

# Non-Hydrodynamic Modes from Linear Response in Kinetic Theory

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Based on SO, Sören Schlichting, [arXiv:2308.04491](https://arxiv.org/abs/2308.04491)



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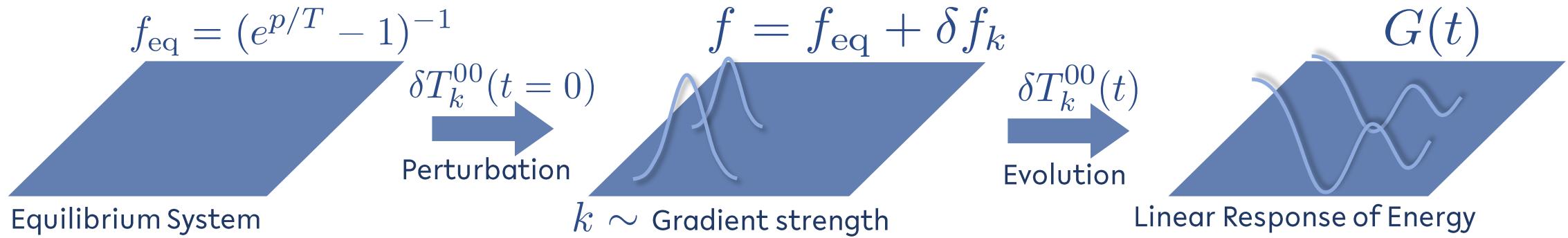


# The Initial Problem

- Heavy-Ion Collisions create Quark-Gluon-Plasma which is initially out of equilibrium and subject to large gradients
- Equilibration allows hydrodynamic description, if system is sufficiently long lived and gradients are sufficiently small
- Time scale of applicability and range of validity for large gradients are still discussed

# Our approach

- Study applicability of hydrodynamics by calculating linear response of a system to an initial energy perturbation in kinetic theory



- Quantified in terms of (retarded) Green's functions of the energy-momentum tensor

$$G(t) = \frac{\delta T_k^{00}(t)}{\delta T_k^{00}(0)}$$

# Hydrodynamic response

- Green's function in first order hydro

$$G(t) = \cos(c_s k t) e^{-\frac{2}{3} \frac{\eta}{sT} k^2 t}$$

- Universal form of response when expressed in scaling variables

$$\bar{k} = \frac{\eta}{sT} k \quad \bar{t} = \frac{sT}{\eta} t$$

$$G(t) = \cos(c_s \bar{k} \bar{t}) e^{-\frac{2}{3} \bar{k}^2 \bar{t}}$$

# Hydro and Non-Hydro Modes

- Hydrodynamic theories describe response in terms of a c.c. pair of sound modes with dispersion relations, e.g.

$$\bar{\omega}_{\text{sound}} = \pm c_s \bar{k} - i \frac{2}{3} \bar{k}^2$$

- Energy response determined by modes

$$G(t) = e^{-i\bar{\omega}_{\text{sound}}\bar{t}} = \cos(c_s \bar{k} \bar{t}) e^{-\frac{2}{3} \bar{k}^2 \bar{t}}$$

- Real part → Phase velocity                      **Imaginary part → Damping rate**
- Beyond first order Hydro, microscopic theories feature additional non-hydrodynamic modes

Higher order hydro : [\[Israel, Stewart, Annals Phys. 118 \(1979\) 341\]](#)

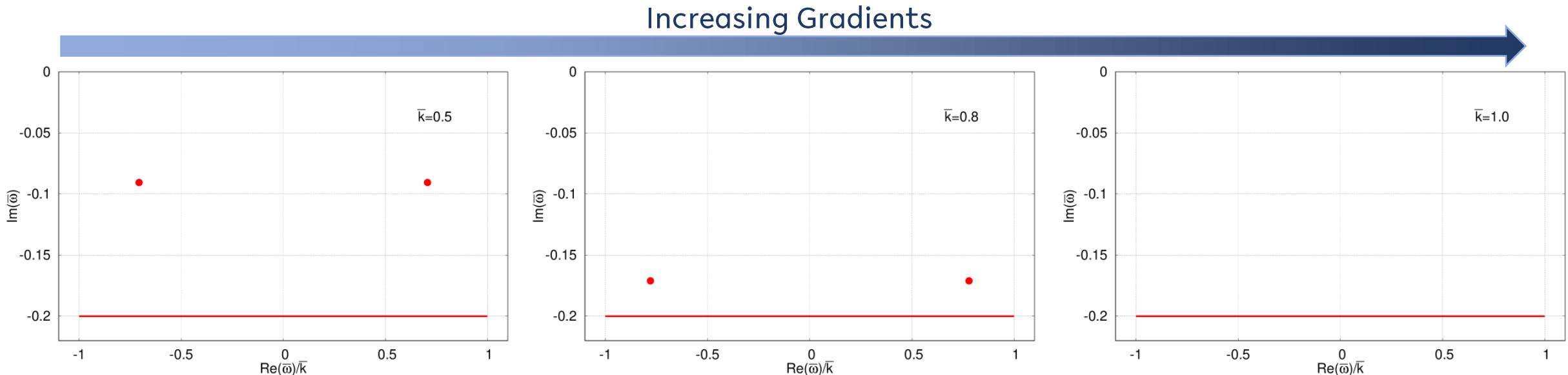
Kinetic theories: [\[Romatschke, Eur. Phys. J. C 76 \(2016\) 352\]](#)

AdS/CFT: [\[Grozdanov, Kaplis, Starinets, JHEP 1607 \(2016\) 151\]](#)

- Influence of non-hydro modes gives insight on validity of hydro

# What Could Non-Hydro Modes look like?

- Analytic calculation possible in Relaxation Time Approximation



- Non-Hydro has stronger decay
- Non-Hydro is a cut instead of singular poles
- Larger gradients let hydro modes vanish

[\[Romatschke, Eur.Phys.J.C 76 \(2016\) 352\]](#)

# How do we analyze modes?

- Use effective kinetic theory for equilibrium background and calculate linear response to perturbations

$$(\partial_t + ik \cos \theta) \delta f_k(\mathbf{p}, t) = C[\delta f_k](\mathbf{p}, t) \quad \text{Linearized Boltzmann Eq.}$$

- $C$  contains underlying microscopic theory
- Find all possible modes  $\omega_i(k)$ , compare their residues  $\mu_i(k)$  (dependent on observable, we analyze energy perturbations)

$$\delta T_k^{\mu\nu}(t) = \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu p^\nu}{p^0} \delta f_k(\mathbf{p}, t) \quad \delta e_k(t) = \delta T_k^{00}(t) = \sum_i \mu_i(k) e^{-i\omega_i(k)t}$$

# How do we find all modes?

- Calculate Green's functions of energy and find singularities?

$$G(t) = \sum_i \mu_i(k) e^{-i\omega_i(k)t} \rightarrow G(\omega) = \sum_i \frac{-\mu_i(k)}{i\omega + \omega_i(k)}$$

- Analytical solutions are rarely an option for complex theories
- Calculate real time Green's functions numerically and Laplace transform? No! Does not work for more than one singularity
- Get creative!

# Eigenvalue Approach

- Eigenvalues of what exactly?

$$\begin{aligned}\partial_t f(t) &= Cf(t) \\ \Leftrightarrow f(t) &= e^{Ct} f(0)\end{aligned}$$

- Solution given by eigenvalues of linear operator  $C$
- Numerically: Operator  $\rightarrow$  Finite dimensional matrix  
Distribution function  $\rightarrow$  Finite dimensional vector
- We will translate Boltzmann equation into matrix vector equation

# Eigenvalue Approach

- Build moments of distribution function on a grid of momenta and angles [\[York, Kunkela, Lu, Moore, Phys.Rev.D 89, 074036 \(2014\)\]](#)

$$\delta f_k(\mathbf{p}, t) \rightarrow \vec{N}(t) \quad N_i(t) = \int \frac{d^3 p}{(2\pi)^3} w_{i_p}(p) w_{i_\theta}(\cos \theta) \delta f_k(\mathbf{p}, t)$$

- Do the same for collision integral

$$C_i(t) = \int \frac{d^3 p}{(2\pi)^3} w_{i_p}(p) w_{i_\theta}(\cos \theta) C[\delta f_k](\mathbf{p}, t)$$

- Calculate matrix as functional derivative

$$C_i(t) = \sum_j C_{ij} N_j(t) = \left( C \vec{N}(t) \right)_i \Rightarrow C_{ij} = \frac{\delta C_i(t)}{\delta N_j(t)}$$

# Eigenvalue Approach

- Solution to Boltzmann Equation in moment space

$$\vec{N}(t) = e^{Ct} \vec{N}(0)$$

- Diagonalization of C

$$C = (\vec{a}_1 \dots \vec{a}_N) \text{diag}(\lambda_1, \dots, \lambda_N) (\vec{b}_1 \dots \vec{b}_N)^T$$

left/right eigenvectors

- Calculate response functions for energy

$$G(t) = \vec{O} \cdot \vec{N}(t) = \sum_i e^{\lambda_i t} (\vec{O} \cdot \vec{a}_i) (\vec{b}_i \cdot \vec{N}(0))$$

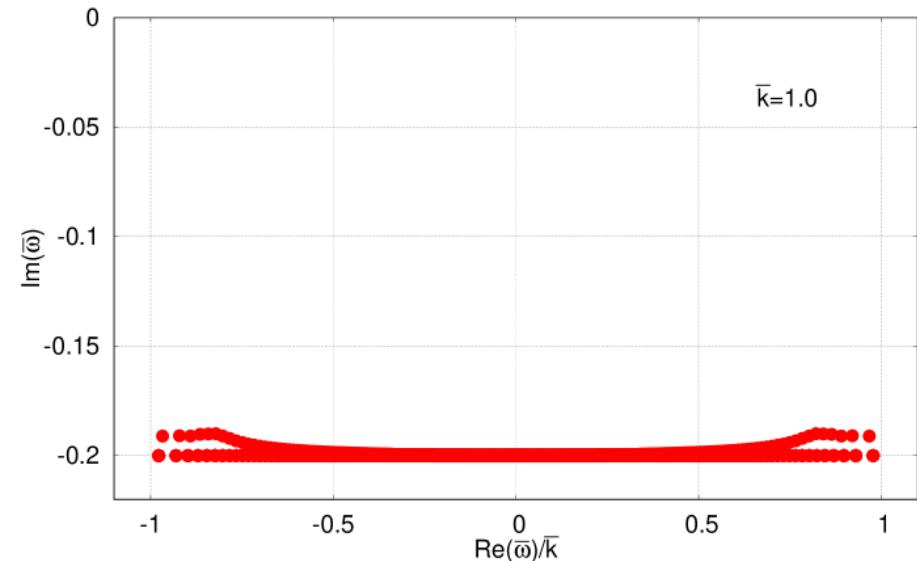
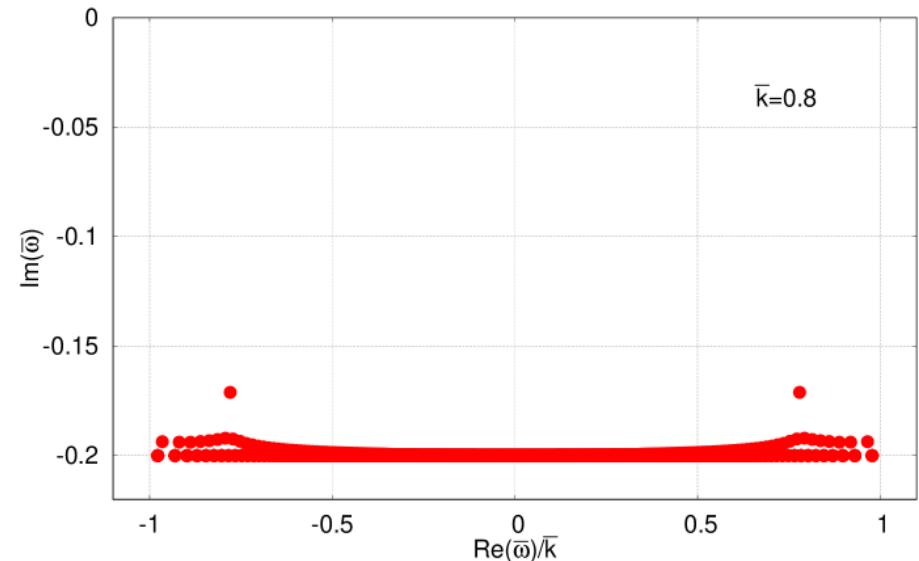
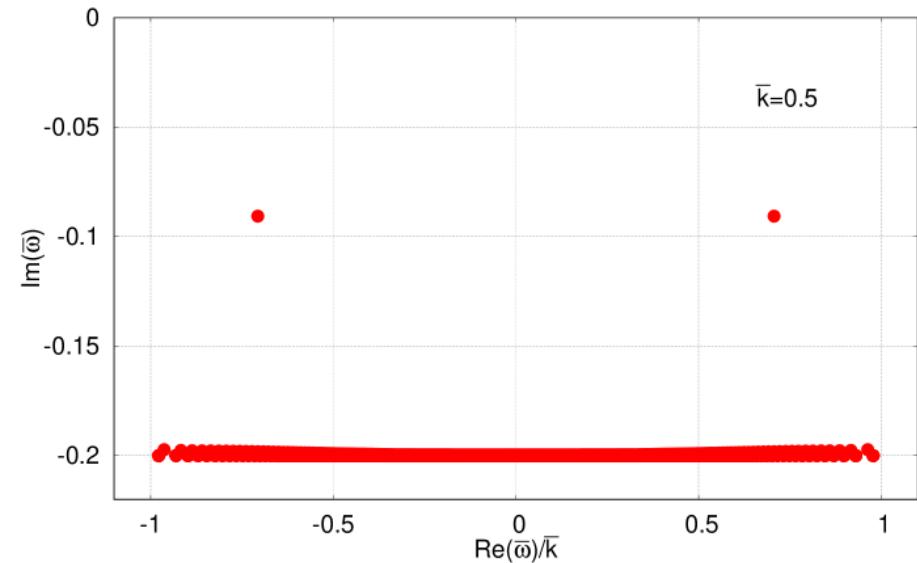
$$\vec{O} = \begin{pmatrix} \vdots \\ p_{i_p} \\ \vdots \end{pmatrix}$$

- Easy translation into frequencies

$$G(\omega) = \sum_i \frac{-\mu_i}{i\omega + \lambda_i}$$

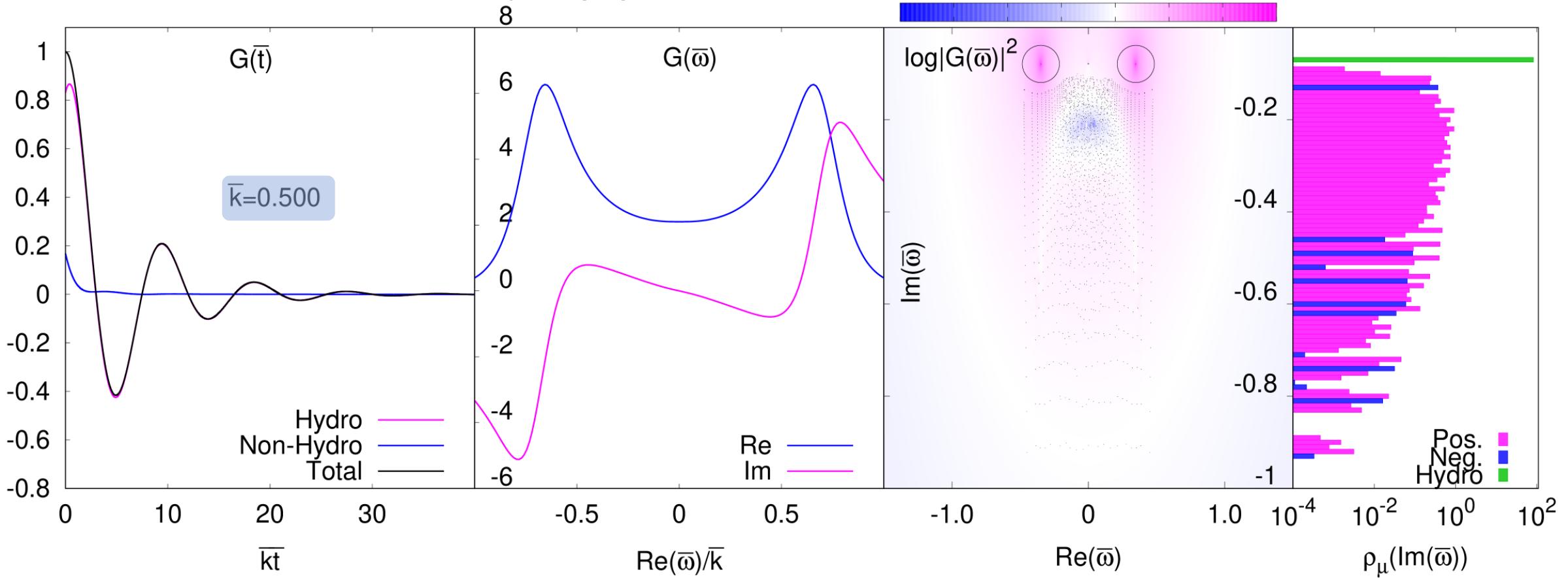
# Benchmark method

- Numerical test with Relaxation Time Approximation
- Cut realized by line of eigenvalues



# Scalar $\phi^4$ Theory

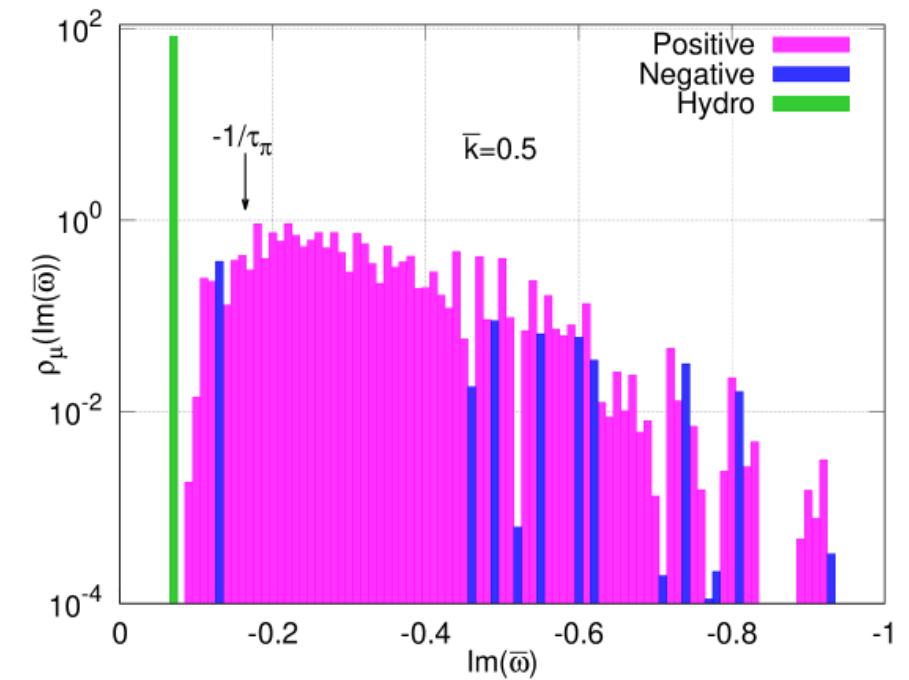
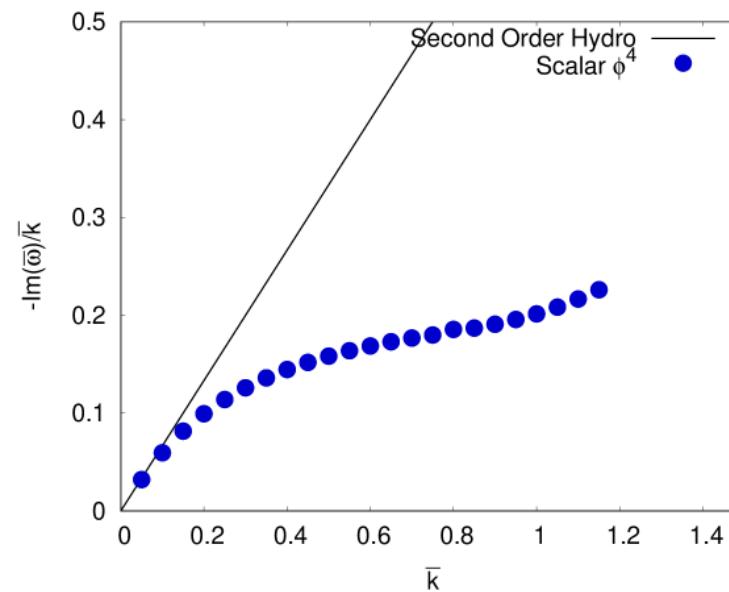
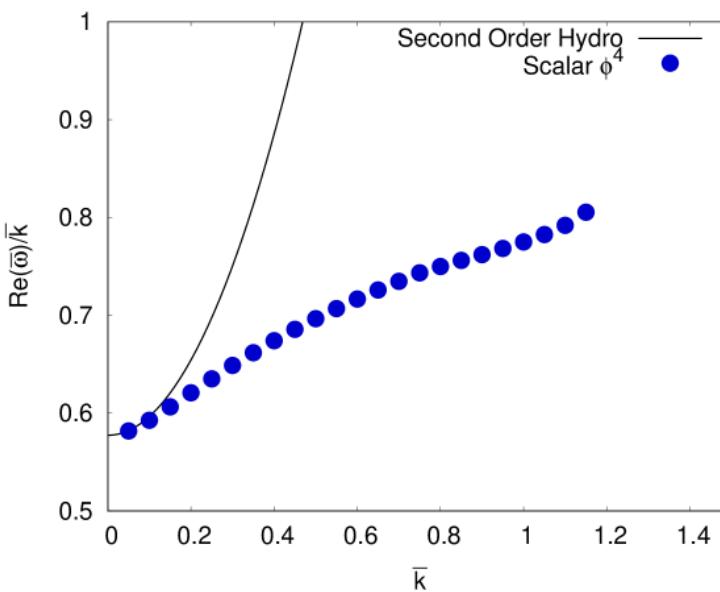
- New results for varying gradients



- Defining Hydro Mode is hard, we use two largest residue eigenvalues

# Scalar $\phi^4$ Theory

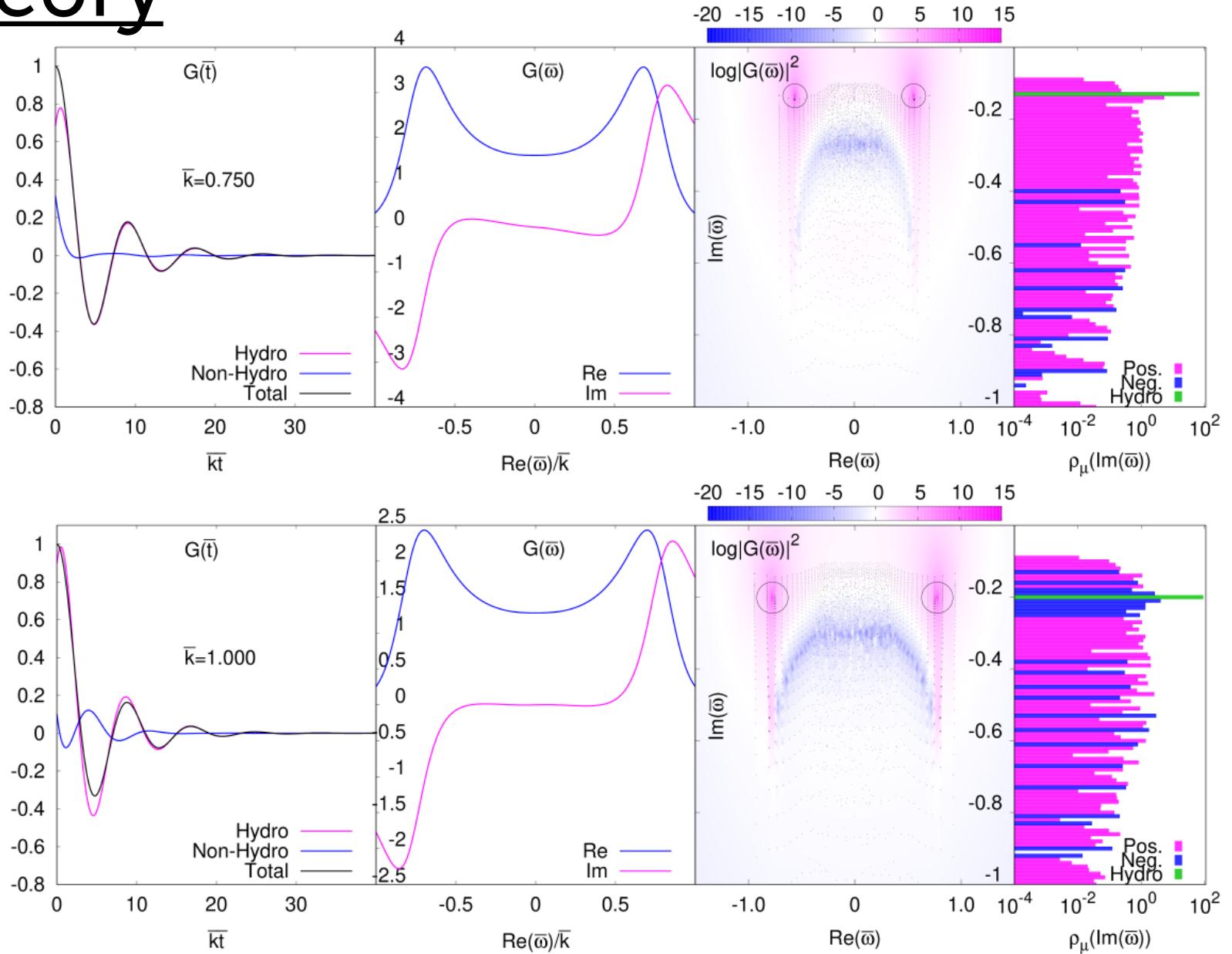
- Hydro mode dominates at small  $\bar{k}$  and satisfies usual dispersion relation up to  $\bar{k} \sim 0.15$



- Continuum of non-hydrodynamic modes
- Continuous distribution of decay rates
  - extends arbitrarily close to real frequency axis but with exponentially small residues
  - distribution for small  $\bar{k}$  peaked around effective relaxation time

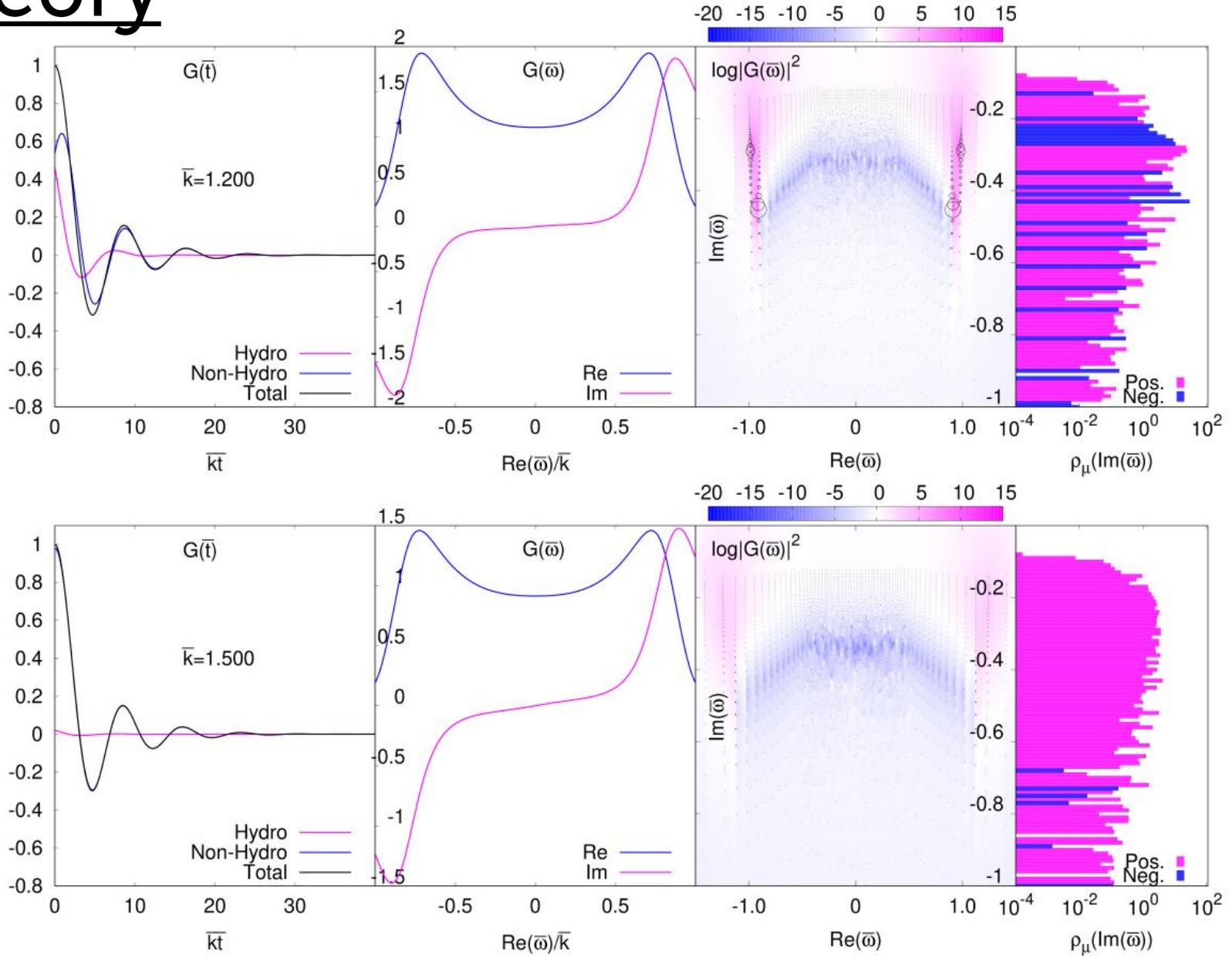
# Scalar $\phi^4$ Theory

- Hydro poles move down in imaginary direction ( $\rightarrow$  faster decay)
- Hydro poles merge with non-Hydro area at  $\bar{k} \sim 1$  ( $\rightarrow$  Breakdown of hydro)



# Scalar $\phi^4$ Theory

- No clear distinction of Hydro modes possible
- Continuum of contributions with different decay rates



# Conclusions

- Understanding non-Hydro modes helps finding the range of applicability of Hydro
- Non-Hydro modes in scalar theory are continuously distributed (more complicated than poles or cuts)
- Hydrodynamic modes merge with non-hydrodynamic area at around  $\bar{k} \sim 1$   
Breakdown of Hydro for length scale  $1/k \sim 0.158 \text{ fm} \left( \frac{200 \text{ MeV}}{T} \right) \left( \frac{\eta/s}{0.16} \right)$
- Work in progress: Extend formalism to QCD kinetic theory,  
also see poster by Xiaojian Du, SO, Sören Schlichting: [arXiv:2306.09094](https://arxiv.org/abs/2306.09094)

# Backup

# Back Transformation from Moments

- One might has to transform back to particle distribution function

$$N_i(t) = \int \frac{d^3 p}{(2\pi)^3} w_{i_p}(p) w_{i_\theta}(\cos \theta) \delta f_k(\mathbf{p}, t) \frac{f_{\text{eq}}(p) e^{p/T}}{f_{\text{eq}}(p) e^{p/T}} = \frac{\delta f_k(\mathbf{p}, t)}{f_{\text{eq}}(p) e^{p/T}} A_i$$

- Gives distribution around node points
- Coefficients  $A$  are calculated and stored beforehand

$$A_i = \int \frac{d^3 p}{(2\pi)^3} w_{i_p}(p) w_{i_\theta}(\cos \theta) f_{\text{eq}}(p) e^{p/T}$$

- Sum every node point

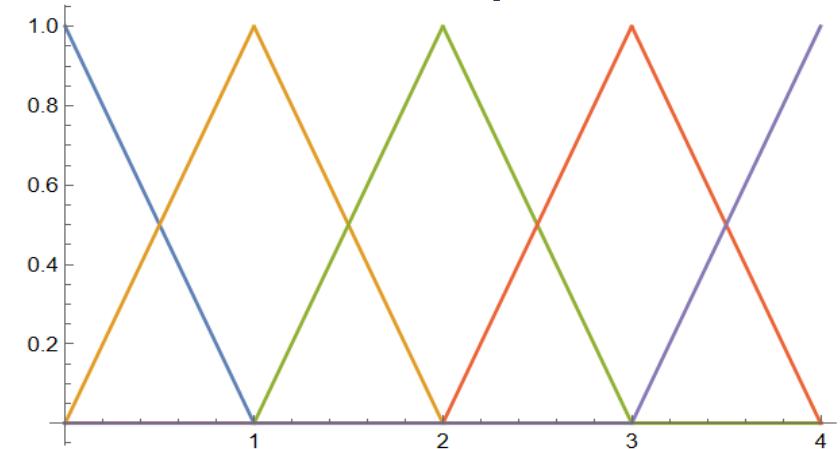
$$\delta f_k(\mathbf{p}, t) = \sum_i w_{i_p}(p) w_{i_\theta}(\cos \theta) N_i(t) \frac{f_{\text{eq}}(p) e^{p/T}}{A_i} = \sum_i K_i(\mathbf{p}) N_i(t)$$

# Wedge Moments

[York, Kurkela, Lu, Moore, PhysRevD (2014)]

- Moments are linear functions, ‘wedges’ around node points

$$w_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \\ 0, & \text{else} \end{cases}$$



- Wedge moments ensure energy conservation

$$\sum_i w_i(x) = \theta(x_{max} - x)\theta(x - x_{min})$$

$$\sum_i x_i w_i(x) = x\theta(x_{max} - x)\theta(x - x_{min})$$

$$\delta n_k(t) = \sum_i N_i(t), \quad \delta e_k(t) = \sum_i p_{i_p} N_i(t), \quad \delta \pi_k(t) = \sum_i p_{i_p} \cos \theta_{i_\theta} N_i(t)$$

# Non-Zero Gradients

- Gradient contributions also expanded into moment matrix

$$(\partial_t + ik \cos \theta) \delta f_k(\mathbf{p}, t) = C[\delta f_k](\mathbf{p}, t) \rightarrow \partial_t \vec{N}(t) = (C + M) \vec{N}(t)$$

$$M_i = -ik \int \frac{d^3 p}{(2\pi)^3} \cos(\theta) w_{i_p}(p) w_{i_\theta}(\cos(\theta)) \delta f_k(\mathbf{p}, t)$$

$$\vec{N}(t) = e^{(C+M)t} \vec{N}(0)$$

# Scalar Theory

- Scalar field Lagrangian

$$\mathcal{L}[\phi, \partial_\mu \phi] = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{24} \phi^4$$

- Collision integral

$$C[f](\mathbf{p}_1, t) = -\frac{1}{2p_1} \frac{1}{2} \int \frac{d^3 p_2 d^3 p_3 d^3 p_4}{(2\pi)^9 2p_2 2p_3 2p_4} (2\pi)^4 \delta^4(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \lambda^2 \\ \times (f(\mathbf{p}_1)f(\mathbf{p}_2)[1 + f(\mathbf{p}_3)][1 + f(\mathbf{p}_4)] - f(\mathbf{p}_3)f(\mathbf{p}_4)[1 + f(\mathbf{p}_1)][1 + f(\mathbf{p}_2)])$$

$$C_{ij} = \frac{\delta C_i}{\delta N_j} = \frac{1}{2} \frac{\lambda^2}{4} \int d\Omega^{2 \leftrightarrow 2} [w_i(p_1, \cos \theta_1) + w_i(p_2, \cos \theta_2) - w_i(p_3, \cos \theta_3) - w_i(p_4, \cos \theta_4)] \\ \times [K_j(p_1)(n_3 n_4 - n_2(1 + n_3 + n_4)) + K_j(p_2)(n_3 n_4 - n_1(1 + n_3 + n_4)) \\ - K_j(p_3)(n_1 n_2 - n_4(1 + n_1 + n_2)) - K_j(p_4)(n_1 n_2 - n_3(1 + n_1 + n_2))]$$

$$n_i = f_{\text{eq}}(p_i)$$