Lee-Yang singularities, series expansions and the critical point

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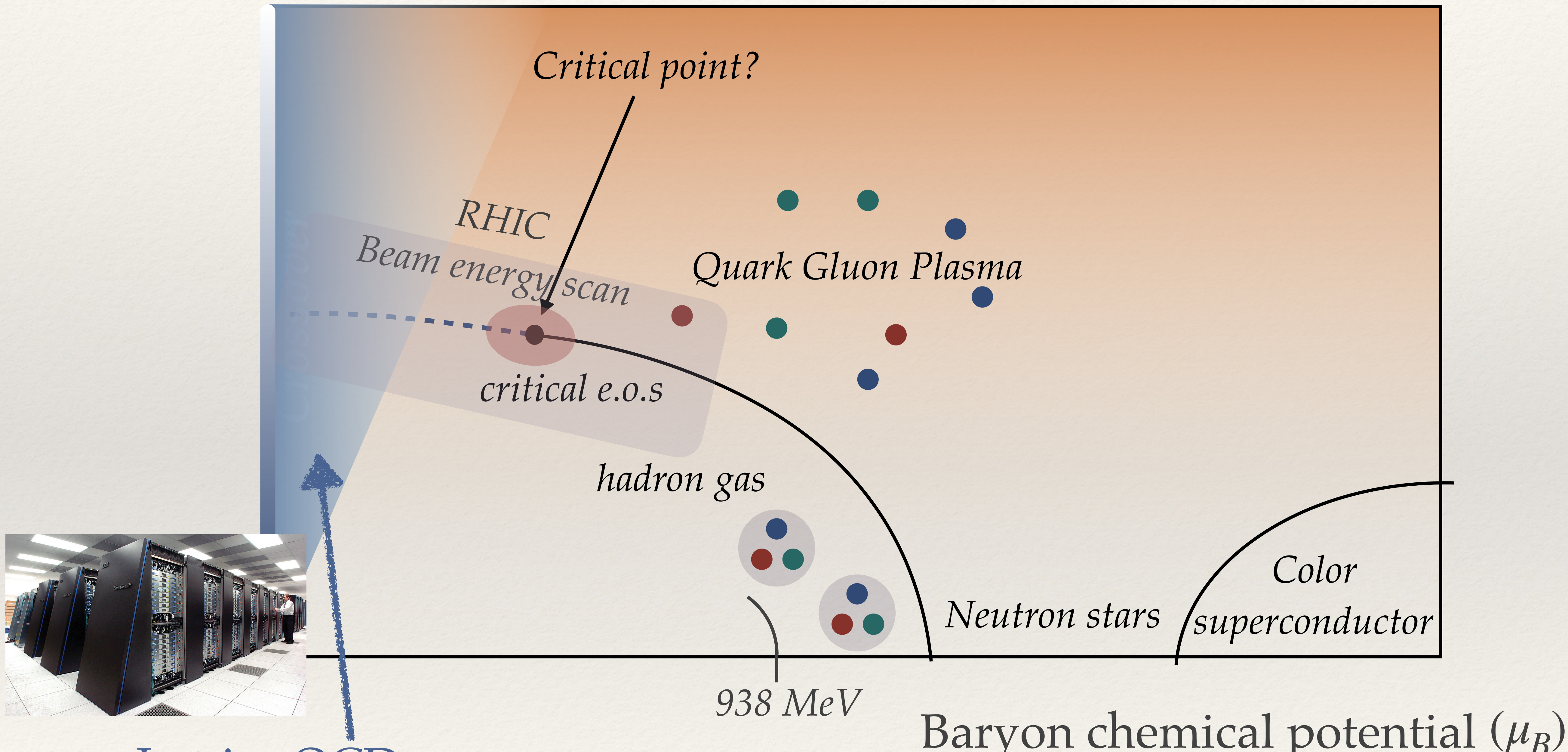
Based on:

GB *PRL* 127 (2021) 17, 171603

GB, G. Dunne (UConn), Z. Yin (UNC → Stanford) *PRD* 105 (2022) 10, 105002

Motivations



Lattice QCD

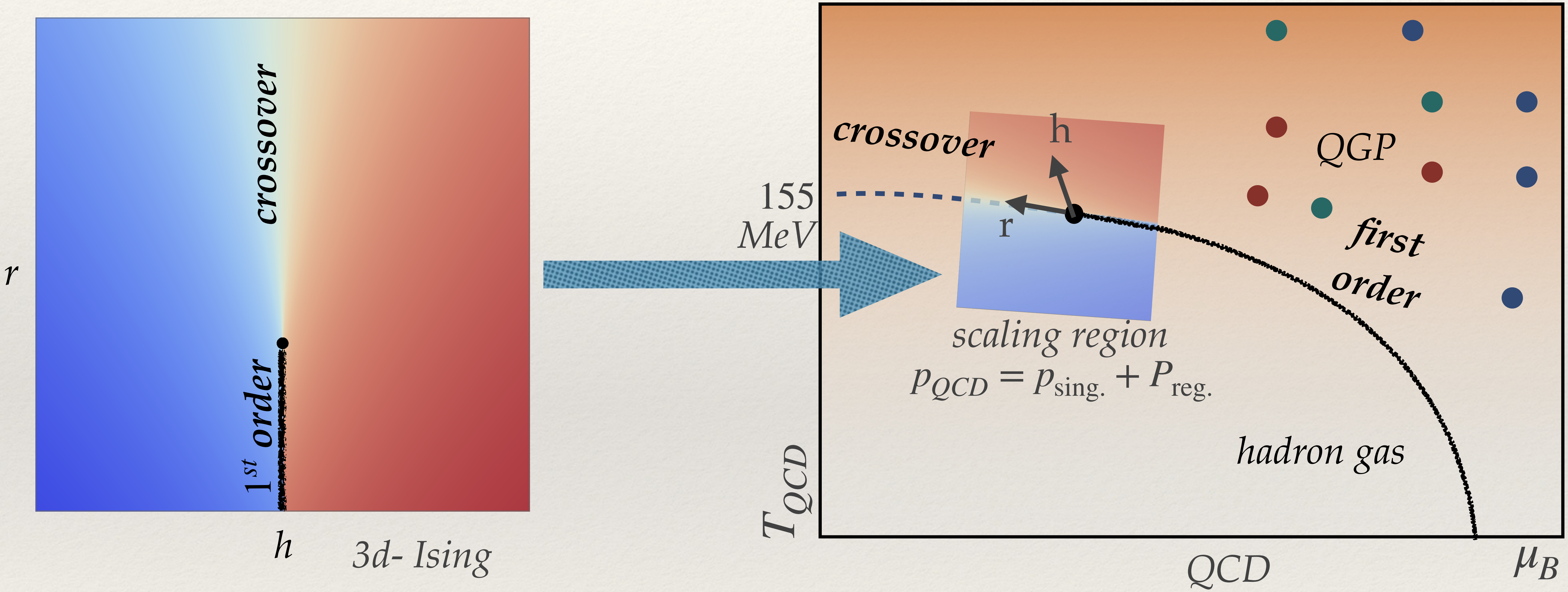
Taylor series around $\mu_B = 0$

Imaginary μ_B

Motivations

BEST COLLABORATION

$$\begin{pmatrix} r \\ h \end{pmatrix} = \begin{pmatrix} r_T & r_\mu \\ h_T & h_\mu \end{pmatrix} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix} := M \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix}$$

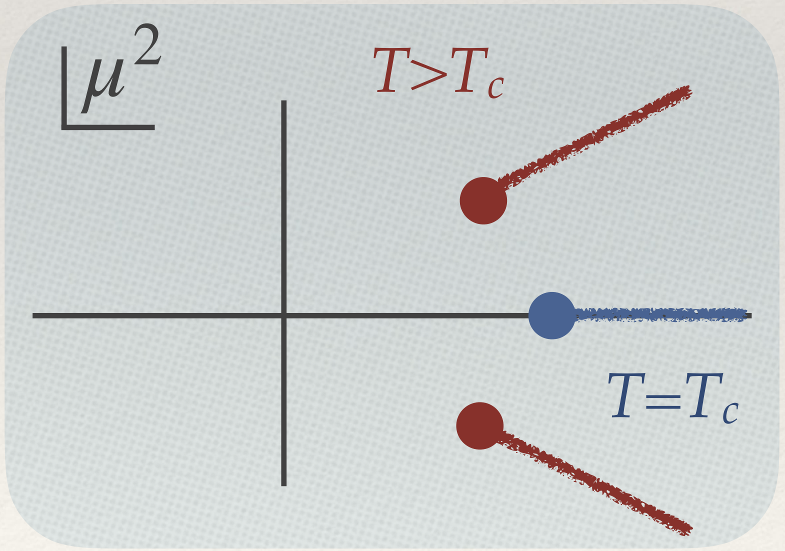


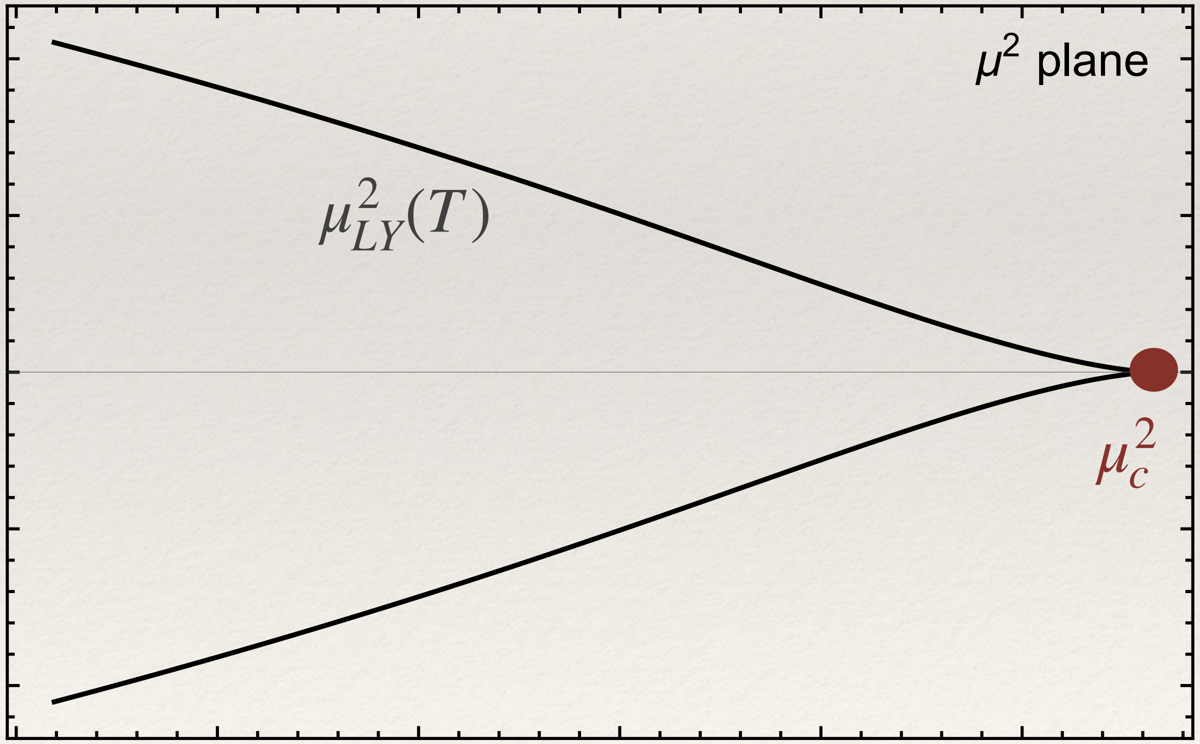
Given the e.o.s. as truncated Taylor series around μ=0, what can we say about *the critical e.o.s* ?

Lee-Yang edge singularities

- The equation of state has complex singularities [Lee-Yang, 52']
- Zeroes of partition function $\mathscr{Z}(\zeta)$ ($\zeta = e^{\mu/T}$: fugacity)
- Coalesce into branch cuts in thermodynamic limit
- Pinch the real axis at a second order transition
- Closest singularity to origin ("extended analyticity conjecture")

[Fonseca, Zamolodchikov '02, An, Mesterházy, Stephanov '17]





[Stephanov, 0603014]

Lee Yang edge singularity

- The scaling e.o.s, $f_s(w)$, has singularities at $w = \pm\, i w_{LY}$ $\quad (w := h r^{-\beta\delta})$

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) \pm i w_{LY} \frac{\left(\det \mathbb{M}\right)^{\beta\delta}}{h_\mu^{\beta\delta+1}}(T - T_c)^{\beta\delta}$$

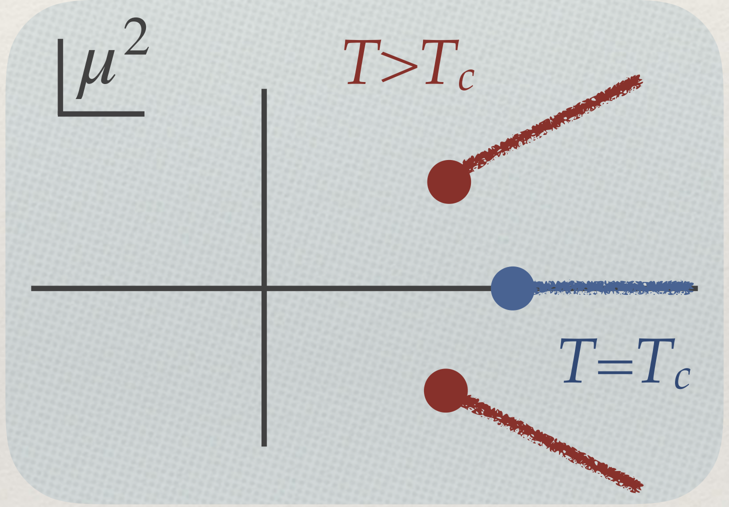
$(\tan \alpha_1)^{-1}$

slope of the crossover line

$\det \mathbb{M} \propto (\tan \alpha_2 - \tan \alpha_1)$

relative angle between r, h axes

see [Pradeep, Stephanov '19]



- The e.o.s. near the LY singularity: $M(w) \sim (w \pm i w_{LY})^{\sigma_{LY}}$, $\quad (M:$ magnetization$)$

$\sigma_{LY,d=3} \approx 0.1, \quad \sigma_{LY,d=6} = 1/2$ (mean field)

[Fisher, '74; An, Stephanov, Mesterházy '16; Connelly, Johnson, Mukherjee, Skokov '20]

When life gives you Taylor series...

Taylor series: $\chi(\mu^2) = \sum_{n=0}^{N} c_{2n}\mu^{2n}$

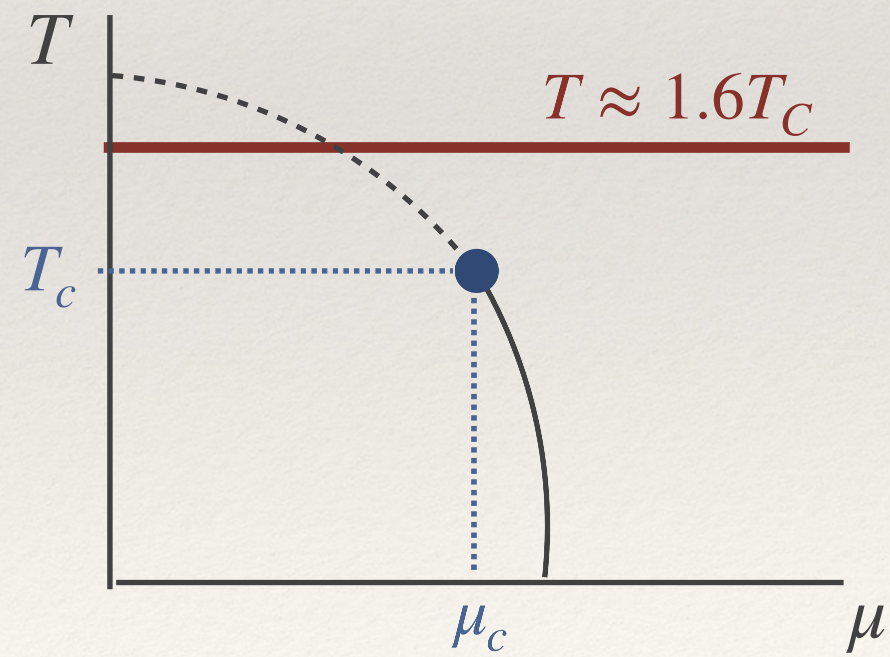
Padé approximant (diagonal) $\mathrm{P}_{[N/2,N/2]}f(\mu^2) = \dfrac{P_{N/2}(\mu^2)}{Q_{N/2}(\mu^2)}$

Singularity of the function $\longleftrightarrow$ *poles/zeroes of Padé*

Let's try this on an exactly solvable model

massive Gross-Neveu model

Preliminary results on QCD at the end..



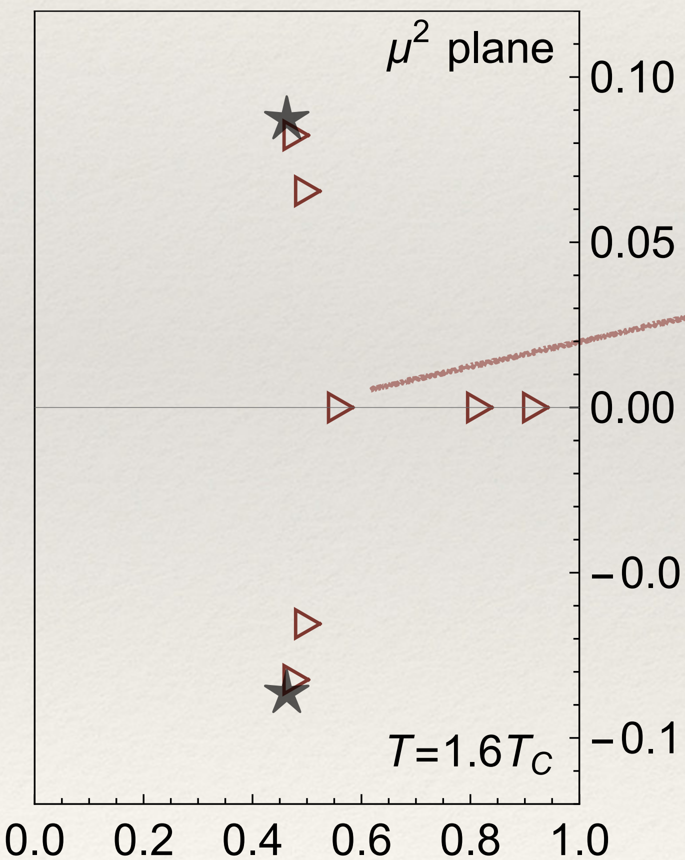
When life gives you Taylor series…

Problem: Padé is fairly good away from the singularity but fails badly near a singularity / branch cut. Not a glitch, a theorem… 🤷‍♂️

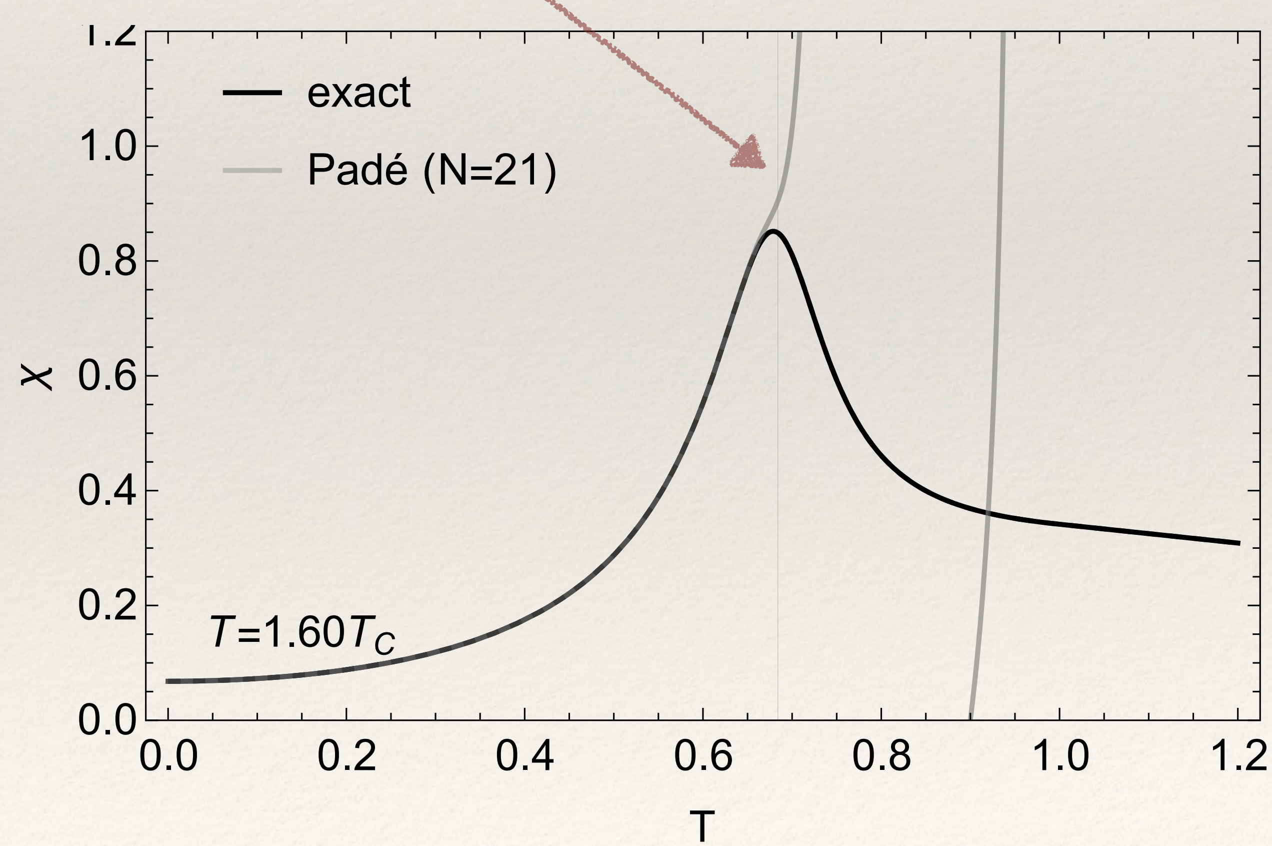
[Stahl' 97, Costin Dunne '20]

GN model

Padé cannot reconstruct the e.o.s. near and beyond the radius of convergence! $(\mu^2 \gtrsim |\mu_{LY}^2|)$



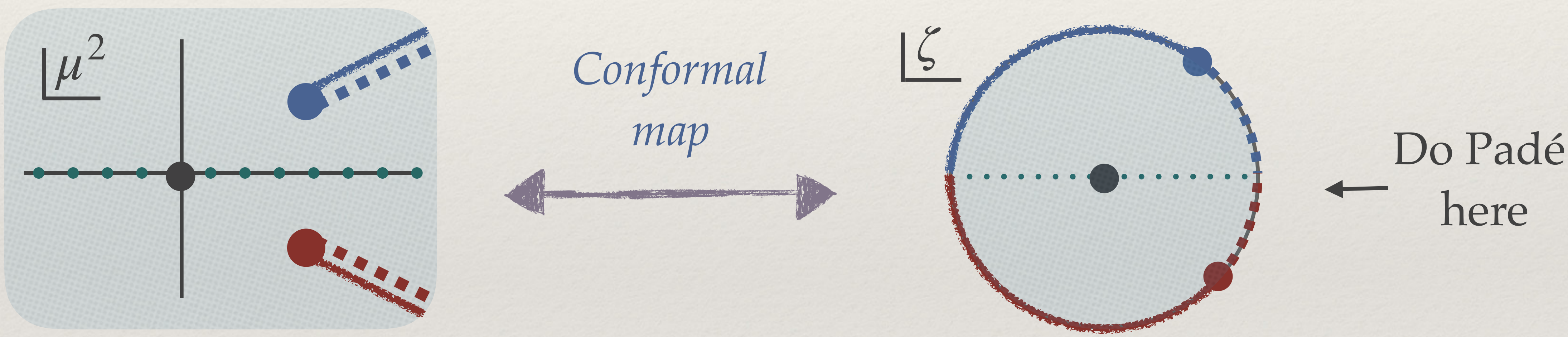
Unphysical poles



Conformal Maps

Solution: Do Padé after a conformal map

- Captures the singular behavior, no unphysical poles along real axis ✓
- Significantly better approximation than Padé ✓



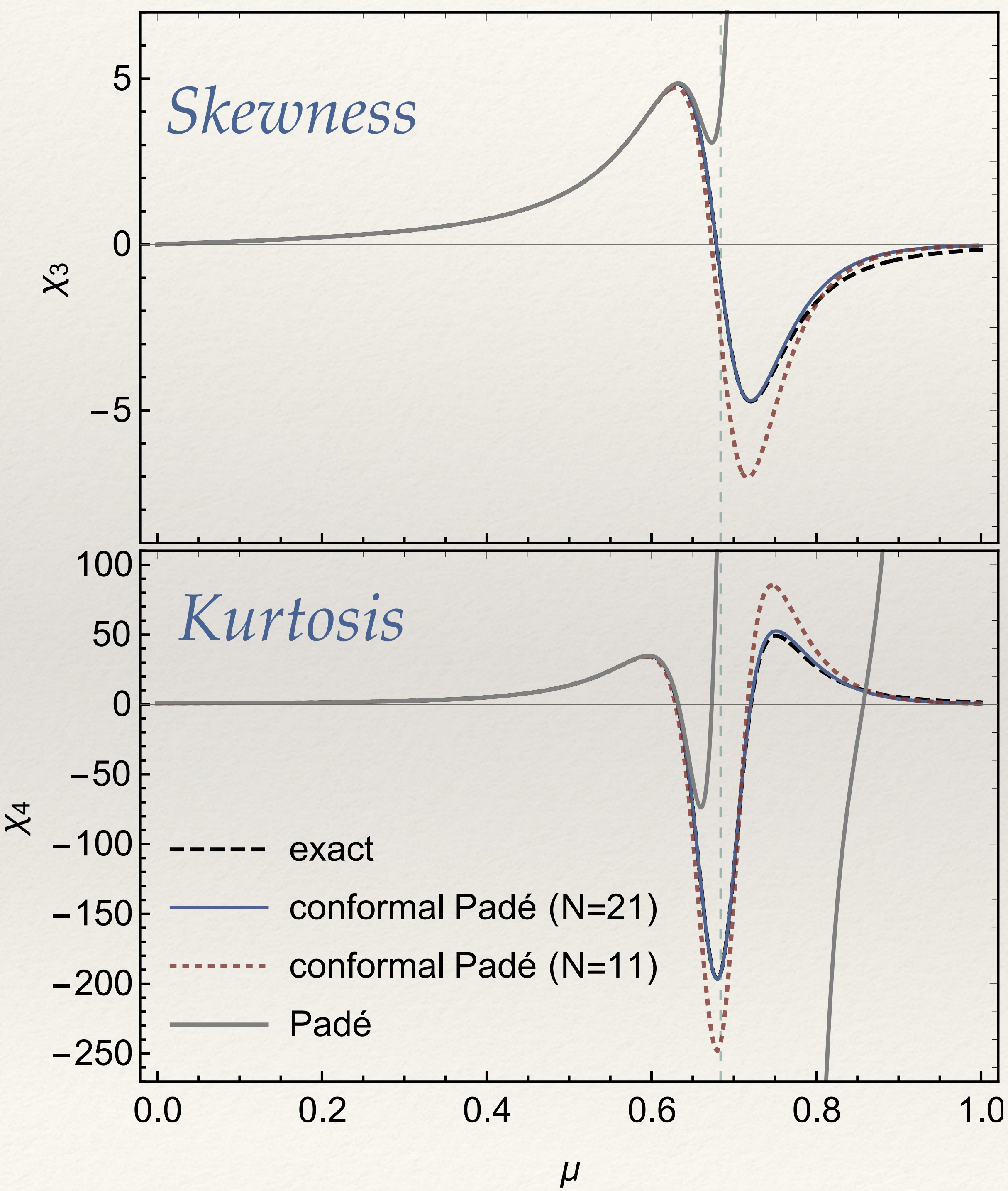
$$\phi(\zeta) = \left(\frac{\theta}{\pi}\right)^{\theta/\pi} \left(1 - \frac{\theta}{\pi}\right)^{1-\theta/\pi} \frac{4\mu_{LY}^2 \zeta}{(1+\zeta)^2} \left(\frac{1+\zeta}{1-\zeta}\right)^{2\theta/\pi}$$

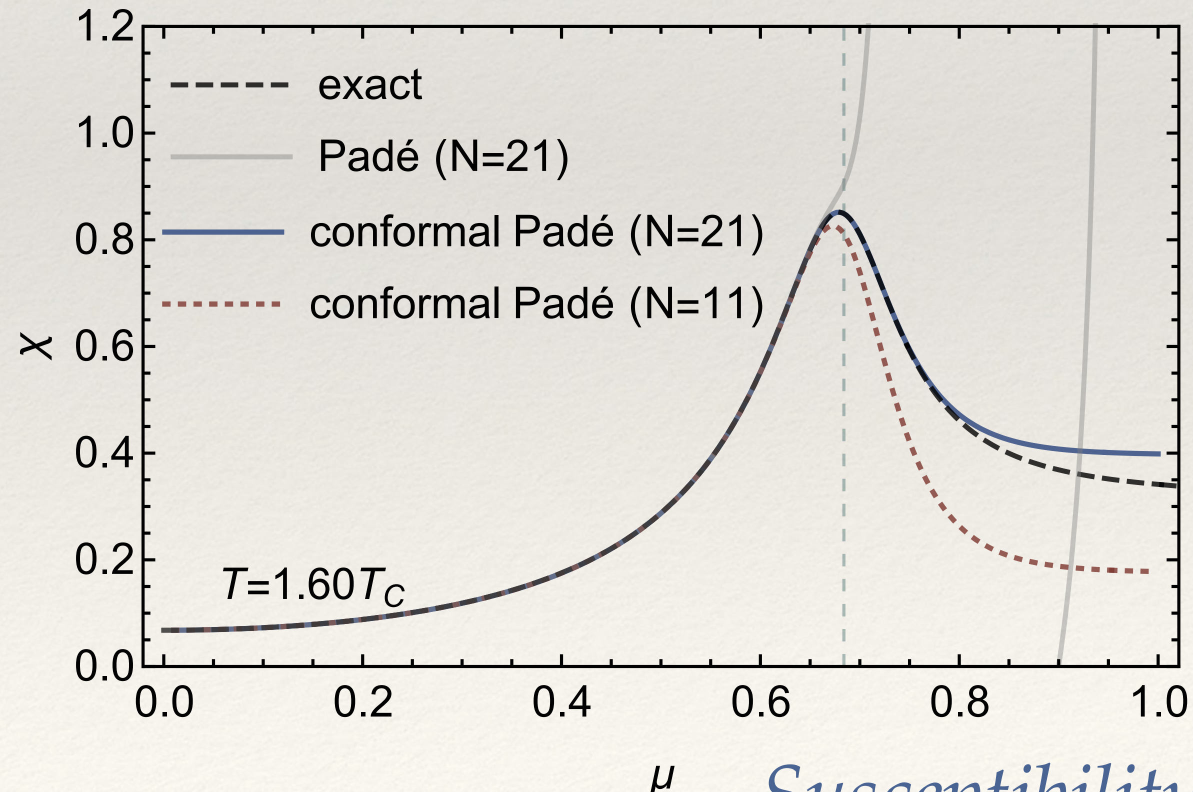
"conformal Padé"

$$\mathrm{P}\chi(T, \phi(\zeta)) = \left.\frac{\tilde{p}_0(T) + \tilde{p}_1(T)\zeta + \ldots + \tilde{p}_{N/2}(T)\zeta^N}{\tilde{q}_0(T) + \tilde{q}_1(T)\zeta + \ldots + \tilde{q}_{N/2}(T)\zeta^N}\right|_{\zeta=\phi^{-1}(\mu^2)}$$

Conformal Maps

- conformal Padé does not introduce unphysical poles on the real axis!
- captures the e.o.s. *beyond the radius of convergence*

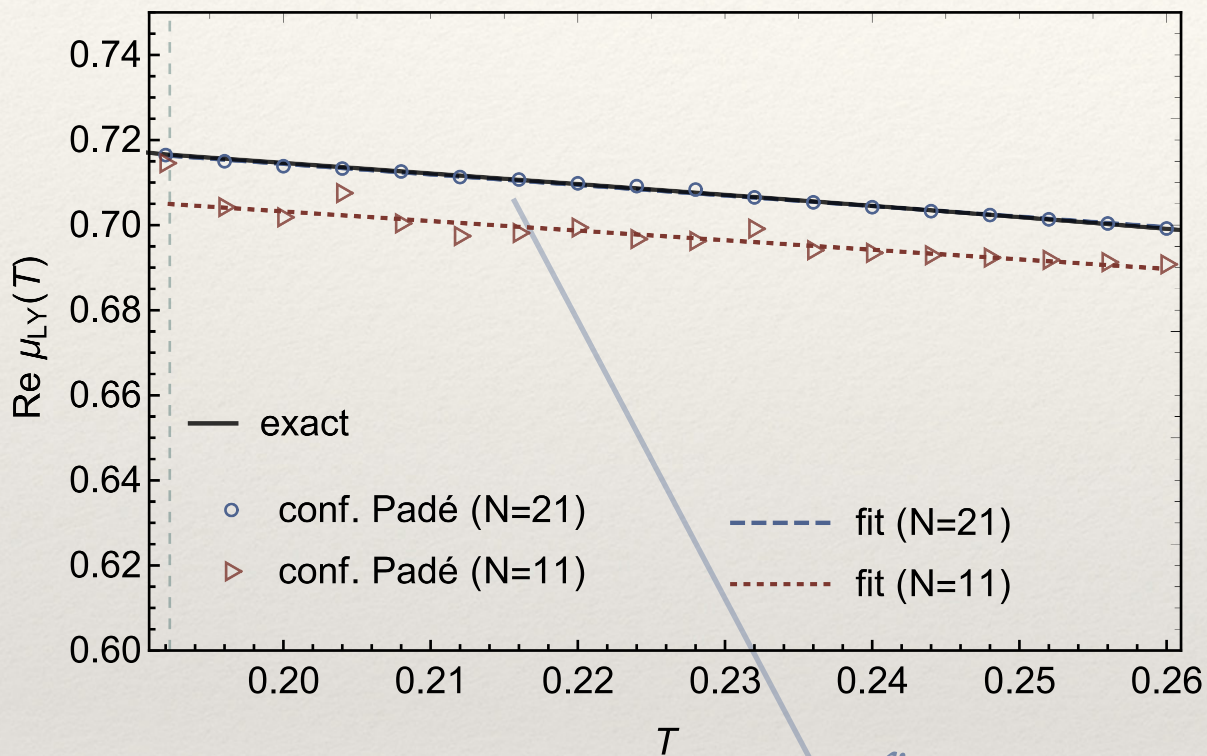


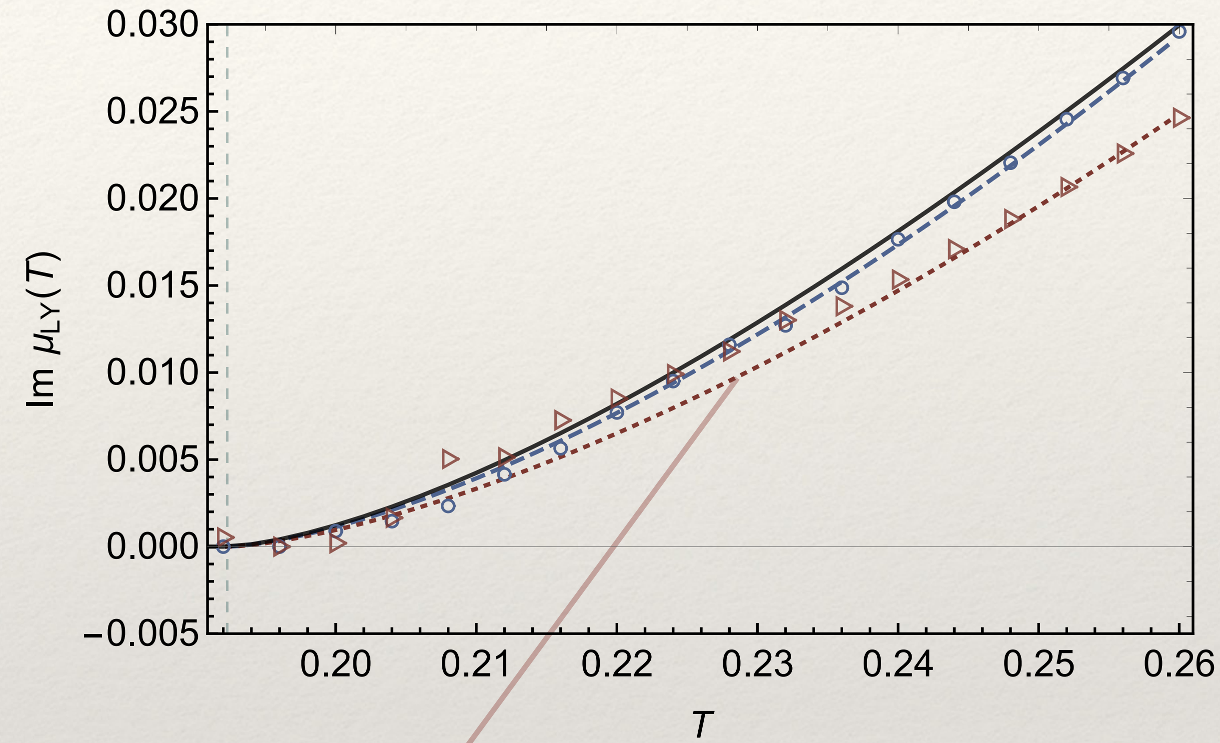


Susceptibility

Lee-Yang trajectory

- Find $\mu_{LY}^2(T)$ from poles of the conformal-Padé (GN model)





fit *fit*

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2} (T - T_c)^{3/2} \qquad w_{LY} = \frac{2}{3\sqrt{3}}$$

- Extract $\mu_c, T_c,$ crossover slope, $\dfrac{h_T}{h_\mu}$, and $\dfrac{r_\mu^{3/2}}{h_\mu}\left(\dfrac{r_T}{r_\mu} - \dfrac{h_T}{h_\mu}\right)^{3/2}$

Ising parameters

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2} (T - T_c)^{3/2}$$

$$w_{LY} = \frac{2}{3\sqrt{3}}$$

	T_c	μ_c	h_T/h_μ	c
exact	0.192	0.717	0.249	4.684
conf. Padé (N=21)	0.195	0.716	0.248	4.323
conf. Padé (N=11)	0.185	0.707	0.225	3.666

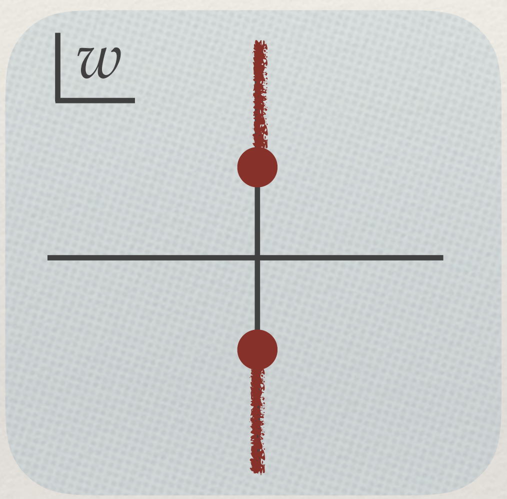
Uniformization Map

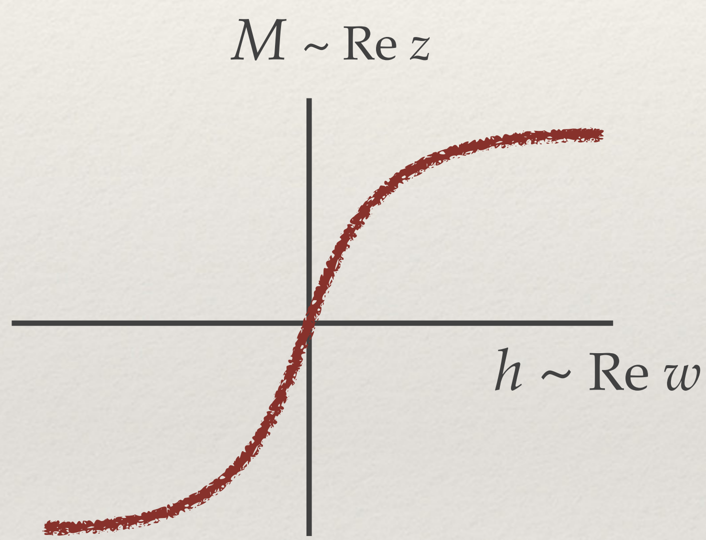
$w = hr^{-\beta\delta}$
$z = Mr^{-\beta}$

Ising model: $\quad w = F(z)$

$F(z) = z + z^3 \quad$ (mean field)

High Temperature (T>T$_c$)





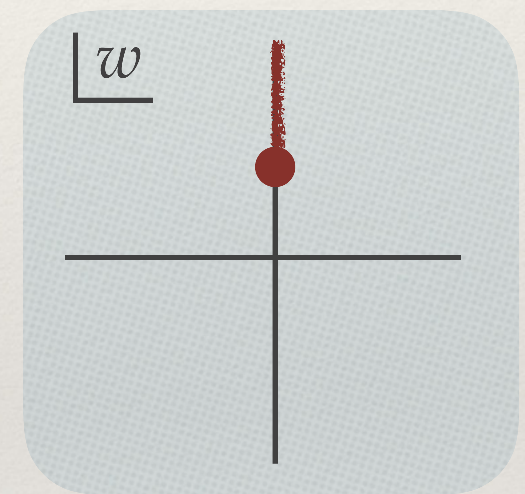
high T sheet
r>0

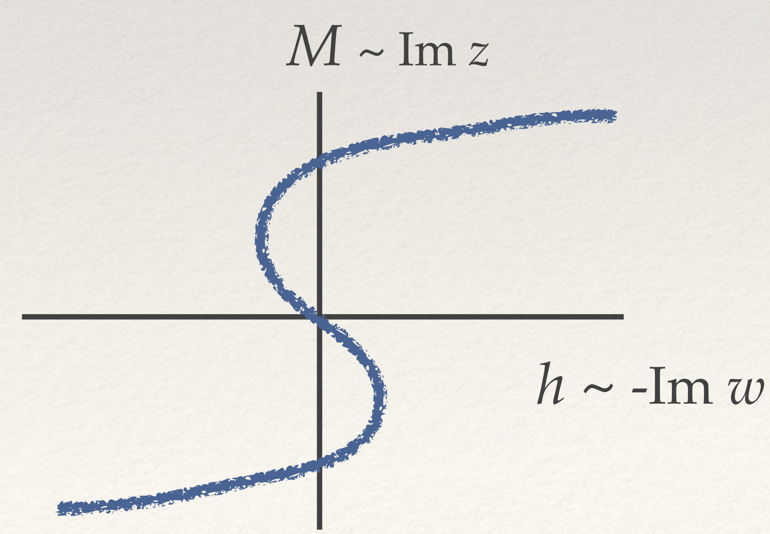
$$z(w) = w - w^3 + 3w^5 - 12w^7 + \ldots$$

high T expansion

Low Temperature (T<T$_c$)

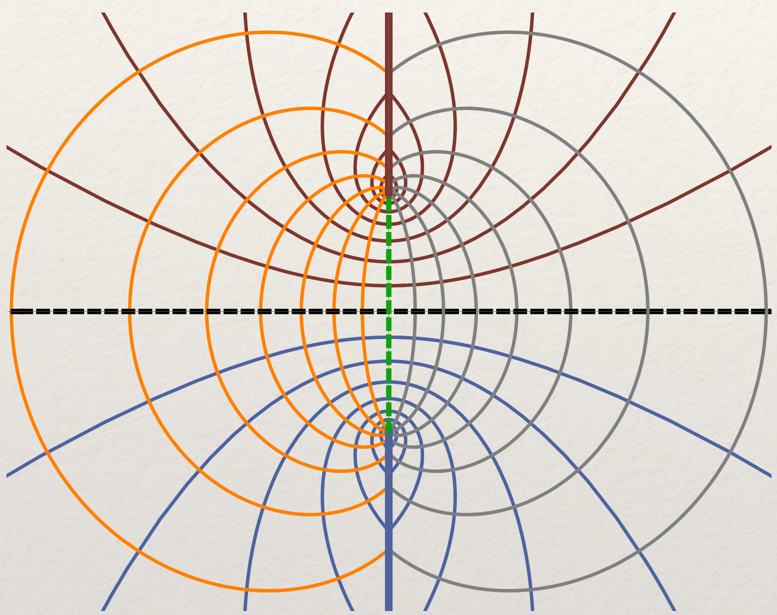
low T sheet
r<0, h>0

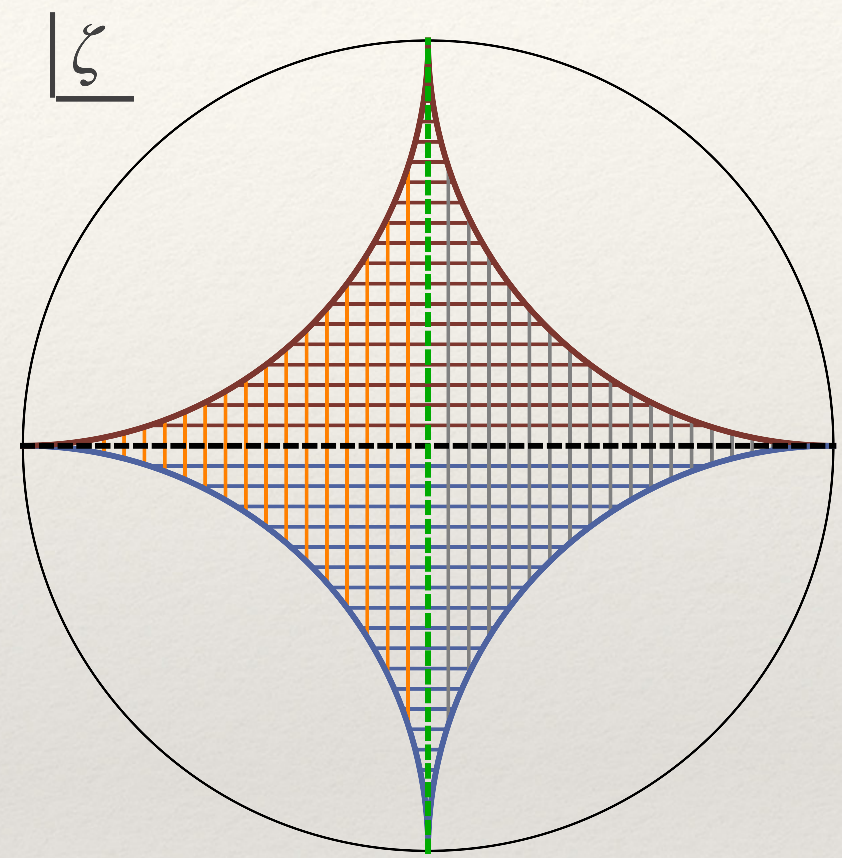




Uniformization: crossing the branch cut

w plane





high T sheet

r>0

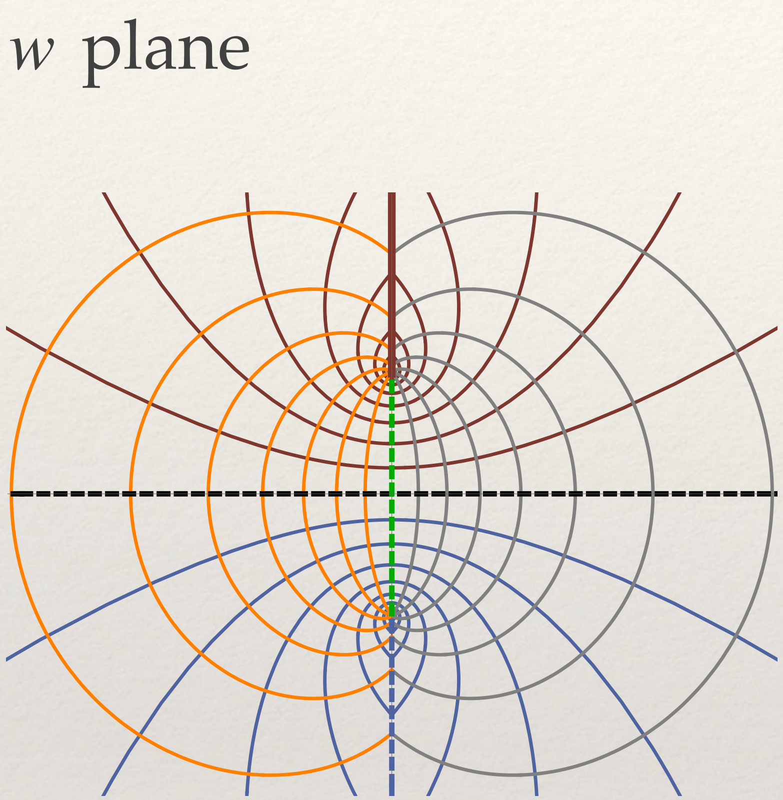
$$w \to w(\tau) = i(-1 + 2\lambda(\tau))$$

$$\tau(\zeta) = i\left(\frac{1 + i\zeta}{1 - i\zeta}\right)$$

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$ (elliptic modular function)

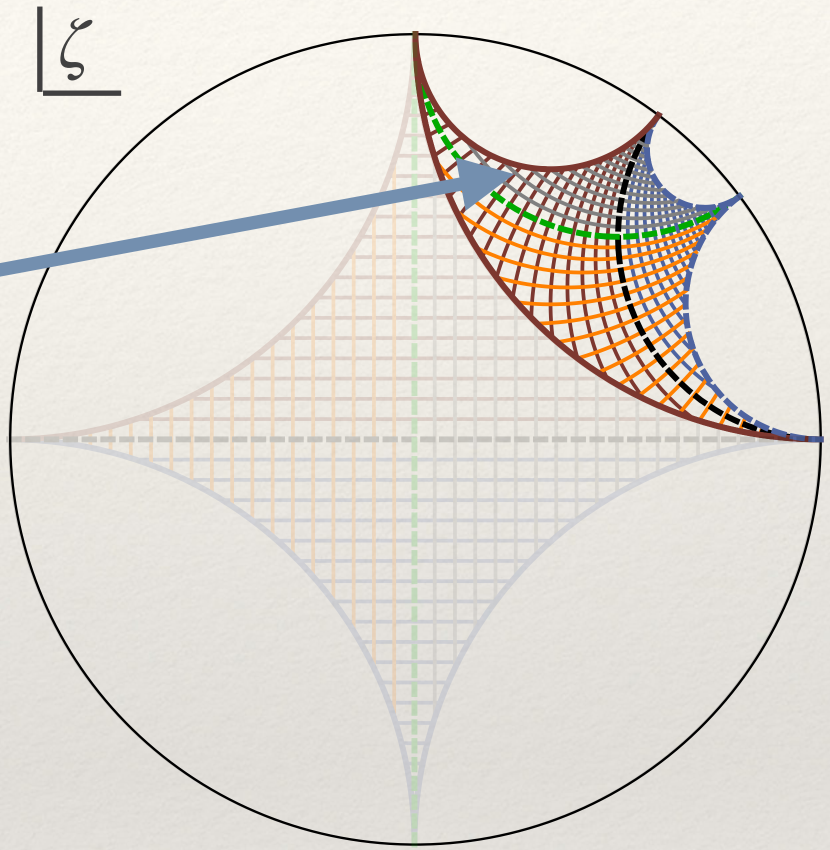
$$\theta_2(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau n^2}$$

Uniformization: crossing the branch cut



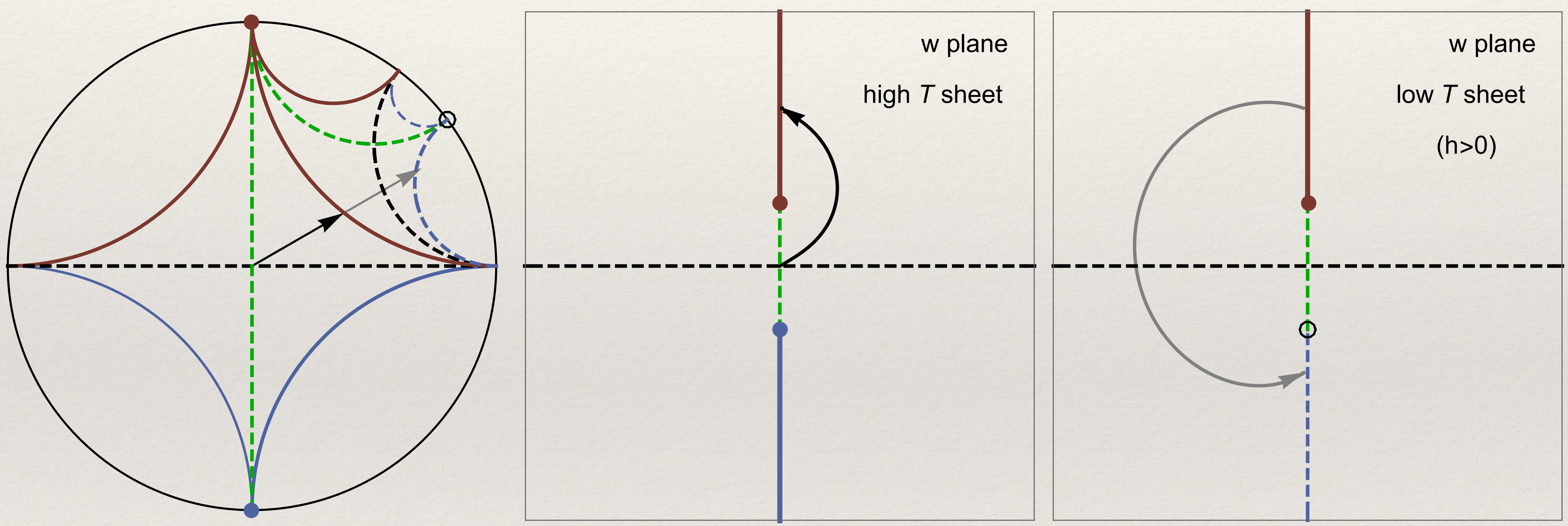
low T sheet

r<0



Low T sheet = Schwartz reflection of the high T sheet
(modular transformation)

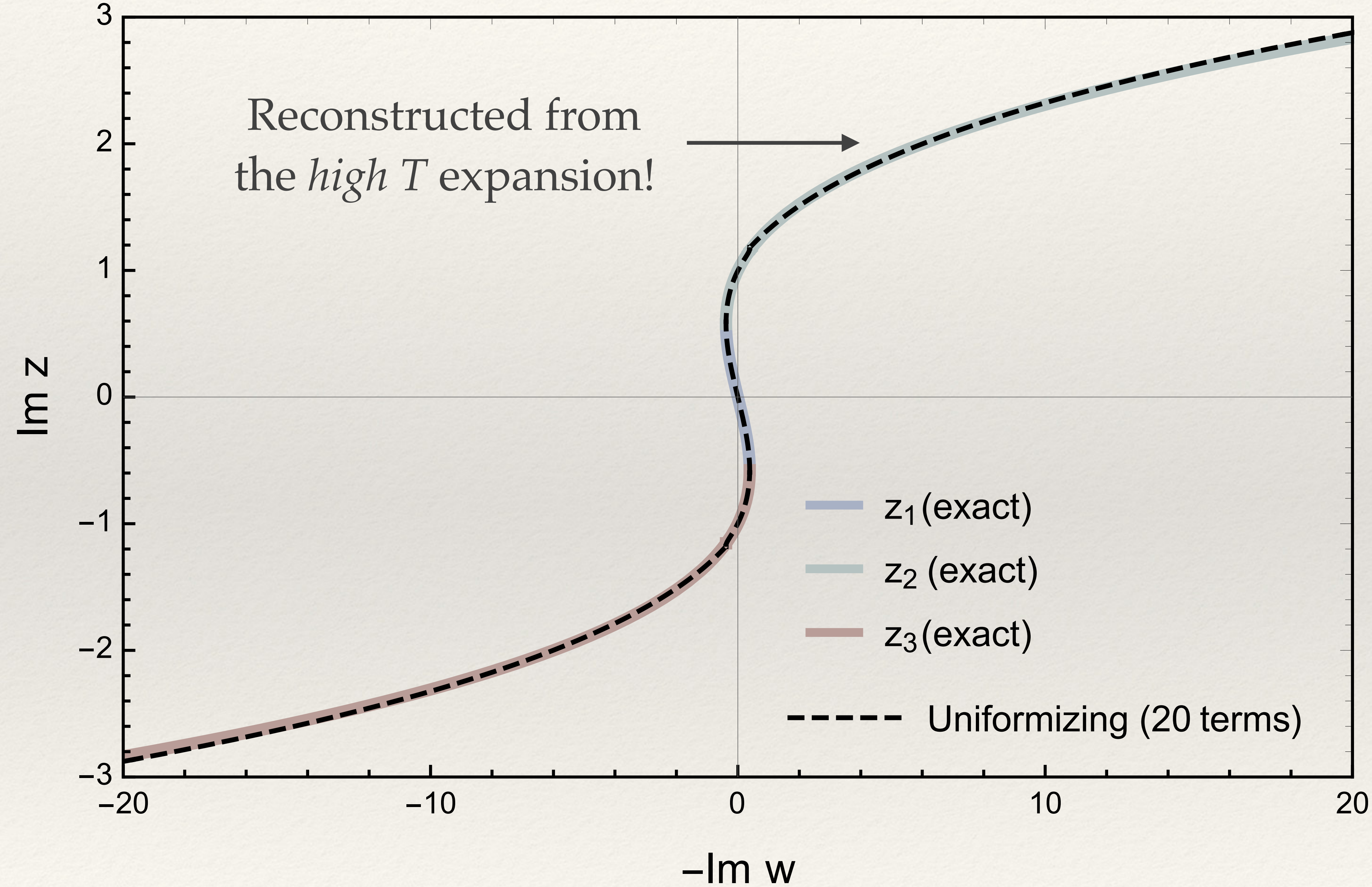
Uniformization: crossing the branch cut



Moving within unit circle (smooth) ⟷ *Jumping through sheets*

Uniformization: crossing the branch cut

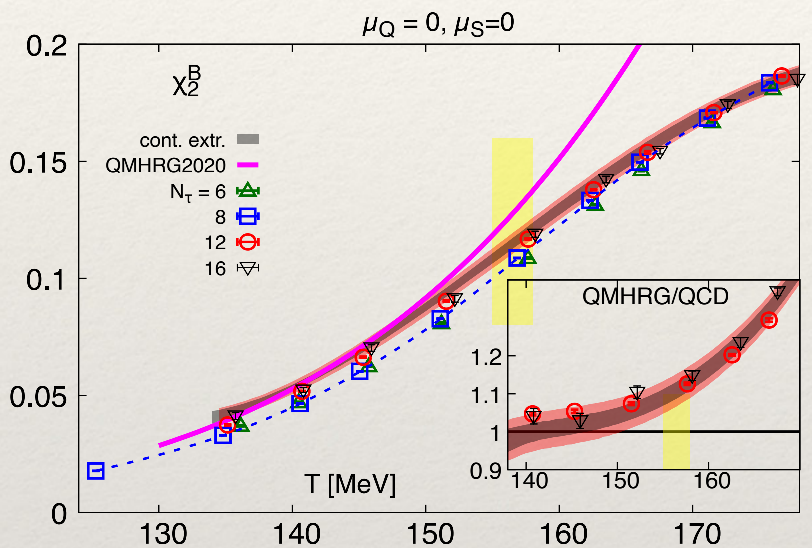
Low T

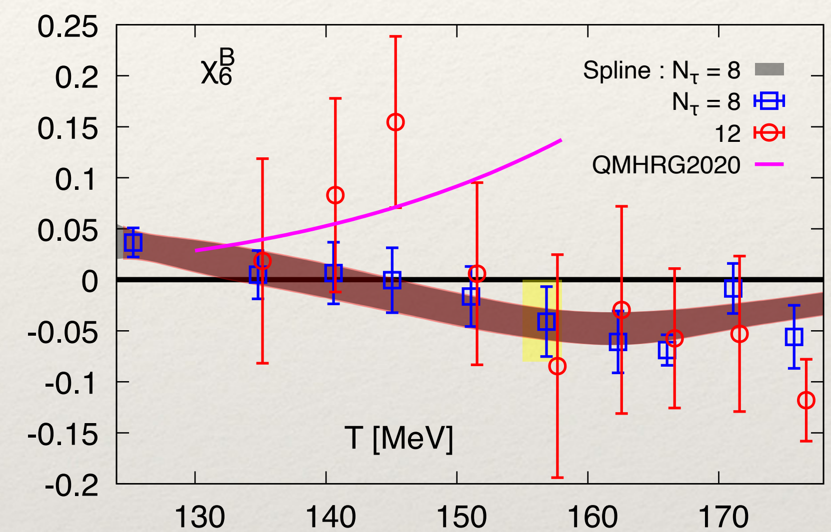


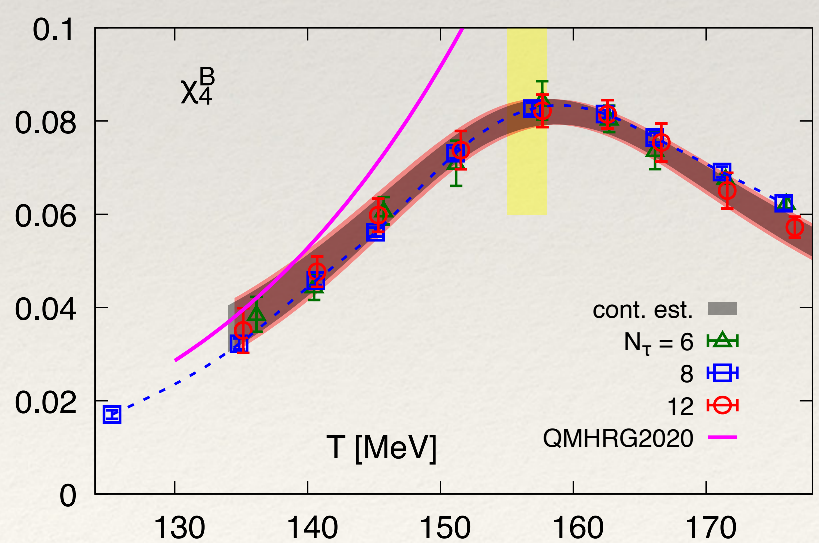
Preliminary Results for QCD

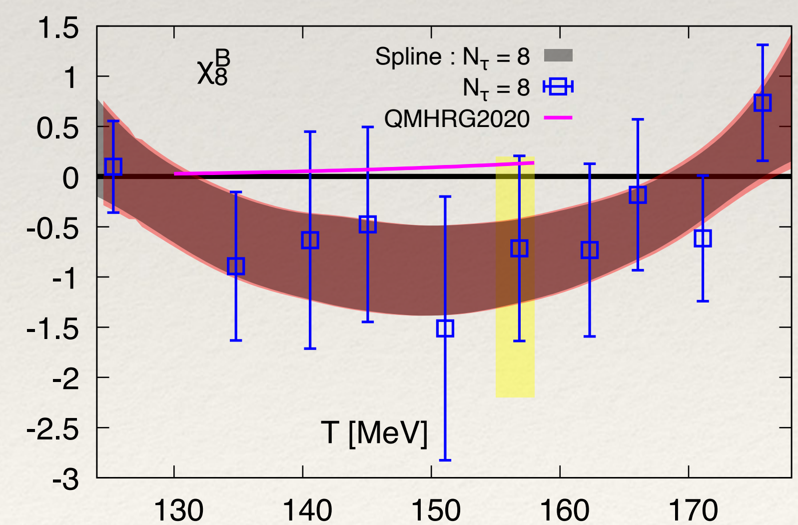
Taylor coefficients from Hot QCD collaboration up to μ_B^8

[Bollweg et al. 2202.09184]



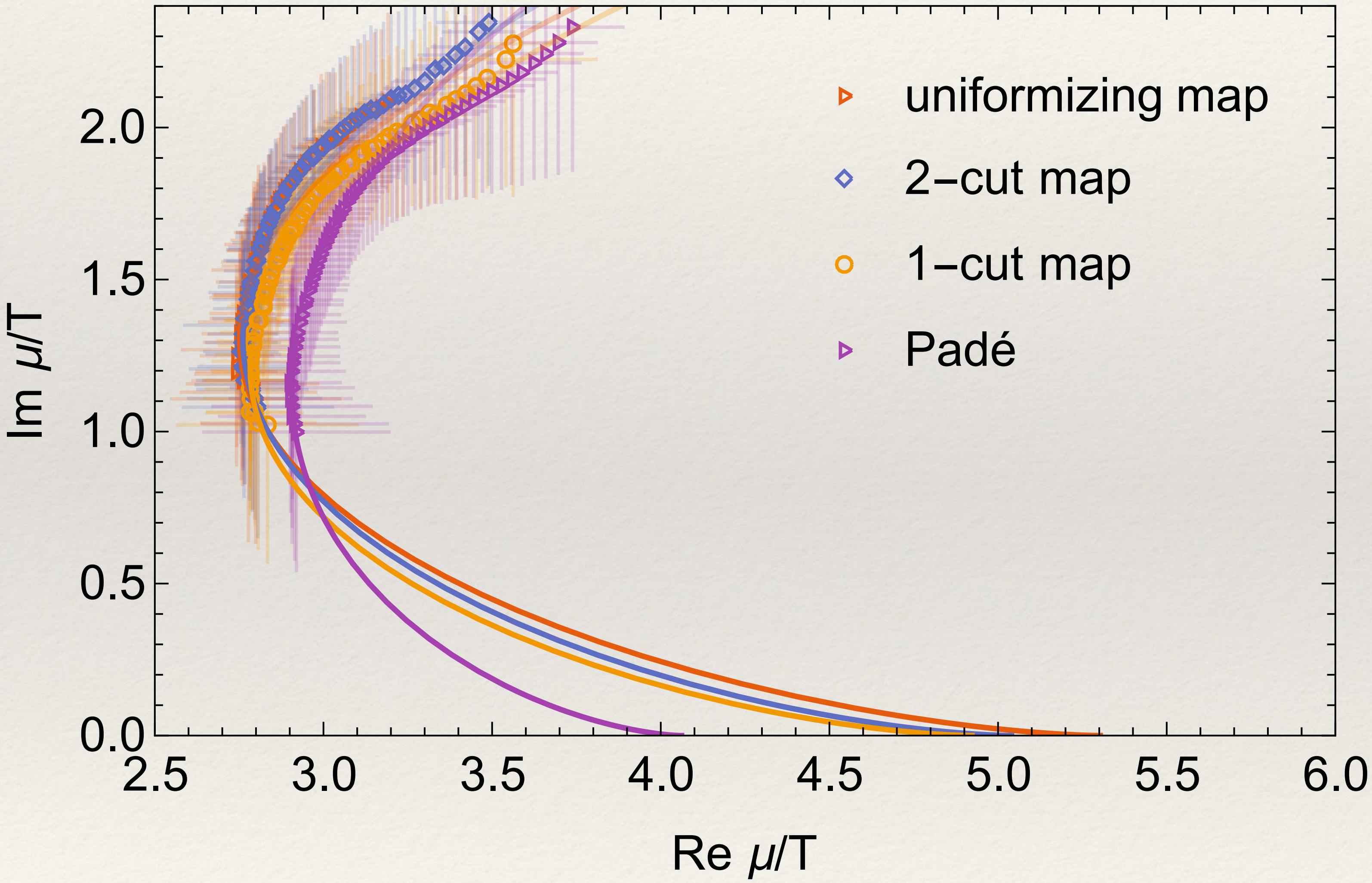






LY Trajectory for QCD

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_c \frac{r_\mu^{\beta\delta}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{\beta\delta}(T - T_c)^{\beta\delta}$$



fits:

$$\mathrm{Re}\mu_{LY}(T) = a(T - T_C)^2 + b(T - T_C) + c$$

$$\mathrm{Im}\mu_{LY}(T) = c w_c (T - T_C)^{\beta\delta}$$

$\beta\delta \approx 1.5631$ (3d Ising)

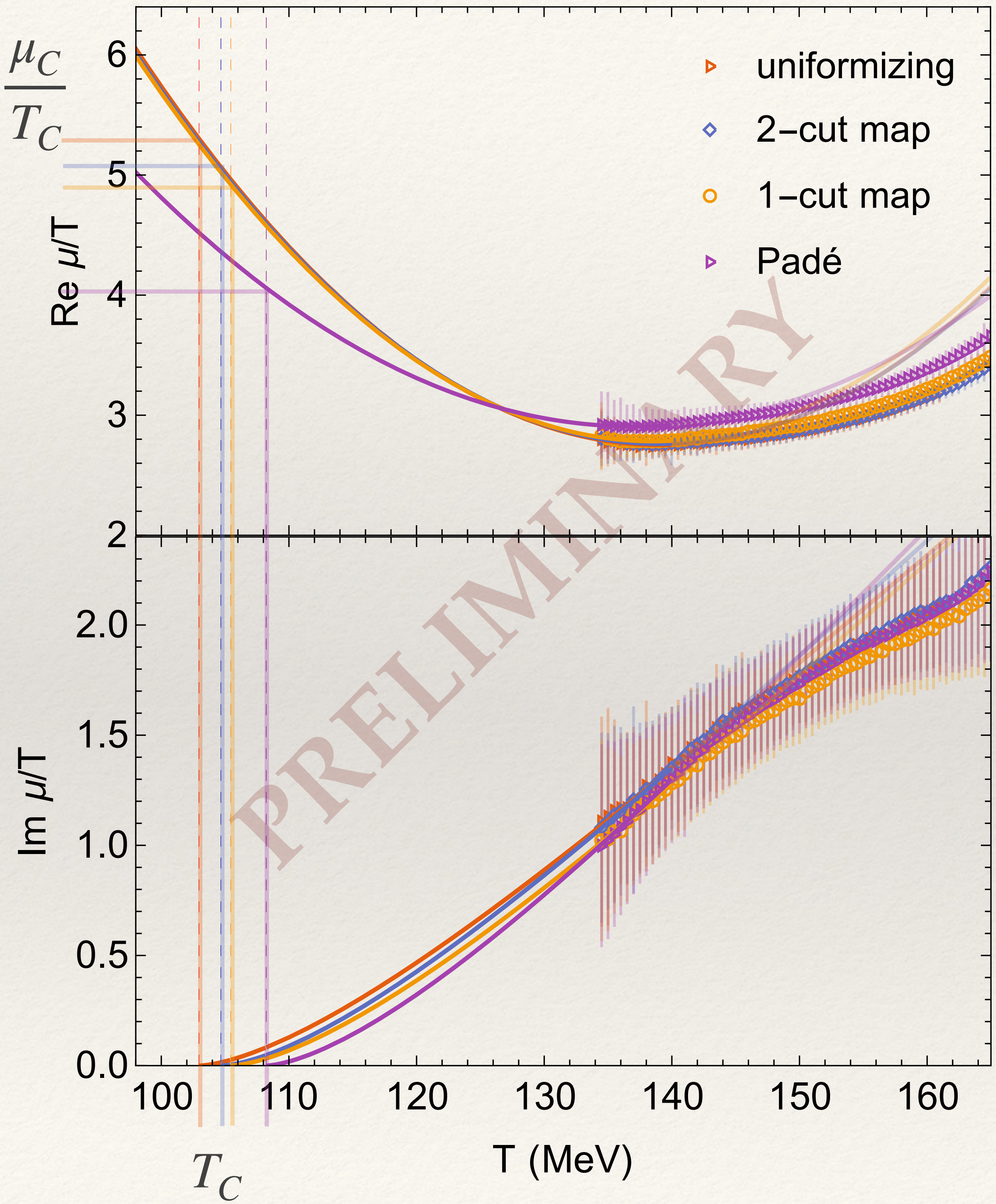
from conformal bootstrap [Simmons-Duffin, 1502.02033]

$w_c = |z_c|^{-\beta\delta} \approx 0.246$

from functional RG [Connelly et al, 2006.12541]

consistent with the HotQCD results [Bollweg et al. 2202.09184]

Estimations for QCD critical point



unif. Padé $T_C = 103\,\mathrm{MeV}$ $\mu_C = 546\,\mathrm{MeV}$ $\alpha_1 = 8.15^\circ$ $c = 2.70$

2-cut conf. Padé $T_C = 105\,\mathrm{MeV}$ $\mu_C = 528\,\mathrm{MeV}$ $\alpha_1 = 6.24^\circ$ $c = 2.92$

1-cut conf. Padé $T_C = 105\,\mathrm{MeV}$ $\mu_C = 519\,\mathrm{MeV}$ $\alpha_1 = 6.43^\circ$ $c = 2.85$

Padé $T_C = 108\,\mathrm{MeV}$ $\mu_C = 495\,\mathrm{MeV}$ $\alpha_1 = 3.32^\circ$ $c = 3.30$

consistent with the Bielefeld-Parma results
$T_C \sim 90\,\mathrm{MeV}$ $\mu_C \sim 600\,\mathrm{MeV}$
[Lattice '23 talk by D. Clarke]

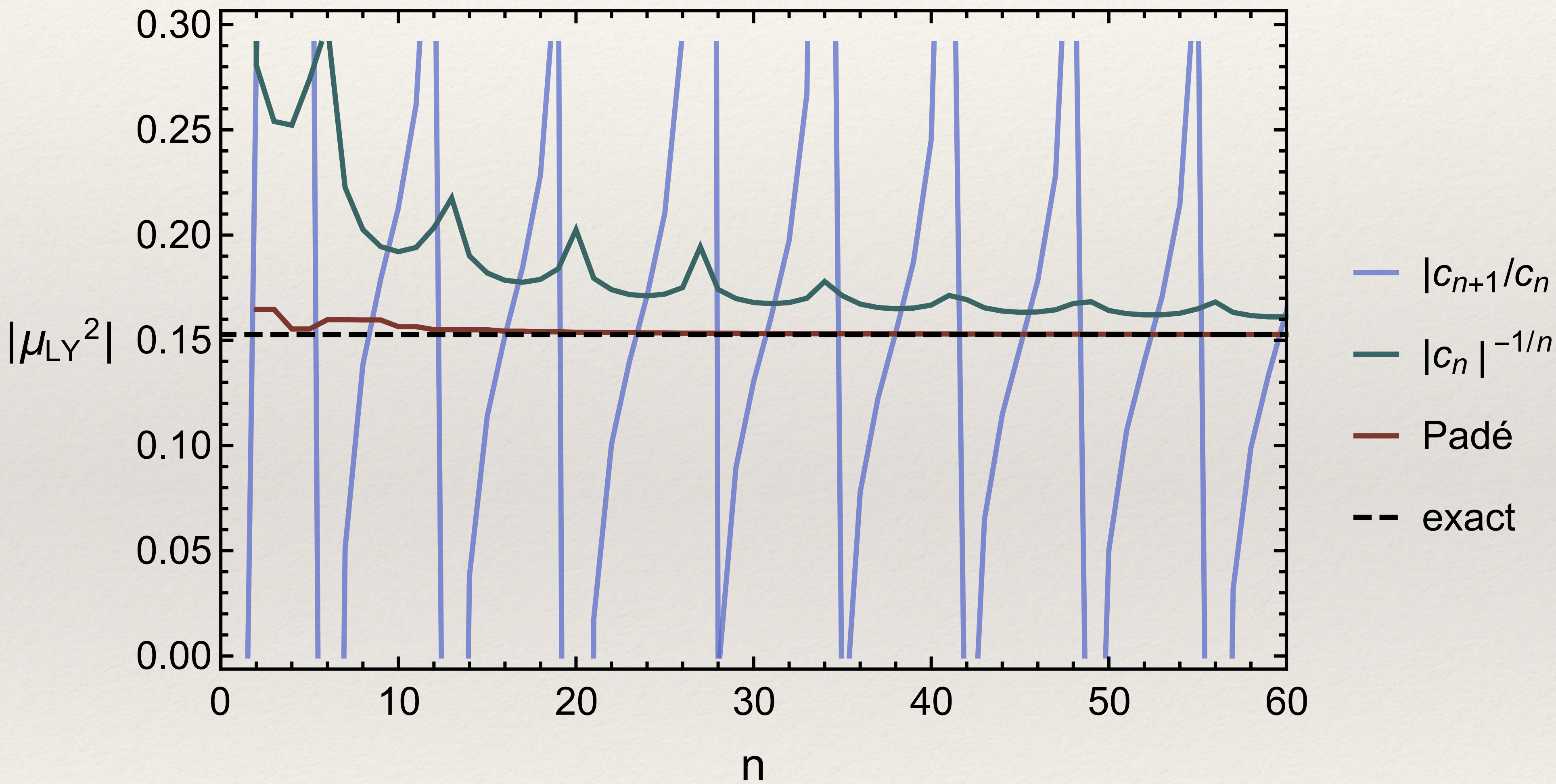
Conclusions and Outlook

- Combined with conformal maps, Padé approximants provide a powerful tool to extract information from truncated Taylor series.
- In the crossover region by using this tool it is possible to pin down the location of the *Lee-Yang edge singularity* and also extract information on the *mapping parameters to critical Ising e.o.s.*
- Conformal Padé gives a significantly better approximation to the e.o.s than than the Taylor series.
- Illustration in Gross-Neveu model, preliminary results for QCD.
- Better error analysis
- Extrapolation from imaginary μ, pairing with other resummation schemes [e.g. Borsanyi et al (2102.06660), Mukherjee et al (2110.02241, 2106.03165), Bielefeld-Parma, '21 (2110.15933), …]
- Singularity elimination

EXTRAS

When life gives you Taylor series...

Random Matrix Model

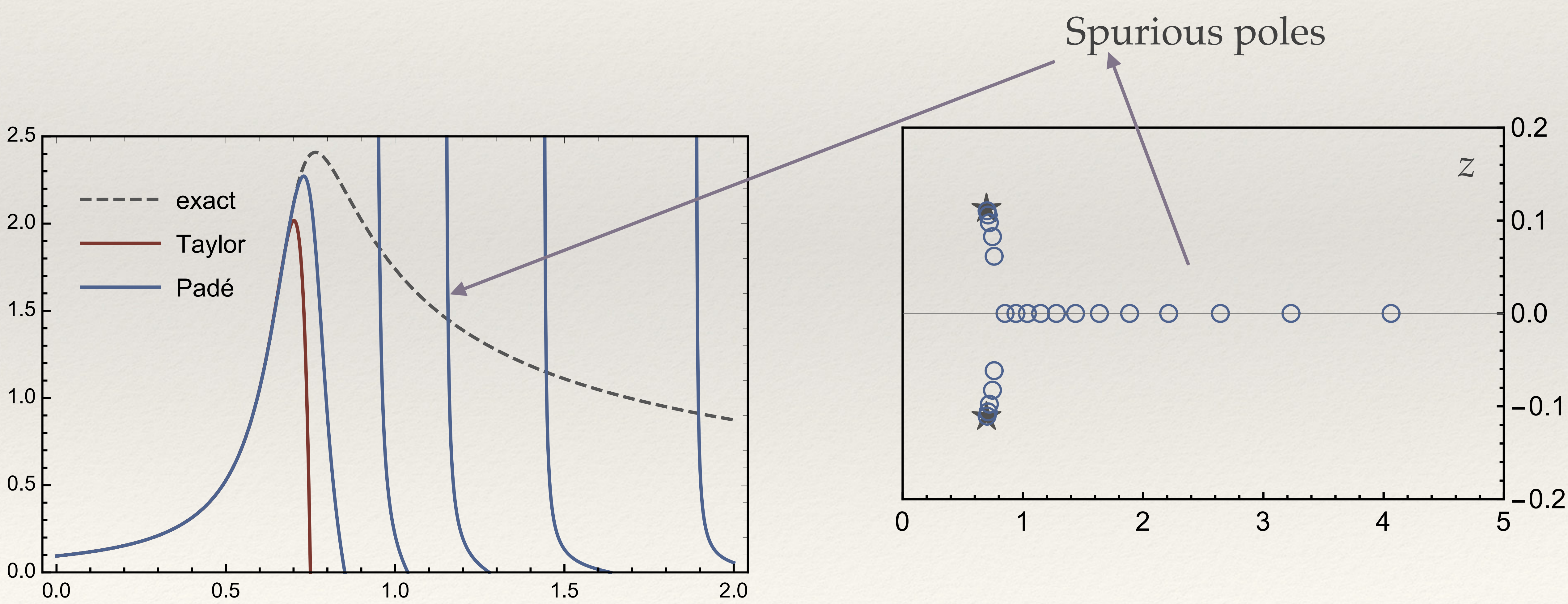


[GB, Dunne, Yin, arXiv: 2112.14269]

When life gives you Taylor series…

Spurious poles are unavoidable in Padé when there are conjugate pair of singularities…

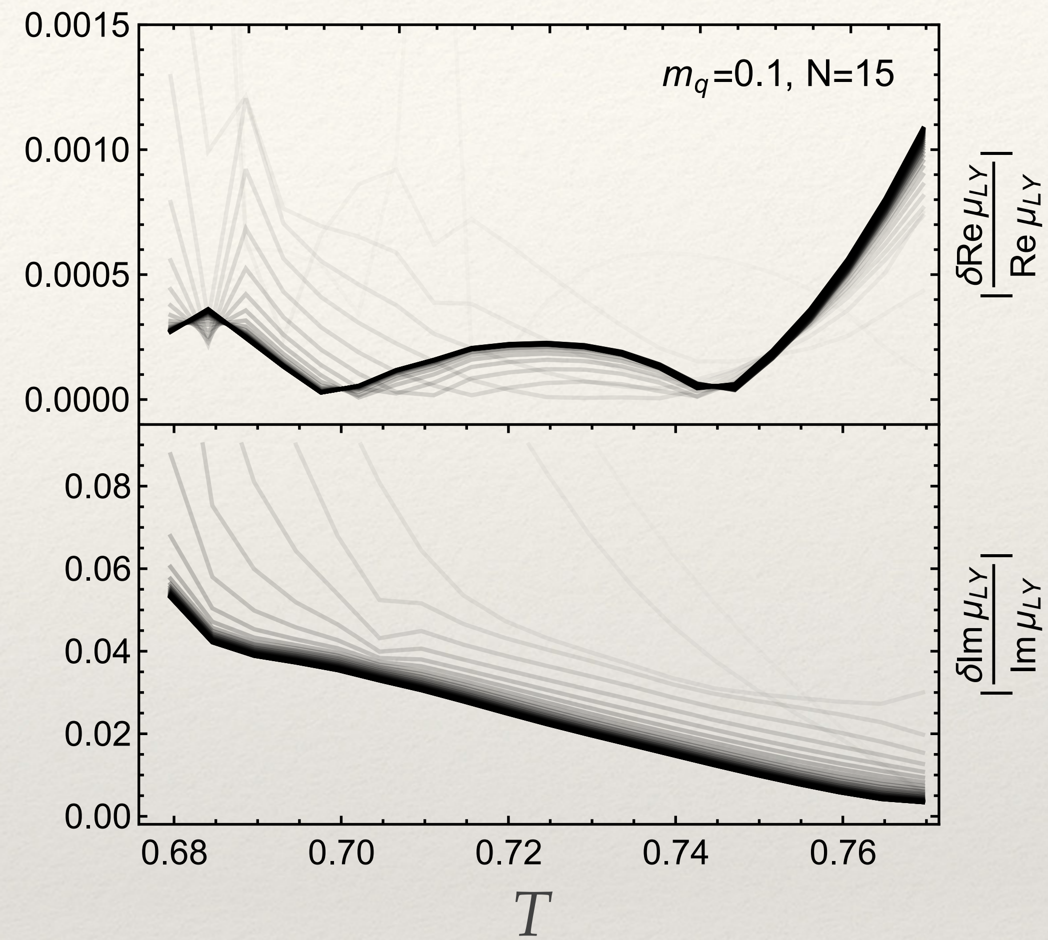
$$\text{e.g.}\quad f(z) = \frac{1}{2}\left(\frac{1}{\sqrt{z - z_c}} + \frac{1}{\sqrt{z - z_c^*}}\right)$$

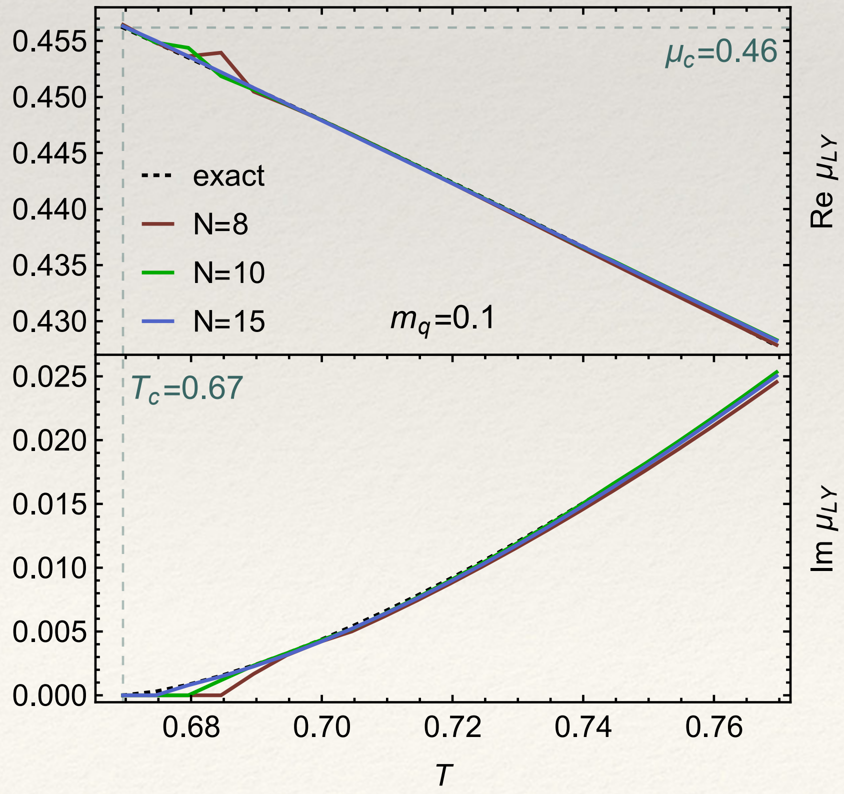


Iterative Algorithm

1. Estimate μ_{LY}^2 from Padé
2. Plug this value into the conformal map
3. Extract μ_{LY}^2 from conformal Padé
4. Plug the new value into the conformal map
5. Repeat.

$$\phi(\zeta) = \left(\frac{\theta}{\pi}\right)^{\theta/\pi} \left(1 - \frac{\theta}{\pi}\right)^{1-\theta/\pi} \frac{4\mu_{LY}^2\zeta}{(1+\zeta)^2} \left(\frac{1+\zeta}{1-\zeta}\right)^{2\theta/\pi}$$

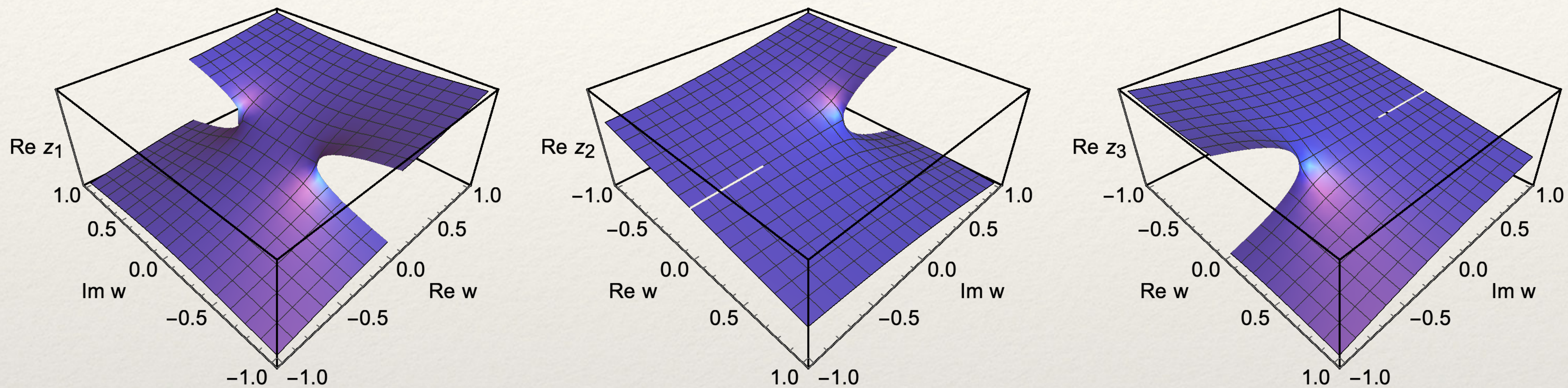




[GB, Dunne, Yin, arXiv: 2112.14269]

Uniformization

$$w = F(z) = z + z^3 \quad \text{(mean field)}$$



$$z_1(w) = -\frac{2i}{\sqrt{3}}\left[{}_2F_1\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}; \frac{1}{2}(1-iw)\right) - \text{c.c.}\right]$$

$$z_2(w) = \frac{2i}{\sqrt{3}}\,{}_2F_1\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}; \frac{1}{2}(1-iw)\right)$$

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \quad \theta_2(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i\tau(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i\tau n^2}$$

$$w(\tau) = i(-1 + 2\lambda(\tau))$$

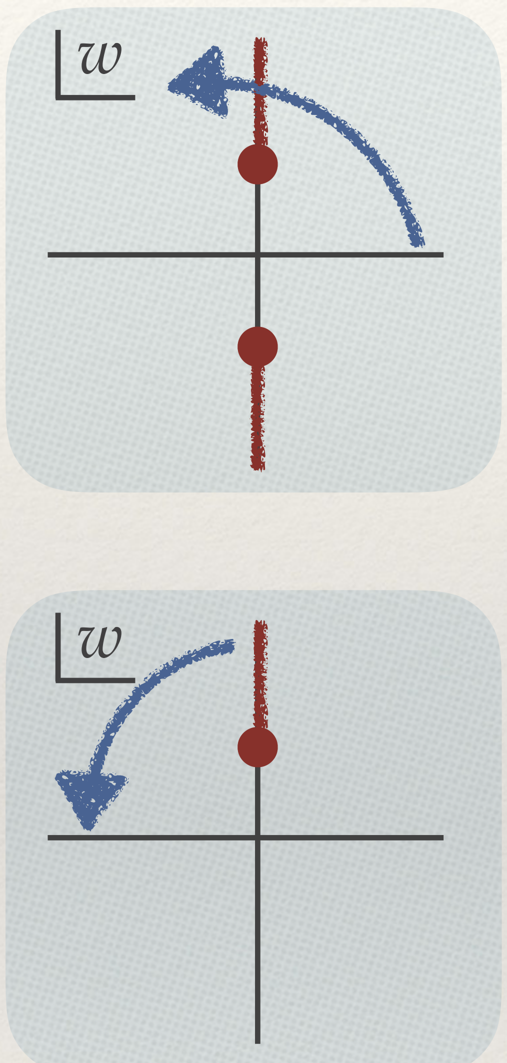
$z(\tau)$: single valued

"uniformization"

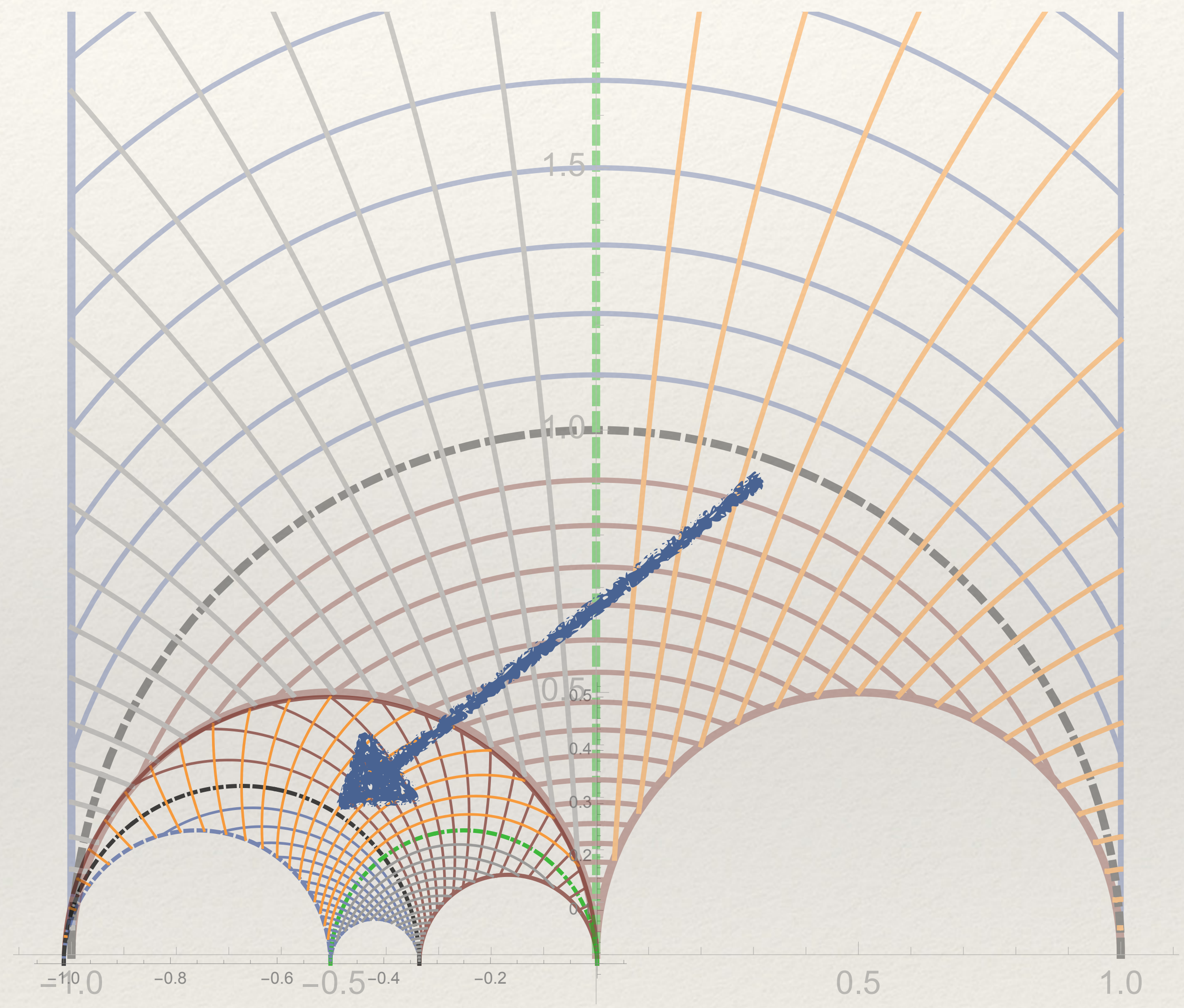
[Bateman, Higher Transcendal Functions I]

$^* w \to 2/(3\sqrt{3})w$

Uniformization



Jumping sheets
in w plane



Smooth in
τ plane

Interactive realization:
https://people.math.osu.edu/costin.9/classes.html

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Uniformization: higher Riemann sheets