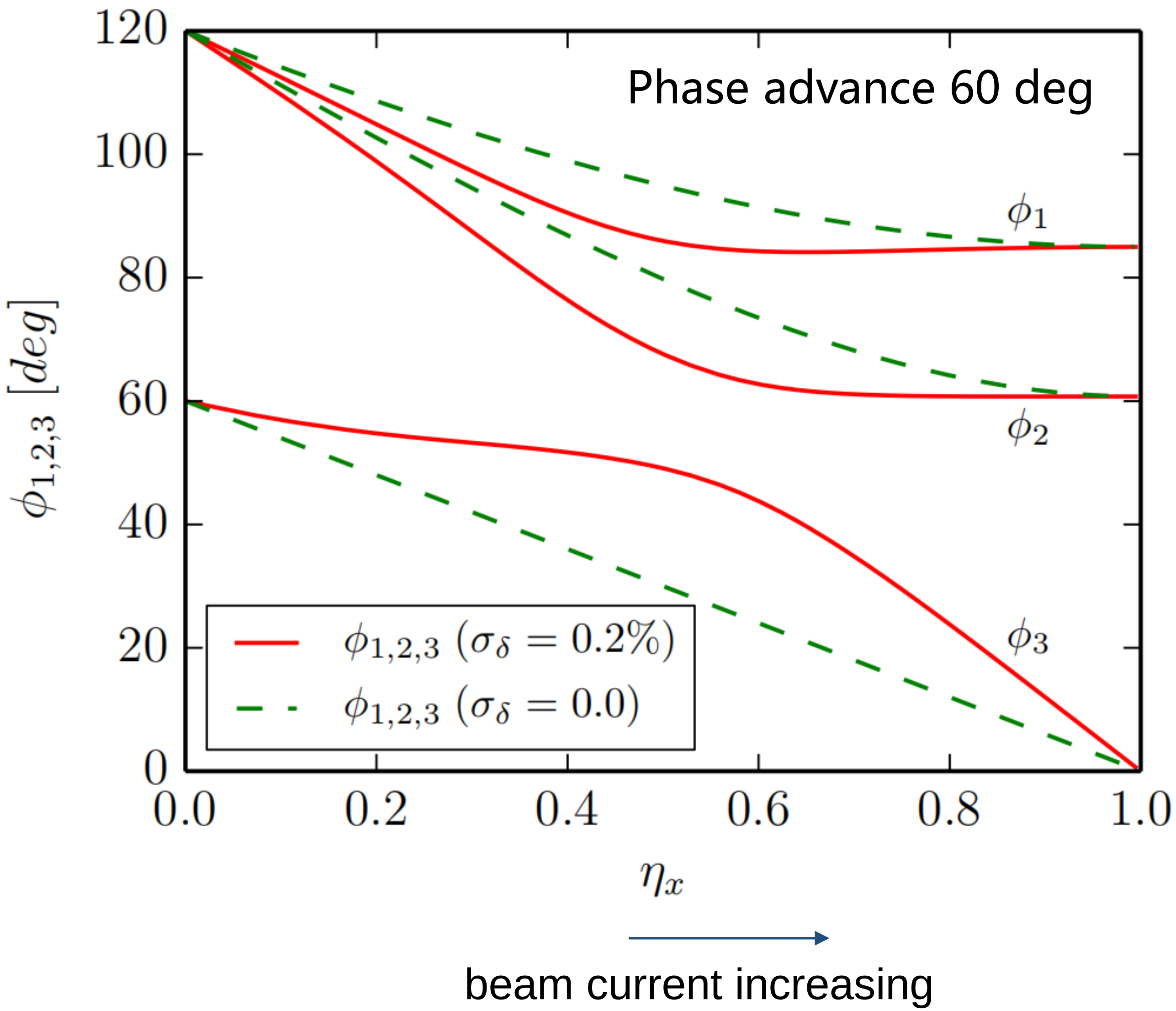
  and dispersion mode ϕ_3

  can be identified from the coefficient matrix of a1 to a5

- The limit of the three modes are equal to that in the case without dispersion

Pioneer works on dispersion mode

M. Ikegami, S. Machida, and T. Uesugi, Phys. Rev. ST Accel. Beams 2, 124201 (1999)

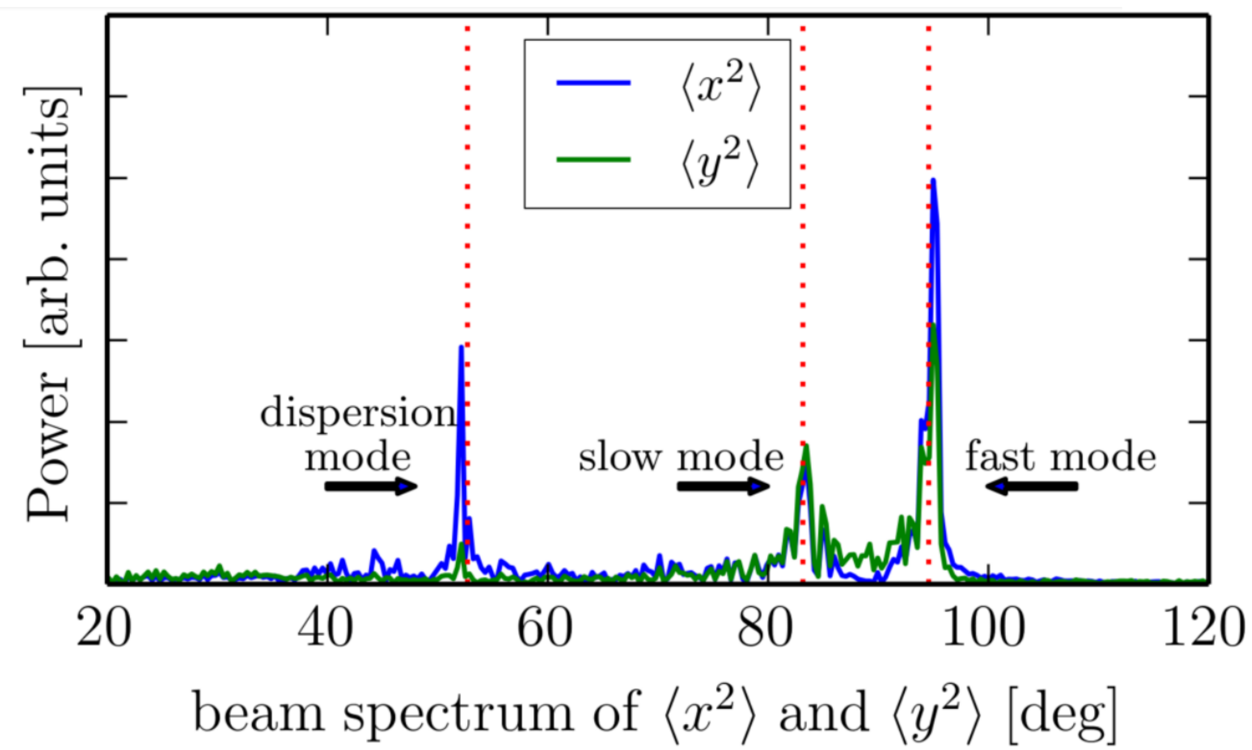


Coherent oscillations and dispersion mode

- PIC simulations are in agreement with the numerical envelope approach

- The frequency of the three modes can be obtained from the beam spectrum of the second order moments

$$\langle x^2 \rangle \text{ and } \langle y^2 \rangle$$



Coherent beam instabilities with dispersion

- For alternating focusing structure, the mismatched oscillaiton can drive instabilites

- The most well-known is the envelope instability (second order even instability)

- Dispersion+space charge
  - modify the envelope instability
  - drive the dispersion instability

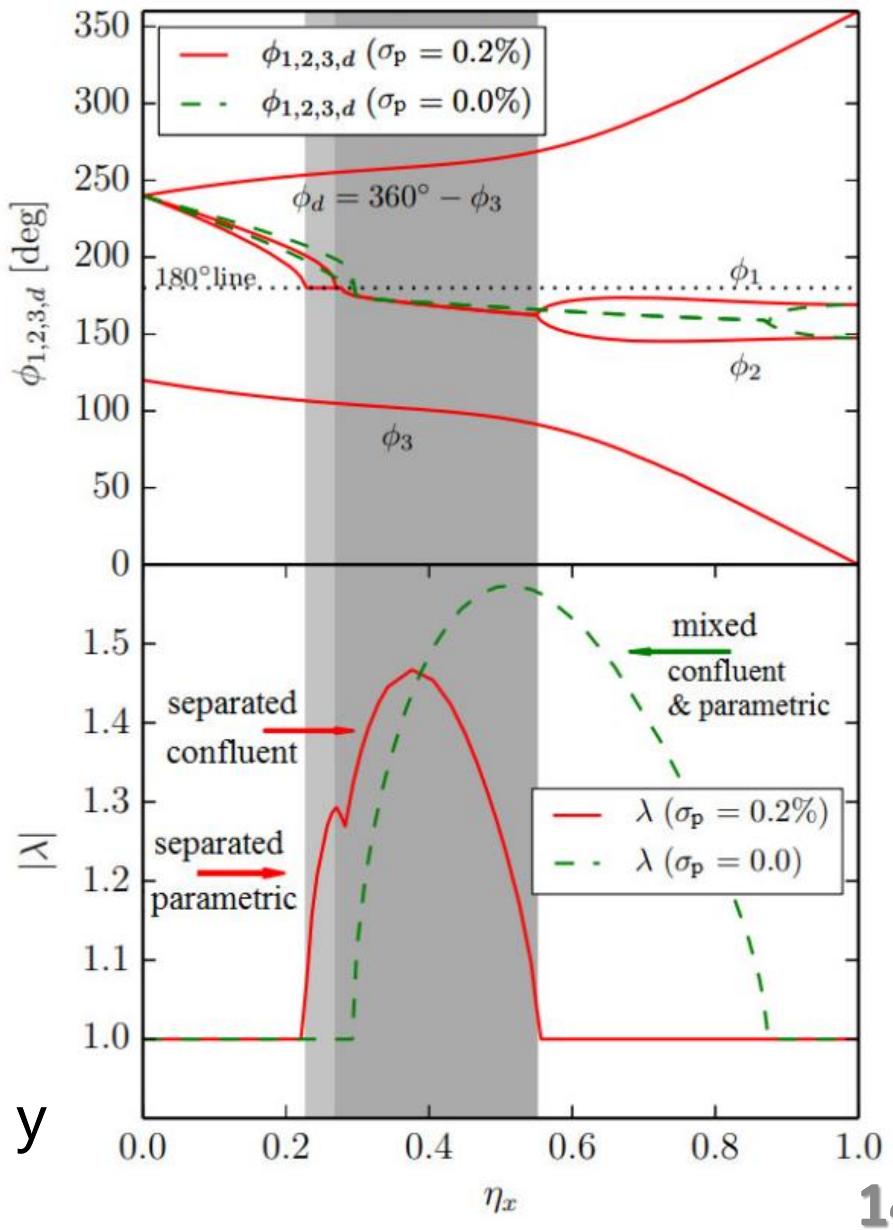
Coherent beam instabilities with dispersion

- Modified 90 envelope instability

  - 90 envelope instability
    - parametric resonance
    - mixed
    - confluent resonance

  - Dispersio-modified 90 envelope instability
    - seperated parametric resonance
    - seperated
    - seperated confluent resonance

  - Dispersion breaks the focusing symmetry of x and y



Coherent beam instabilities with dispersion

- Dispersion-induced instability
  - Dispersion mode is on resonacen with periodic focusing structure
  - criteria: phase advance > 120 degree

- Compared with: Envelope instability
  - Envelope modes are resonant with periodic focusing structure
  - Criteria: phase advance > 90 degree

3-D bunch case

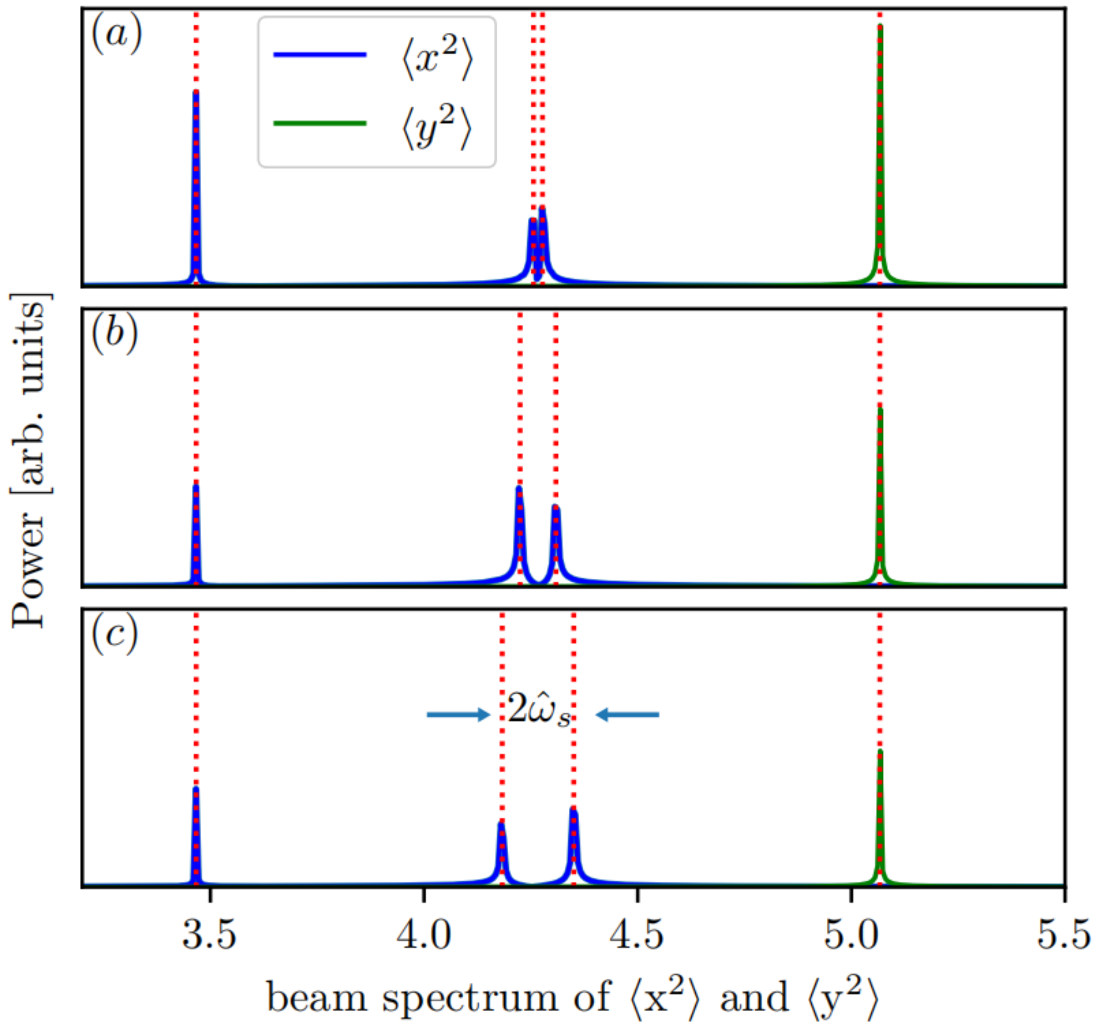
➢ For 3D bunches, dispersion mode will be split due to the synchrotron motion

➢ Analysis:

$$\begin{cases} \xi'' + a_0\xi + a_1\eta + a_2 d_x = 0 \\ \eta'' + a_1\xi + a_3\eta + a_4 d_x = 0 \\ d_x'' + \dfrac{a_2}{\sigma_p^2}\xi + \dfrac{a_4}{\sigma_p^2}\eta + a_5 d_x = 0 \end{cases}$$

$$X^2 = \langle x^2 \rangle = (\sigma_x + \xi)^2 + (D_x + d_x)^2 \sigma_p^2$$

$$Y^2 = \langle y^2 \rangle = (\sigma_y + \eta)^2$$



3-D bunch case

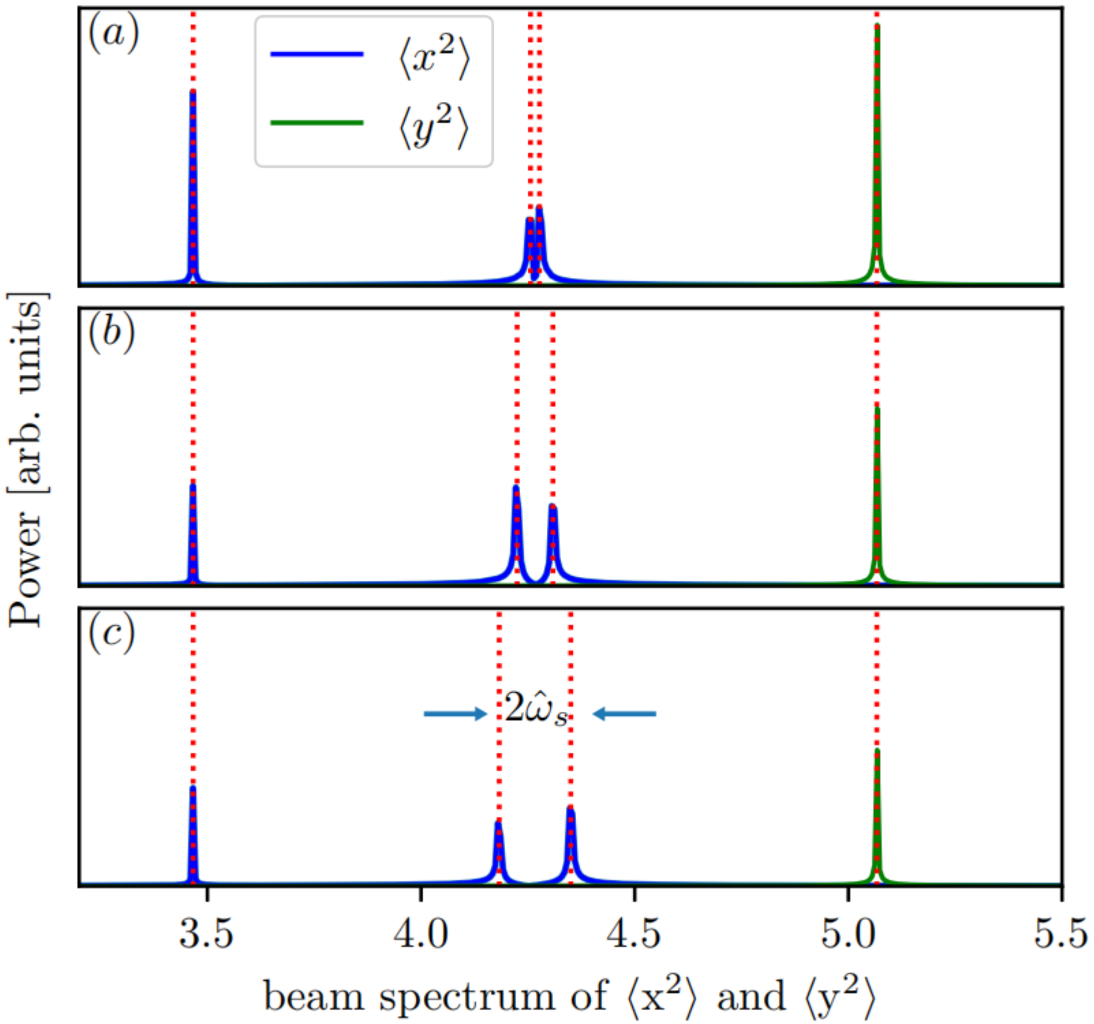
➢ For 2-D coasting beams

$$\sigma_p = \text{const} \qquad (D_{x0} + d_x)\sigma_{p0}$$

➢ For 3-D bunched beams

$$\sigma_p = \sigma_{p0} + \Delta\sigma_p(t) \qquad (D_{x0} + d_x)[\sigma_{p0} + \Delta\sigma_p(t)]$$

➢ The width of the split "gap" depends on the synchrotron frequency



3-D bunch case

- Sidebands appear around the envleope modes

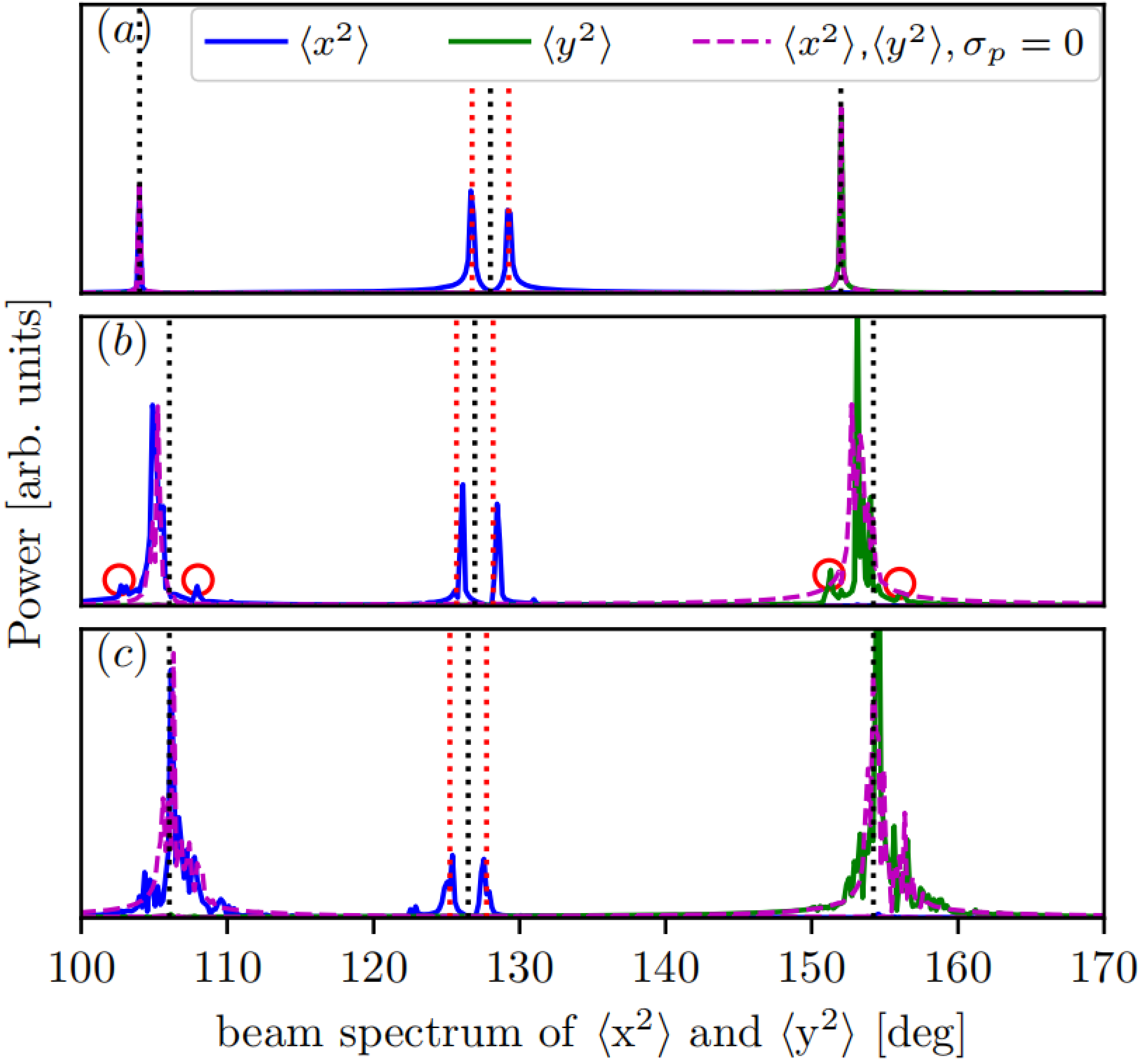
- In the presence of space charge, the split of dispersion mode is coupled to the envelope modes

- space-charge coupling between

$$(D_{x0} + d_x)[\sigma_{p0} + \Delta\sigma_p(t)]$$

and

$\xi, \quad \eta$



Conclusion

- Combined effect of space charge and dispersion has been investigated by using the envelope equation set including dispersion

- For 2-D coasting beams,
  - Envelope instabilies are modified
  - Disperison-induced instability
- For 3-D bunced beams, dispersion mode is split because of synchrotron motion

In the future

- Chromoticity effect on the dispersion mode

Thanks for your attention!