Calculation of the Polarized Bethe-Heitler Cross Section for the Electron Ion Collider

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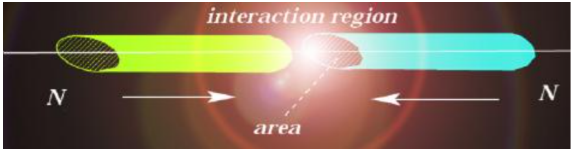
Luminosity



Instantaneous Luminosity
for identical Gaussian beams:

$$\mathcal{L} = \frac{f\,N^2}{4\pi\sigma_x\sigma_y}$$

The EIC will measure lots of absolute cross sections

$$\sigma_{meas} = \frac{N_{meas}}{L}$$

Luminosity is a crucial normalization
for all cross sections

f = crossing frequency
N = number particles per beam
σ_x, σ_y = Gaussian widths.

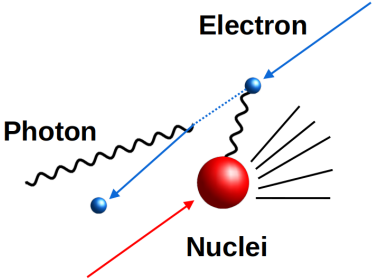
- Problem: σ_x and σ_y are not precisely known during injection
- Van der Meer scans typically used in hadron colliders, but they are not done in real time (special fills)
- Best collider $\mathcal{L}$ measurements in the world are accurate to about 1%

Measuring Luminosity at HERA/EIC

Measure it in real-time using a precisely calculable process in QED

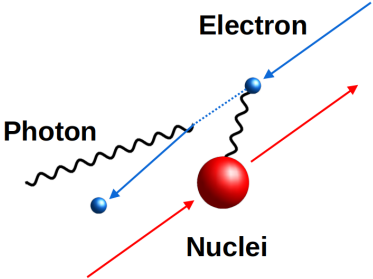
Nuclear DIS

Not precisely calculable nor "easy" to measure.



Bremsstrahlung

Precisely calculable and "easy" to measure



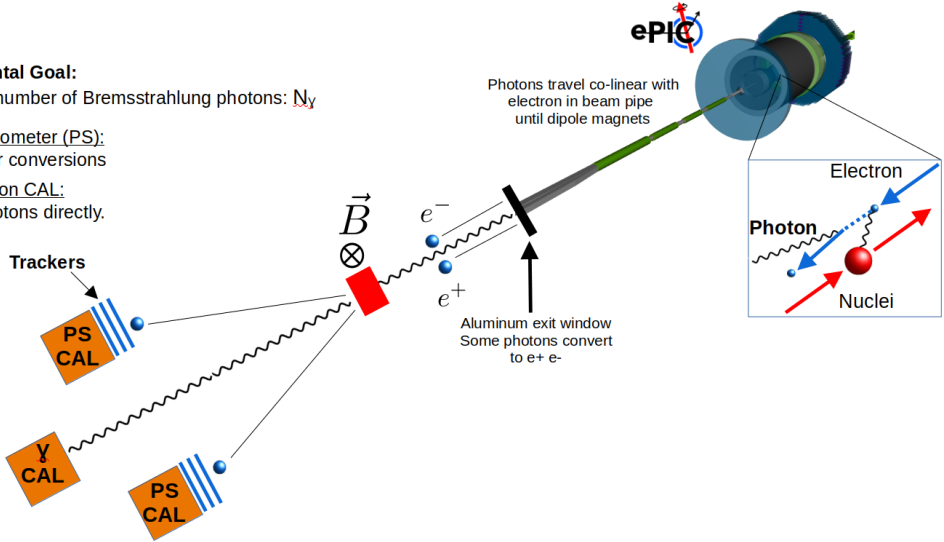
Task: Measure the collider luminosity by counting the rate of produced photons and dividing by the calculated Bremsstrahlung cross section: $\mathcal{L} = R_{meas}/\sigma_{Brem}$

Measuring Luminosity with ePIC

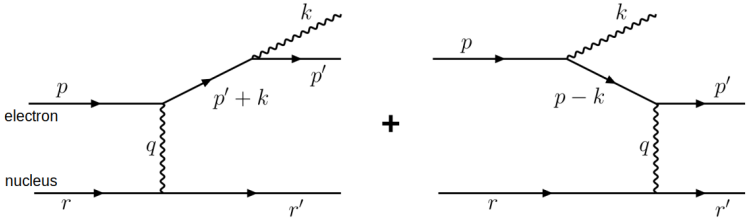
Experimental Goal:
Count the number of Bremsstrahlung photons: N_γ

Pair Spectrometer (PS):
Counts pair conversions

Direct photon CAL:
Counts photons directly.

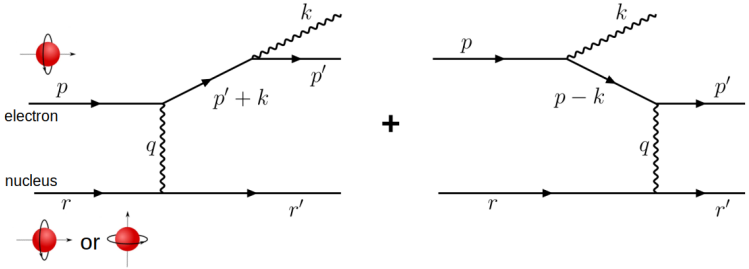


What Has Been Calculated



- Bethe and Heitler calculated the **unpolarized** leading order Bremsstrahlung cross section in 1934: Proc.Roy.Soc.Lond.A 146 (1934) 83-112
- Higher orders contribute at the 0.1% level (very perturbative): Eur.Phys.J.C 71 (2011) 1574
- The Bethe-Heitler cross section was successfully used at HERA to measure luminosity.

What Needs to be Calculated for the EIC

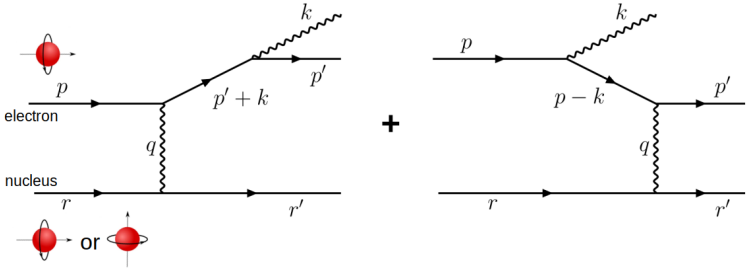


- The upcoming Electron Ion Collider (EIC) will collide **polarized** electrons and **polarized** ions.
- An analytic calculation of polarized cross section is presented here.

<u>3 references used:</u>

1. **Landau and Lifshitz (QED textbook, sections 93-97)**
2. Matthew Schwartz (QFT textbook)
3. Gluckstern and Hull (Phys. Rev. 90 1953)

Feynman Diagrams



$$p = (\varepsilon, \mathbf{p}) \qquad p' = (\varepsilon', \mathbf{p}') \qquad k = (\omega, \mathbf{k})$$

$q^2 \sim m_e^2$: due to the small polar angles of photon in Bethe-Heitler process, $\theta_k \sim m_e/\varepsilon$

$\rightarrow$ *Suitable Approximations*

- Structure of hadron can be neglected
- Amplitude scales with Z for larger ions (coherent scattering)
- Neglect energy transfer to nucleus: $q = (q_0, \mathbf{q}) \approx (\frac{\mathbf{q}^2}{2m_p}, \mathbf{q}) \approx (0, \mathbf{q})$

Amplitude and Amplitude Squared

Amplitude from Feynman Rules:

$$M = e^3 \frac{g_{\mu\nu}}{q^2} \epsilon^*_\sigma(k) \left[\bar{u}(r')\gamma^\mu u(r)\right] \left[\bar{u}(p') \left[\gamma^\sigma \frac{\not{p}'+\not{k}+m_e}{2p'k}\gamma^\nu + \gamma^\nu \frac{\not{p}-\not{k}+m_e}{-2pk}\gamma^\sigma\right] u(p)\right]$$

Amplitude Squared:

$$|M|^2 = -\frac{e^6}{4q^4} Tr[u(r')\bar{u}(r')\gamma^\mu u(r)\bar{u}(r)\gamma^\alpha]\ Tr[u(p')\bar{u}(p')Q^\sigma_\mu u(p)\bar{u}(p)\bar{Q}_{\sigma\alpha}]$$

$$Q^\sigma_\mu = \gamma^\sigma \frac{\not{p}'+\not{k}+m_e}{p'k}\gamma_\mu - \gamma_\mu \frac{\not{p}-\not{k}+m_e}{pk}\gamma^\sigma$$

$$\sum_{pol} \epsilon^*_a(k)\epsilon_b(k) \to -g_{ab} \quad \text{(sum over } \gamma \text{ polarizations in final state)}$$

Tensor form of Amplitude Squared:

$$|M|^2 \equiv -\frac{e^6}{4q^4} W^{\mu\alpha}\, w_{\mu\alpha}$$

Particle Polarizations

Polarized part

$$W^{\mu\alpha} = \mathcal{U}^{\mu\alpha} + \mathcal{P}^{\mu\alpha}$$

$$w_{\mu\alpha} = \mathfrak{u}_{\mu\alpha} + \mathfrak{p}_{\mu\alpha}$$

Unpolarized part

- $\mathcal{U}^{\mu\alpha}$ and $\mathfrak{u}_{\mu\alpha}$ are symmetric tensors
- $\mathcal{P}^{\mu\alpha}$ and $\mathfrak{p}_{\mu\alpha}$ are antisymmetric
- Thus, only $\mathcal{U}^{\mu\alpha}\mathfrak{u}_{\mu\alpha}$ and $\mathcal{P}^{\mu\alpha}\mathfrak{p}_{\mu\alpha}$ survive
- $\mathcal{U}^{\mu\alpha}\mathfrak{u}_{\mu\alpha} \rightarrow$ Bethe-Heitler expression
- $\mathcal{P}^{\mu\alpha}\mathfrak{p}_{\mu\alpha} \rightarrow$ **to be calculated**

Unmeasured spins in the final state

$$\sum_{spin} u(p')\bar{u}(p') = \not{p}' + m_e$$

$$\sum_{spin} u(p)\bar{u}(p) = \not{p} + m_e$$

Measured beam polarizations

$$u(p)\bar{u}(p) = \frac{1}{2}(\not{p} + m_e)(1 - \gamma^5 \not{a}^{(e)})$$

$$u(r)\bar{u}(r) = \frac{1}{2}(\not{r} + m_p)(1 - \gamma^5 \not{a}^{(p)})$$

Spin 4-vectors (longitudinal beam polarizations)

Target Rest Frame & ultrarelativistic limit

$$a^{(e)} = 2\mathbb{P}_e \frac{E_e E_p}{m_e m_p}(-1, 0, 0, +1)$$

$$a^{(p)} = \mathbb{P}_p (0, 0, 0, +1)$$

- $\mathbb{P}_e$ and $\mathbb{P}_p$ are electron and proton beam polarizations along the respective momentum in lab frame. E_e and E_p are the beam energies in lab frame.

Fully-Differential Cross Section

$$
\begin{aligned}
\mathcal{P}^{\mu\nu} p_{\mu\nu} &= -32 m_e m_p \Bigg[\frac{1}{(pk)^2} \left(\frac{q^2}{2} k a^{(e)} (q a^{(p)} - 2 p' a^{(p)}) \right) \\
&+ \left(\frac{1}{(pk)^2} + \frac{1}{(p'k)^2} \right) \left(q a^{(e)} q a^{(p)} m_e^2 - a^{(e)} a^{(p)} q^2 m_e^2 \right) \\
&+ \frac{1}{(pk)(p'k)} \Big(-q^4 a^{(e)} a^{(p)} - 2 q a^{(e)} q a^{(p)} m_e^2 + \frac{q^2}{2} (2 p' a^{(e)} (p' a^{(p)} - p a^{(p)}) \\
&\qquad + k a^{(e)} (3 q a^{(p)} + 2 p a^{(p)}) + 4 m_e^2 a^{(e)} a^{(p)} \Big) \\
&+ \left(\frac{1}{p'k} - \frac{1}{pk} \right) \left((q a^{(e)} + k a^{(e)}) q a^{(p)} - 2 q^2 a^{(e)} a^{(p)} \right) \Bigg]
\end{aligned}
$$

Polarized Component to Differential Cross Section

$$
d\sigma_{\mathcal{P}} = \frac{-\alpha\, r_e^2\, m_e^2}{4(4\pi)^2 q^4\, m_p^2} \mathcal{P}^{\mu\alpha} \mathfrak{p}_{\mu\alpha} \frac{|\mathbf{p}'| \omega d\omega}{|\mathbf{p}|} d\Omega' d\Omega_k \qquad \alpha = \frac{e^2}{4\pi} \qquad r_e = \frac{e^2}{4\pi m_e}
$$

The challenge is to integrate analytically over the scattered electron and photon angular phase space: $d\Omega' d\Omega_k$

Scattered Electron Integration

For longitudinal beam polarizations, all 4-vector products involving p' can be expressed in terms of $(1 - \mathbf{p'a})$ and $(1 - \mathbf{p'b})$ and constants of integration.
For instance, $p'k = \varepsilon'\omega(1 - \mathbf{p'b})$ $\qquad \mathbf{b} = \frac{\mathbf{k}}{\varepsilon'\omega}$
The integrals over the scattered electron angles can be put into the form:

$$I_{m,n} = \int d\Omega'(1 - \mathbf{p}'\mathbf{a})^{-m}(1 - \mathbf{p}'\mathbf{b})^{-n}$$

Gluckstern and Hull, Phys. Rev. 90 (1953)

The array of integrals present:
$I_{0,0}, I_{0,1}, I_{1,0}, I_{-1,1}, I_{1,-1}, I_{1,1}, I_{2,0}, I_{0,2}, I_{2,1}, I_{1,2}, I_{2,-1}, I_{2,-2}, I_{2,2}$

Non-trivial one are $I_{1,1}, I_{2,1}, I_{1,2}, I_{2,2}$
Steps to Integrate these:

1. Merge denominators using a Feynman parameter
2. Trivial integration over azimuthal angles
3. Look up remaining Feynman parameter integral in a table of integrals

Assembly of integrated pieces and subsequent algebraic simplification done with Mathematica

Double-Differential Cross Section

Boost from target-rest-frame to lab-frame, especially simple here because photons are almost colinear to beam electron: $\theta_k \sim m_e/\varepsilon$

$$\omega \to \omega \frac{2\epsilon_p}{m_p} \qquad \varepsilon \to \varepsilon \frac{2\epsilon_p}{m_p} \qquad \varepsilon' \to \varepsilon' \frac{2\epsilon_p}{m_p}$$

Cross section differential in photon energy and polar angle

$$\begin{aligned}\frac{d\sigma_{\mathcal{P}}}{d\omega d\delta} &= \mathbb{P}_e \mathbb{P}_p \frac{\alpha\, r_e^2\, m_e^2\, \delta}{\omega \varepsilon^3 \varepsilon' \varepsilon_p (1+\delta^2)^2} \Bigg[\delta_0^2 \Big(4L_1 \varepsilon \varepsilon'^2 + 2\omega \Big((L_1 + L_2 - L_\theta)\varepsilon^2 + L_1 \varepsilon \varepsilon' - L_2 \varepsilon'^2 \Big) - 2\omega^3 - 5\varepsilon' \omega^2 - 6\varepsilon'^2 \omega - 4\varepsilon'^3 \Big) \\ &\quad + 2\varepsilon \Big((1 + 2L_\theta)\varepsilon'^2 + (1 - L_1 - L_2 + L_\theta)\omega^2 + \varepsilon' \omega \Big(-L_1 - L_2 + 2L_\theta + \frac{1}{1+\delta_0^2} \Big) \Big) \Bigg]\end{aligned}$$

$$\delta = \theta \frac{\varepsilon}{m_e} \qquad L_1 = 2 \ln \frac{4\varepsilon' \varepsilon_p}{m_e m_p} \qquad L_2 = \ln \frac{4\,\varepsilon\,\varepsilon'\,\varepsilon_p}{\omega\, m_e m_p} \qquad L_\theta = \ln(1+\delta^2)$$

- The spectrum is peaked near $\delta \sim 1$, which means $\theta_k \lesssim \frac{m_e}{\varepsilon}$

Single-Differential Cross Section: **Main Result**

Integration over the photon polar angle, δ_k, is elementary but leads to tedious algebra. Mathematica is used for this integration and subsequent simplification:

Cross section differential in photon energy

$$\frac{d\sigma_{\mathcal{P}}}{d\omega} = \mathbb{P}_e \mathbb{P}_p \frac{4\alpha r_e^2}{\omega} \frac{\varepsilon'}{\varepsilon} \frac{m_e^2}{\varepsilon\, \varepsilon_p} \left(F_1 + \frac{\varepsilon}{4\varepsilon'} F_2 + \frac{\varepsilon'}{8\varepsilon} F_3 + \frac{\varepsilon^2}{2\varepsilon'^2} F_4 \right)$$

$$\begin{aligned}
F_1 &= \frac{1}{8}\left(7 + L_2(2 - 4L_3) - 4L_3 + L_1(-2 + 4L_3)\right), \\
F_2 &= -3 + L_1 + 2L_2 + L_3(1 - 2L_2 + 2L_3), \\
F_3 &= (-1 + 2L_2)(-1 + 2L_3), \\
F_4 &= (-1 + L_3)(-2 + L_1 + L_2 - L_3).
\end{aligned}$$

$$L_1 = \ln \frac{4\varepsilon'\, \epsilon_p}{m_e m_p} \qquad L_2 = \ln \frac{4\varepsilon\, \varepsilon'\, \epsilon_p}{\omega\, m_e m_p} \qquad L_3 = \ln \frac{\pi\varepsilon\, \epsilon_p}{m_e m_p}$$

Cross Section Visualization: **Main Result**

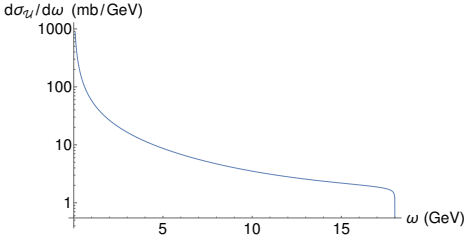
Full 1D cross section

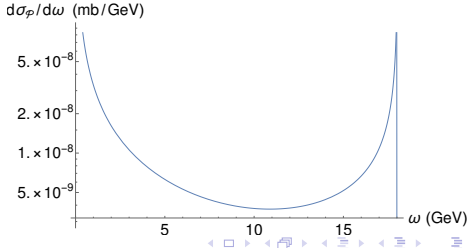
$$\frac{d\sigma}{d\omega} = \frac{4\alpha r_e^2}{\omega}\frac{\varepsilon'}{\varepsilon}\left[\mathcal{U} + \mathcal{P}\right]$$

Unpolarized Bethe-Heitler term $\mathcal{U} = \left(\frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon} - \frac{2}{3}\right)\left(L_2 - \frac{1}{2}\right)$

New Polarized term $\mathcal{P} = \mathbb{P}_e \mathbb{P}_p \frac{m_e^2}{\varepsilon\,\varepsilon_p}\left(F_1 + \frac{\varepsilon}{4\varepsilon'}F_2 + \frac{\varepsilon'}{8\varepsilon}F_3 + \frac{\varepsilon^2}{2\varepsilon'^2}F_4\right)$

$\mathbb{P}_e = \mathbb{P}_p = 1$ and top EIC energies: $\varepsilon = 18$ GeV, $\varepsilon_p = 275$ GeV





Summary and Additional Remark

- An analytic expression for the polarized Bremsstrahlung cross section has been calculated
- The calculation is applicable at low q^2, as needed for the luminosity program at the upcoming EIC.
- The polarized component is highly suppressed ($\frac{m_e^2}{\varepsilon\,\varepsilon_p}$) wrt the unpolarized Bethe-Heitler term.
- This work has been posted recently: arXiv:2307.16245

Additional Remark

The EIC will accelerate ions as heavy as uranium ($Z = 92$) For high Z, the Bethe-Heitler expression is known to be inaccurate. Bethe & Maximon provided a more general solution that the EIC should use instead:

Bethe-Maximon equation Phys Rev 93, 768 & 788 (1954)

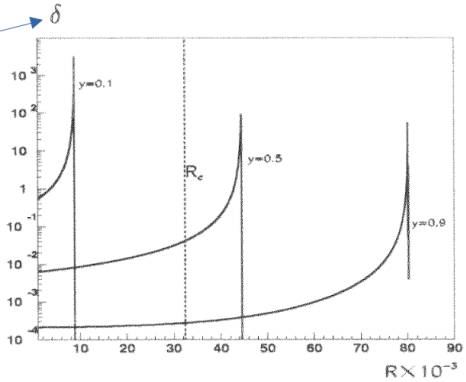
$$\frac{d\sigma_{\rm BM}}{d\omega} = \frac{4\alpha r_e^2}{\omega}\frac{\varepsilon'}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon} - \frac{2}{3}\right)\left(\ln\frac{4\varepsilon\,\varepsilon'\,\varepsilon_p}{\omega\, m_p m_e} - \frac{1}{2} - f(\alpha Z)\right)$$

$$f(\alpha Z) = (\alpha Z)^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+(\alpha Z)^2)} \qquad \text{for } \alpha Z \ll 1 \quad f(\alpha Z) \approx 1.2(\alpha Z)^2$$

Extra Slides

Sharp drop at upper energy threshold

normalized cross section



Akushevich et al. (1998)
arXiv:hep-ph/9804361

Figure 6. Integrand of radiative correction to the cross section (36) over R normalized to the Born one in the case of combination of the cross section (43). The broken line shows the used cut of photon energy ($E_\gamma < 10$ GeV).

R is proportional to the photon energy