Applicability of higher-order hydrodynamics in heavy-ion collisions

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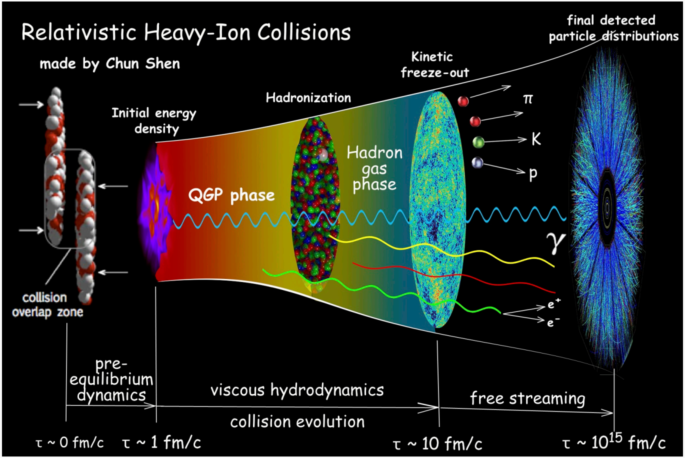
Heavy-ion collision

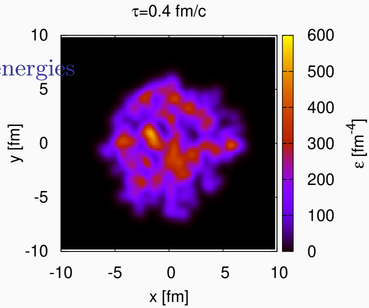


Multistage simulations of heavy-ion collisions based on hydrodynamic models explains observed data. However..

Hydrodynamic simulation of HIC

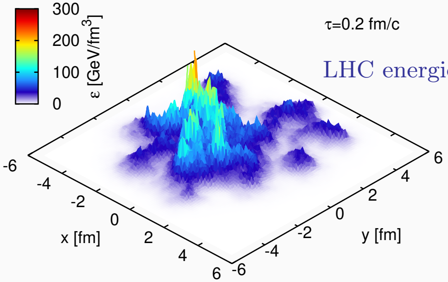
Hydrodynamics is applied in regime of large gradients.

RHIC energies



Schenke, Jeon, Gale, PRL **106** (2011), 042301

LHC energies



Schenke, Tribedy, Venugopalan, PRL **108** (2012), 252301

Simulations like these explains data ("unreasonable effectiveness of hydrodynamics")

$\Longrightarrow$ Nearly thermalized medium formed at $\tau \lesssim 1$ fm/c (?)

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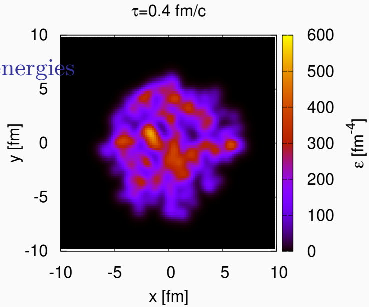

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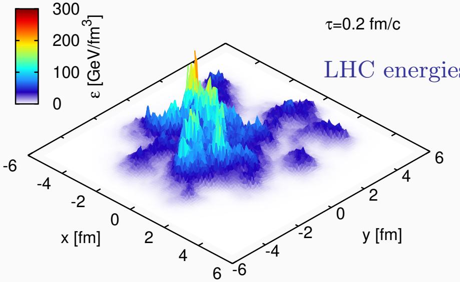

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What is the domain of hydrodynamics?

- Usual picture: Requires system to be close to local equilibrium.
  - Microscopic degrees of freedom relax quickly towards local equilibrium. Long wavelength modes, associated to conservation laws, relax on longer time scales. Separation of scales: $\lambda_{\rm mfp} \ll L \sim \{\partial T,\, \partial\mu,\, \partial u^\mu\} \ll 1$.

- Viscous hydrodynamics is formulated as an expansion in gradients of the equilibrium fields $(T,\, \mu,\, u^\mu)$.

$$T^{\mu\nu} = T^{\mu\nu}_{\rm ideal} + \Pi^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}$$

  - Landau frame choice: $T^{\mu\nu}u_\nu = \epsilon u^\mu, \quad \epsilon = \epsilon_{\rm eq}$. $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$. Vanishing chemical potential – no net conserved charge.

- $1^{\rm st}$ order hydrodynamics: Navier-Stokes:
  Eckart, Phys. Rev.58 (1940), Landau and Lifshitz, "Fluid mechanics" (1987)

$$\pi^{\mu\nu} = \eta\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha\right) = 2\,\eta\,\sigma^{\mu\nu}, \qquad \Pi = -\zeta\,\partial_\mu u^\mu.$$

  - However, Navier-Stokes eqs. imposes instantaneous response of dissipative fluxes to dissipative forces – Acausal + Instabilities! Hiscock and Lindblom (1983, 1985)

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Müller-Israel-Stewart theory

Müller, Z. Phys, 198, 329 (1967), Israel and Stewart, Ann. Phys. 100, 310 (1976)

Phenomenological Israel-Stewart theory: causal and stable theory

- Starting point:

$$S^\mu \equiv S^\mu\left(T, \mu, u^\mu, N^\mu, T^{\mu\nu}\right) \equiv S^\mu\left(T, \mu, u^\mu, \Pi, \pi^{\mu\nu}, V^\mu\right)$$

Here, N^μ is conserved current, V^μ is particle diffusion current.

- Expand S^μ in powers of the dissipative currents around a fictitious equilibrium state

$$S^\mu = \frac{P}{T}u^\mu + \frac{1}{T}u_\nu T^{\mu\nu} - \frac{\mu}{T}N^\mu - X^\mu\left(\delta N^\mu, \delta T^{\mu\nu}\right)$$

- Expanding X^μ to second-order

$$S^\mu = su^\mu - \frac{\mu}{T}V^\mu - \frac{u^\mu}{2}\left(\delta_0\Pi^2 - \delta_1 V_\alpha V^\alpha + \delta_2\pi_{\alpha\beta}\pi^{\alpha\beta}\right) - \gamma_0\Pi V^\mu - \gamma_1\pi^\mu_\nu V^\nu + \mathcal{O}(\delta^3)$$

- Demand entropy divergence is positive

$$
\begin{aligned}
\partial_\mu S^\mu = \tfrac{\Pi}{T} & \underbrace{\left(-\theta - T\delta_0\dot{\Pi} - \frac{T}{2}\Pi\dot{\delta}_0 - \frac{T}{2}\delta_0\Pi\theta - T\gamma_0\partial_\mu V^\mu - T(1-r)V^\mu\nabla_\mu\gamma_0\right)}_{\Omega_\Pi \Pi} \\
& + V_\mu \underbrace{\left(-\nabla^\mu\left(\frac{\mu}{T}\right) + \delta_1\dot{V}^{\langle\mu\rangle} + \frac{V^\mu}{2}\dot{\delta}_1 + \frac{\delta_1}{2}V^\mu\theta - \gamma_0\nabla^\mu\Pi - r\Pi\nabla^\mu\gamma_0 - \gamma_1\partial_\nu\pi^{\mu\nu} - y\pi^{\mu\nu}\nabla_\nu\gamma_1\right)}_{-\Omega_V V^\mu} \\
& + \tfrac{\pi^{\mu\nu}}{T} \underbrace{\left(\sigma^{\mu\nu} - T\delta_2\dot{\pi}^{\langle\mu\nu\rangle} - \frac{T}{2}\pi^{\mu\nu}\dot{\delta}_2 - \frac{T}{2}\delta_2\pi^{\mu\nu}\theta - T\gamma_1\nabla^{\mu\langle}V^{\rangle\nu} - T(1-y)V^{\langle\mu}\nabla^{\nu\rangle}\gamma_1\right)}_{\Omega_\pi \pi_{\mu\nu}}
\end{aligned}
$$

Here, $\Omega_\Pi\,, \Omega_V\,, \Omega_\pi \geq 0$. Co-moving derivative $\dot{A} \equiv u^\mu\partial_\mu A$.

- Relaxation type equations for dissipative stresses

$$
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\Omega_\pi}{T\delta_2}\pi^{\mu\nu} = \frac{1}{T\delta_2}\sigma^{\mu\nu} + \cdots
$$

- Causal and stable. Π, $\pi^{\mu\nu}$, V^μ : new fields → dynamical degrees of freedom.
  Many variants of this theory: second-order hydro, aHydro, vaHydro, ME-Hydro, ..
  – Israel-Stewart-Like (ISL) hydro.

  Used in hydrodynamic simulations of heavy-ion.

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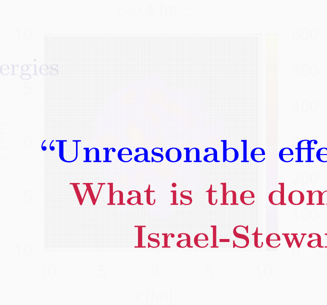

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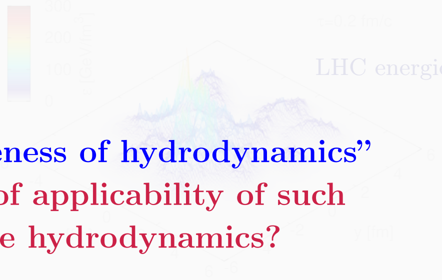

“Unreasonable effectiveness of hydrodynamics”

What is the domain of applicability of such Israel-Stewart-like hydrodynamics?

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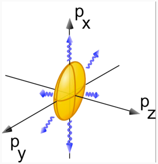
A simplified system

- Ultra-relativistic heavy-ion collisions admits a weakly coupled description of the matter at early times (assume).

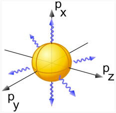


- The very fast logitudinal expansion of the matter tends to drive the momentum distribution to a very flat distribution.

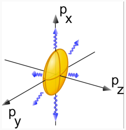
- Translates into the existence of two different pressures: longitudinal (P_L) and transverse (P_T).

- Approach to equilibrium: competition between

Collisions $\Rightarrow$



Expansion $\Rightarrow$



- Bjorken flow [J. D. Bjorken, PRD 27, 140 (1983)]: homogeneity in the transverse (x, y) plane, boost invariance along the z (beam) direction, and reflection symmetry $z \to -z$. Appropriate description of early-time dynamics.

Set of special moments of distribution function

- Non-conformal Boltzmann equation in RTA approx undergoing Bjorken expansion:
$$\left(\frac{\partial}{\partial\tau} - \frac{p_z}{\tau}\frac{\partial}{\partial p_z}\right) f(\tau,p) = -\frac{f(\tau,p) - f_{\rm eq}(p_0/T)}{\tau_R(\tau)}$$

- Consider the moments:
$$\mathcal{L}_n \equiv \int_p p_0^2\, P_{2n}(p_z/p_0)\, f(\tau,p), \qquad \mathcal{M}_n \equiv m^2 \int_p P_{2n}(p_z/p_0)\, f(\tau,p)$$
where $\int_p \equiv \frac{d^3p}{(2\pi)^3\, p_0}$ and P_{2n} is the Legendre polynomial of order $2n$.

Blaizot and Yan, PLB **780** (2018) SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

- Only three moments are hydro quantities: $(\mathcal{L}_0 = \varepsilon,\ \mathcal{L}_1,\ \mathcal{M}_0 = T^\mu_\mu)$
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$$\frac{\partial\mathcal{M}_n}{\partial\tau} = -\frac{1}{\tau}\left(a'_n\mathcal{M}_n + b'_n\mathcal{M}_{n-1} + c'_n\mathcal{M}_{n+1}\right) - \frac{(\mathcal{M}_n - \mathcal{M}_n^{\rm eq})}{\tau_R}$$

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Fixed point structure

- Equation of $\mathcal{L}_n$ moments are decoupled from $\mathcal{M}_n$ moments $\implies$ evolution of energy density ($\mathcal{L}_0$) does not depend on $\mathcal{M}_n$ evolution.

- Consider the quantity: $g_0 \equiv \frac{\tau}{\mathcal{L}_0}\frac{\partial \mathcal{L}_0}{\partial \tau}$. In the regimes where the energy density behave as power law, g_0 is the exponent in that power law.

- Define $\beta(g_0, w) \equiv w \frac{\mathrm{d}g_0}{\mathrm{d}w}$ where $w = \tau/\tau_R$. Equation for $\mathcal{L}_n$ becomes:

$$-\beta(g_0, w) = g_0^2 + g_0\left(a_0 + a_1 + w\right) + a_0 a_1 - c_0 b_1 + a_0 w - c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0} - \frac{c_0}{2} w \left(1 - 3\frac{P}{\epsilon}\right)$$

- Zeros of $\beta(g_0, w)$ gives fixed points.

- Free-streaming fixed points ($w \ll 1$):
  - Exact fixed point: $g_0 = -1$ (stable: $P_L = 0$) and $g_0 = -2$ (unstable).

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- Consider the quantity: $g_0 \equiv \frac{\tau}{\mathcal{L}_0}\frac{\partial \mathcal{L}_0}{\partial \tau}$. In the regimes where the energy density behave as power law, g_0 is the exponent in that power law.

- Define $\beta(g_0, w) \equiv w\frac{\mathrm{d}g_0}{\mathrm{d}w}$ where $w = \tau/\tau_R$. Equation for $\mathcal{L}_n$ becomes:
$$-\beta(g_0, w) = g_0^2 + g_0\left(a_0 + a_1 + w\right) + a_0 a_1 - c_0 b_1 + a_0 w - c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0} - \frac{c_0}{2} w \left(1 - 3\frac{P}{\epsilon}\right)$$

- Zeros of $\beta(g_0, w)$ gives fixed points.

- Free-streaming fixed points ($w \ll 1$):
  - Exact fixed point: $g_0 = -1$ (stable: $P_L = 0$) and $g_0 = -2$ (unstable).

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    $g_0 = -0.93$ (stable) and $g_0 = -2.21$ (unstable). Captures FP structure.

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- Hydrodynamic fixed point ($w \gg 1$) : $g_* = -1 - P/\epsilon$ (governed by EoS).

Three-moment truncation

- Equation of three moments:

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau}\left(a_0\mathcal{L}_0 + c_0\mathcal{L}_1\right), \qquad \frac{\partial \mathcal{L}_1}{\partial \tau} = -\frac{1}{\tau}\left(a_1\mathcal{L}_1 + b_1\mathcal{L}_0 + c_1\,\mathcal{L}_2\,\right) - \frac{\left(\mathcal{L}_1 - \mathcal{L}_1^{\mathrm{eq}}\right)}{\tau_R},$$

$$\frac{\partial \mathcal{M}_0}{\partial \tau} = -\frac{1}{\tau}\left(a_0'\mathcal{M}_0 + c_0'\,\mathcal{M}_1\,\right) - \frac{\left(\mathcal{M}_0 - \mathcal{M}_0^{\mathrm{eq}}\right)}{\tau_R}.$$

- Different truncation schemes for $\mathcal{L}_2$ and $\mathcal{M}_1$ leads to variants of ISL theory:

  Grad 14-moment truncation and Chapman-Enskog approx. $\rightarrow$ second-order hydro

  Denicol et.al., arXiv:1202.4551 (2012); Jaiswal, arXiv:1305.3480 (2013)

  Using Romatschke-Strickland form of distribution function $\rightarrow$ anisotropic hydro

  Romatschke, Strickland, Martinez, Heinz, Florkowski, Ryblewski, ...

  Using maximum entropy distribution $\rightarrow$ ME-hydro

  Chattopadhyay, Heinz and Schaefer, arXiv:2307.10769 (2023).

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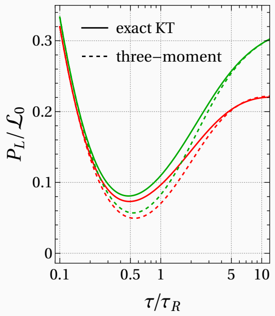
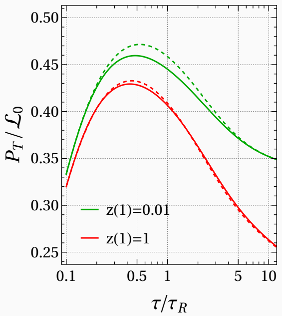
$$\frac{\partial \mathcal{M}_0}{\partial \tau} = -\frac{1}{\tau}\left(a_0'\mathcal{M}_0 + c_0'\cancel{\mathcal{M}_1}\right) - \frac{\left(\mathcal{M}_0 - \mathcal{M}_0^{\rm eq}\right)}{\tau_R}.$$

- Considering three lowest moments ($\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{M}_0$) is enough to approximately capture the exact evolution.





$z = m/T$

Isotropic IC

Constant τ_R

Second-order hydrodynamics from moments

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- Second-order hydro equations is obtained by expanding $\mathcal{L}_2$ and $\mathcal{M}_1$ till first-order in gradients. However, there are inherent ambiguities in definition of some second-order transport coefficients. SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022). SJ, Blaizot; in prep.

- Relaxation-type structure inherent in moments equations – necessary for causality and extending domain in free-streaming regime.

- Time derivative of $\mathcal{L}_1$ and $\mathcal{M}_0$, and correspondingly, $\pi \equiv -\frac{2}{3}\left(\mathcal{L}_1 + \frac{\mathcal{M}_0}{2}\right)$ and $\Pi \equiv \left(\mathcal{L}_0 - 3P - \mathcal{M}_0\right)/3$ in ISL hydro, captures approximately some of the features of the collisionless regime.

Illustration ⇒

Second-order hydrodynamics from moments

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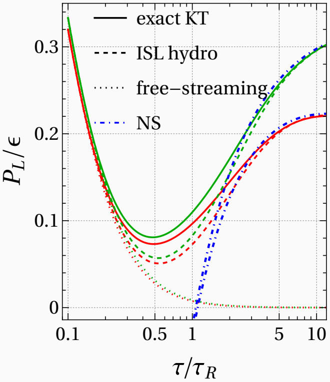
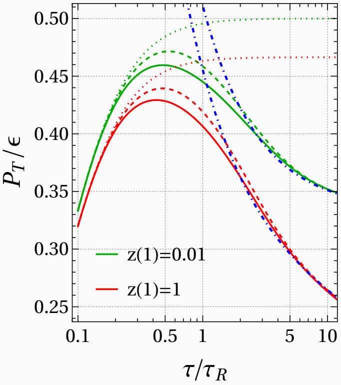
Illustration ⇒

Second-order hydrodynamics captures free-streaming!

SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

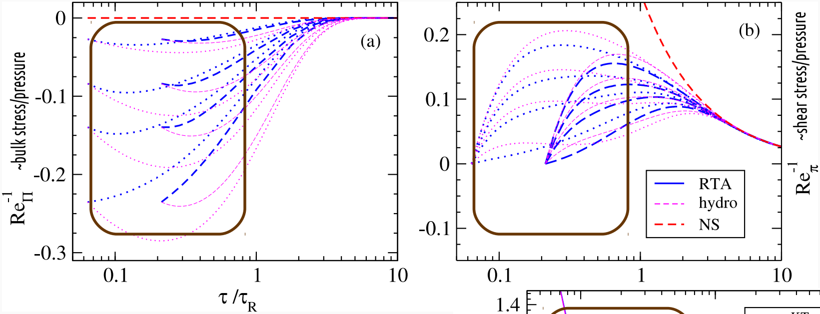
Isotropic initial conditions.

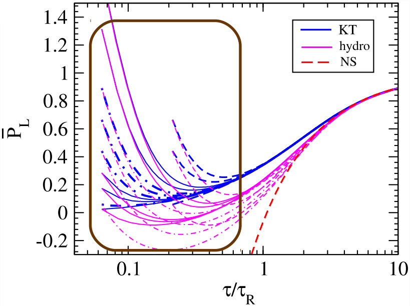




Short free-streaming regime (dotted curves) seen in both the kinetic theory and second-order hydrodynamic. There is nothing typically "hydrodynamic" here; hydrodynamics becomes a valid description only for times $\tau \gtrsim \tau_R$.

Collisionless and near-equilibrium regime in ISL hydro





Chattopadhyay, SJ, Du, Heinz, Pal, PLB 824, 136820 (2021)

SJ, Chattopadhyay, Du, Heinz, Pal, PRC 105, 024911 (2022)

The second-order hydro solutions are not very bad even in the far-off-equilibrium regime. Note that hydrodynamics as a gradient expansion (NS solution) diverges in this regime.

Hydrodynamics is applied in regime of large gradients.

RHIC energies

LHC energies

Schenke, Jeon, Gale, PRL **106** (2011), 042301

Schenke, Tribedy, Venugopalan, PRL **108** (2012), 252301

- "Unreasonable effectiveness of hydrodynamics": The success of ISL hydro in allowing early-time description of matter expansion has nothing to do with near-equilibrium hydrodynamic theory. It results from a subtle property of IS equations that mimic the early time, collisionless, regime.
- Nearly thermalized medium formed at $\tau \lesssim 1$ fm/c (?): Success of such simulations does not imply formation of nearly equilibrated medium at early times.

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Extras

Ambiguity of second-order transport coefficients

SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

- Equation of $\mathcal{L}_n$ moments are decoupled from $\mathcal{M}_n$ moments $\implies$ evolution of energy density ($\mathcal{L}_0$) does not depend on $\mathcal{M}_n$ evolution.

- Since only $\Pi - \pi = c_0(\mathcal{L}_1 - \mathcal{L}_1^{\mathrm{eq}})$ enters in evolution of ϵ, similar decoupling in the hydrodynamic equations expected. Such decoupling holds in the ISL hydro iff
$$\delta_{\Pi\Pi} + \frac{2}{3}\lambda_{\pi\Pi} = \lambda_{\Pi\pi} + \frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}$$
Not satisfied by transport coefficients derived in A. Jaiswal et. al., PRC 90 (2014) 044908

- New transport coefficients derived following a different truncation for $\mathcal{L}_2$ and $\mathcal{M}_1$ appearing in the equation for $\mathcal{L}_1$ and $\mathcal{M}_0$. Coefficients of the gradient series of Π and π unchanged.