

Far-from-equilibrium attractors for massive kinetic theory in the relaxation time approximation

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Outline

- ▶ Motivation
- ▶ Exact solution to the RTA Boltzmann equation
- ▶ Evaluation of the moments
- ▶ Results
- ▶ Conclusions

Motivation

- ▶ There are now a large number of papers on the existence of hydrodynamic attractors in conformal theories.
- ▶ Do hydrodynamic attractors exist for non-conformal kinetic theories?
- ▶ What is the effect of a realistic mass- and temperature-dependent relaxation time, $\tau_{\text{eq}}(T, m)$, on the attractor behavior?
- ▶ How does the variation in initial momentum-space anisotropy and initialization time influence the time evolution of integral moments?

Exact solution to the RTA Boltzmann equation

- ▶ Consider the Boltzmann equation in relaxation time approximation (RTA):

$$p^\mu \partial_\mu f(x, p) = \frac{p \cdot u}{\tau_{\text{eq}}} (f_{\text{eq}} - f)$$

- ▶ The exact solution to the 0+1d RTA Boltzmann equation is :

[W. Florkowski, R. Ryblewski and M. Strickland, arXiv: 1304.0665](#)
[W. Florkowski, R. Ryblewski and M. Strickland, arXiv: 1305.7234](#)
[W. Florkowski, E. Maksymiuk, R. Ryblewski and M. Strickland, arXiv: 1402.7348](#)

$$f(\tau, w, p_T) = D(\tau, \tau_0) f_0(w, p_T) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\text{eq}}(\tau')} D(\tau, \tau') f_{\text{eq}}(\tau', w, p_T)$$

Damping function: $D(\tau_2, \tau_1) = \exp \left[- \int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\text{eq}}(\tau'')} \right]$

- ▶ We will assume a Boltzmann distribution as equilibrium distribution

$$f_{\text{eq}} = \exp \left(- \frac{p \cdot u}{T} \right)$$

- ▶ Initial distribution function in spheroidally-deformed form:

[P. Romatschke and M. Strickland, arXiv : \[hep-ph/0304092\].](#)
[P. Romatschke and M. Strickland, arXiv : \[hep-ph/0406188\].](#)

$$f_0(w, p_T) = \exp \left[- \frac{\sqrt{(1 + \xi_0)w^2 + (m^2 + p_T^2)\tau_0^2}}{\Lambda_0 \tau_0} \right]$$

Relaxation Time for a Massive Gas

- ▶ The relaxation time for a massive gas is:

$$\tau_{\text{eq}}(T, m) = \frac{5\bar{\eta}}{T} \gamma(\hat{m})$$

with

$$\gamma(\hat{m}) \equiv \frac{3}{\kappa(\hat{m})} \left(1 + \frac{\varepsilon}{P} \right)$$

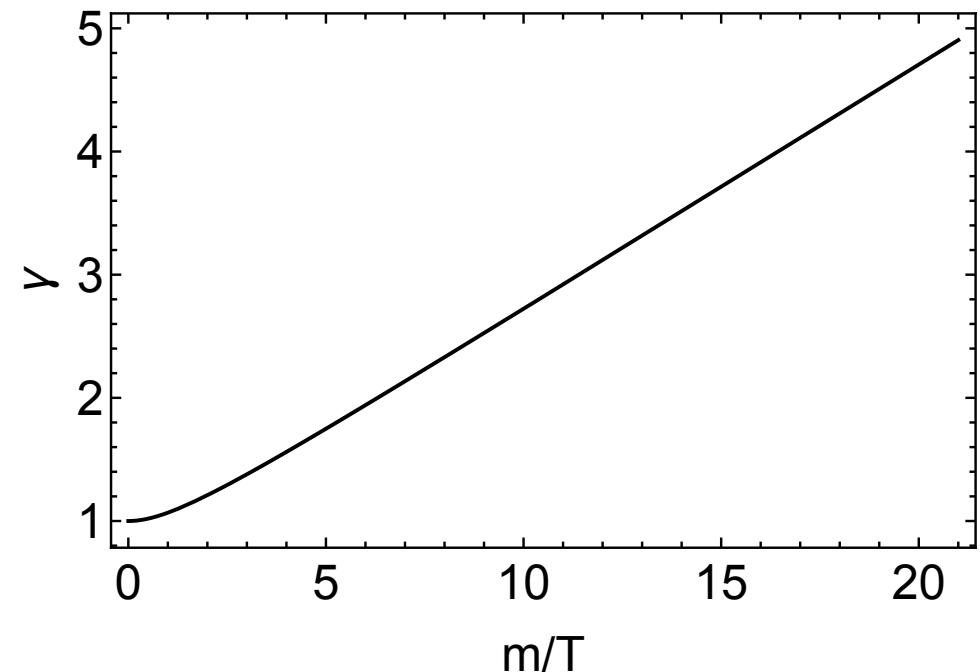
and

$$\kappa(x) \equiv x^3 \left[\frac{3}{x^2} \frac{K_3(x)}{K_2(x)} - \frac{1}{x} + \frac{K_1(x)}{K_2(x)} - \frac{\pi}{2} \frac{1-xK_0(x)L_{-1}(x)-xK_1(x)L_0(x)}{K_2(x)} \right]$$

where $\hat{m} \equiv m/T$, $K_n(x)$ is modified Bessel functions of the second kind, and $L_n(x)$ is modified Struve functions.

- ▶ The behavior of $\gamma(\hat{m})$ indicates slower relaxation to equilibrium compared to a massless gas.
- ▶ The strong enhancement of the relaxation time at low temperatures modifies the asymptotic approach to equilibrium.

J. Anderson and H. Witting, Relativistic quantum transport coefficients, Physica 74 (1974) 489.
W. Czyz and W. Florkowski, Kinetic Coefficients for Quark - Anti-quark Plasma, Acta Phys. 26.Polon. B 17 (1986) 819.



The integral equation obeyed by all moments

- ▶ The moments of the one-particle distribution function

$$\mathcal{M}^{nl}[f] \equiv \int dP (p \cdot u)^n (p \cdot z)^{2l} f(\tau, w, p_T)$$

- ▶ Some specific cases of \mathcal{M}^{nl} map to familiar quantities:

$$n = 1, l = 0 \quad \rightarrow \quad \text{Number density: } \mathcal{N} = \mathcal{M}^{10},$$

$$n = 2, l = 0 \quad \rightarrow \quad \text{Energy density : } \varepsilon = \mathcal{M}^{20},$$

$$n = 0, l = 1 \quad \rightarrow \quad \text{Longitudinal pressure: } P_L = \mathcal{M}^{01}.$$

- ▶ Transverse pressure P_T can be obtained using

$$P_T = \mathcal{M}^{20} - \mathcal{M}^{01} - m^2 \mathcal{M}^{00}$$

The integral equation obeyed by all moments (cont'd)

- ▶ Taking a general moment of the exact solution for the one-particle distribution function one obtains:

$$\mathcal{M}^{nl}(\tau) = D(\tau, \tau_0) \mathcal{M}_0^{nl}(\tau) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\text{eq}}(\tau')} D(\tau, \tau') \mathcal{M}_{\text{eq}}^{nl}(\tau')$$

- ▶ Evaluating the necessary integrals, we arrive at the following expression:

$$\begin{aligned}\mathcal{M}^{nl} &= \frac{D(\tau, \tau_0) \Lambda_0^{n+2l+2}}{(2\pi)^2} \tilde{H}^{nl} \left(\frac{\tau_0}{\tau \sqrt{1 + \xi_0}}, \frac{m}{\Lambda_0} \right) \\ &\quad + \frac{1}{(2\pi)^2} \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\text{eq}}(\tau')} D(\tau, \tau') T^{n+2l+2}(\tau') \tilde{H}^{nl} \left(\frac{\tau'}{\tau}, \frac{m}{T(\tau')} \right)\end{aligned}$$

Landau matching condition

- ▶ Specializing to the case $n = 2$ and $l = 0$ and requiring conservation of energy

$$\varepsilon(\tau) = \varepsilon_{\text{eq}}(T)$$

we obtain the integral equation:

$$\begin{aligned} & 2T^4(\tau) \hat{m}^2 \left[3K_2\left(\frac{m}{T(\tau)}\right) + \hat{m}K_1\left(\frac{m}{T(\tau)}\right) \right] \\ &= D(\tau, \tau_0) \Lambda_0^4 \tilde{H}^{20}\left(\frac{\tau_0}{\tau\sqrt{1+\xi_0}}, \frac{m}{\Lambda_0}\right) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\text{eq}}(\tau')} D(\tau, \tau') T^4(\tau') \tilde{H}^{20}\left(\frac{\tau'}{\tau}, \frac{m}{T(\tau')}\right). \end{aligned}$$

- ▶ Solving this integral equation numerically provides a self-consistent determination of the temperature profile $T(\tau)$ as a function of the proper time τ .

Results

- ▶ General moments

- 1) holding the initial energy density and initialization time fixed, while varying the initial momentum anisotropy.

$$\tau_f = 100 \text{ fm/c}$$

$$m = 0.2, 5 \text{ GeV}$$

$$T_0 = 1 \text{ GeV}$$

$$\tau_0 = 0.1 \text{ fm/c}$$

$$\alpha_0 = 1/\sqrt{1 + \xi_0} \in \{0.12, 0.25, 0.5, 1, 2\}$$

} **Late-time attractor**

- 2) holding the initial energy density and the initial momentum anisotropy fixed, while varying the initialization time.

$$\tau_f = 100 \text{ fm/c}$$

$$m = 0.2, 5 \text{ GeV}$$

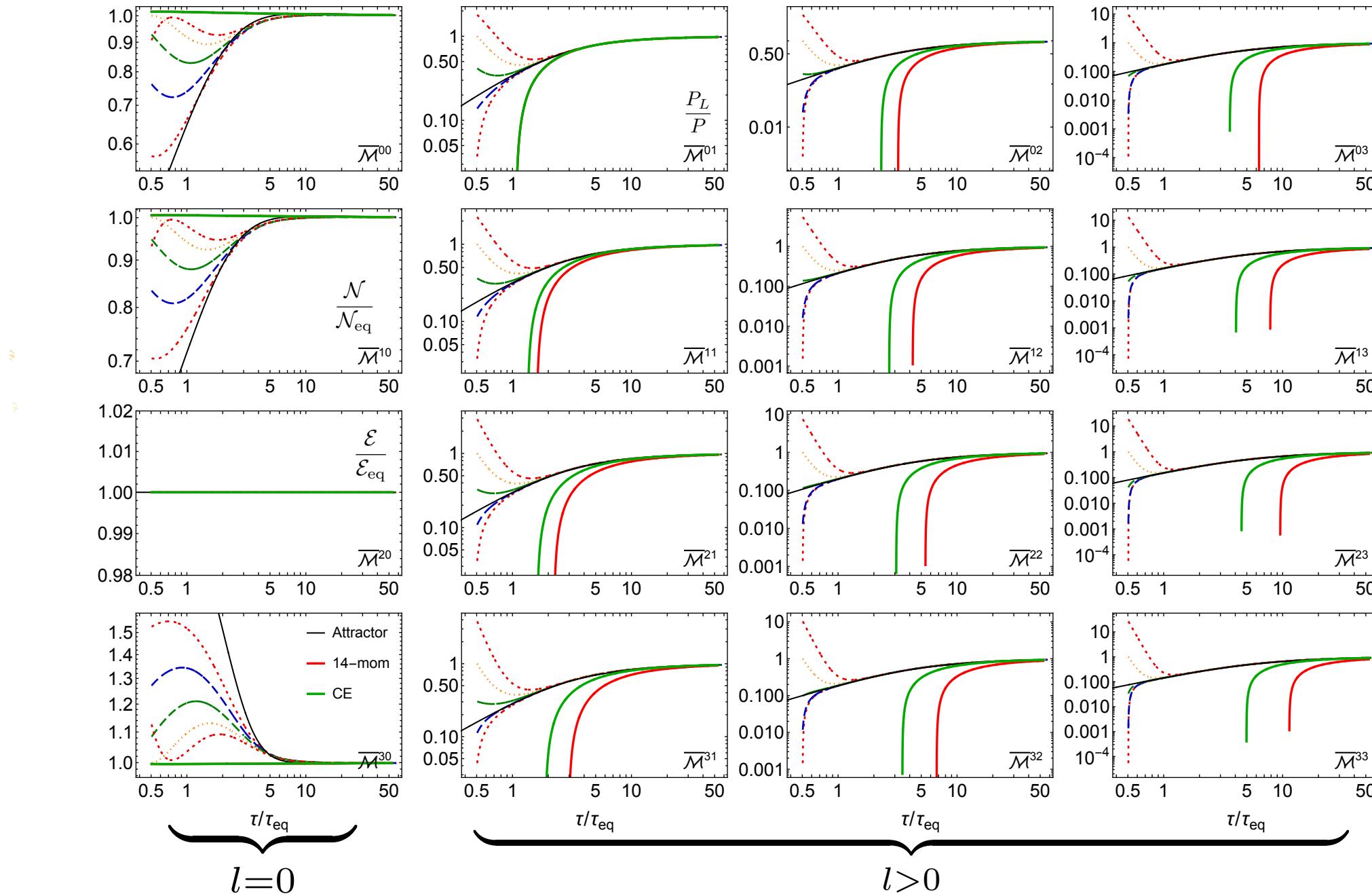
$$T_0 = 1 \text{ GeV}$$

$$\xi_0 = 0$$

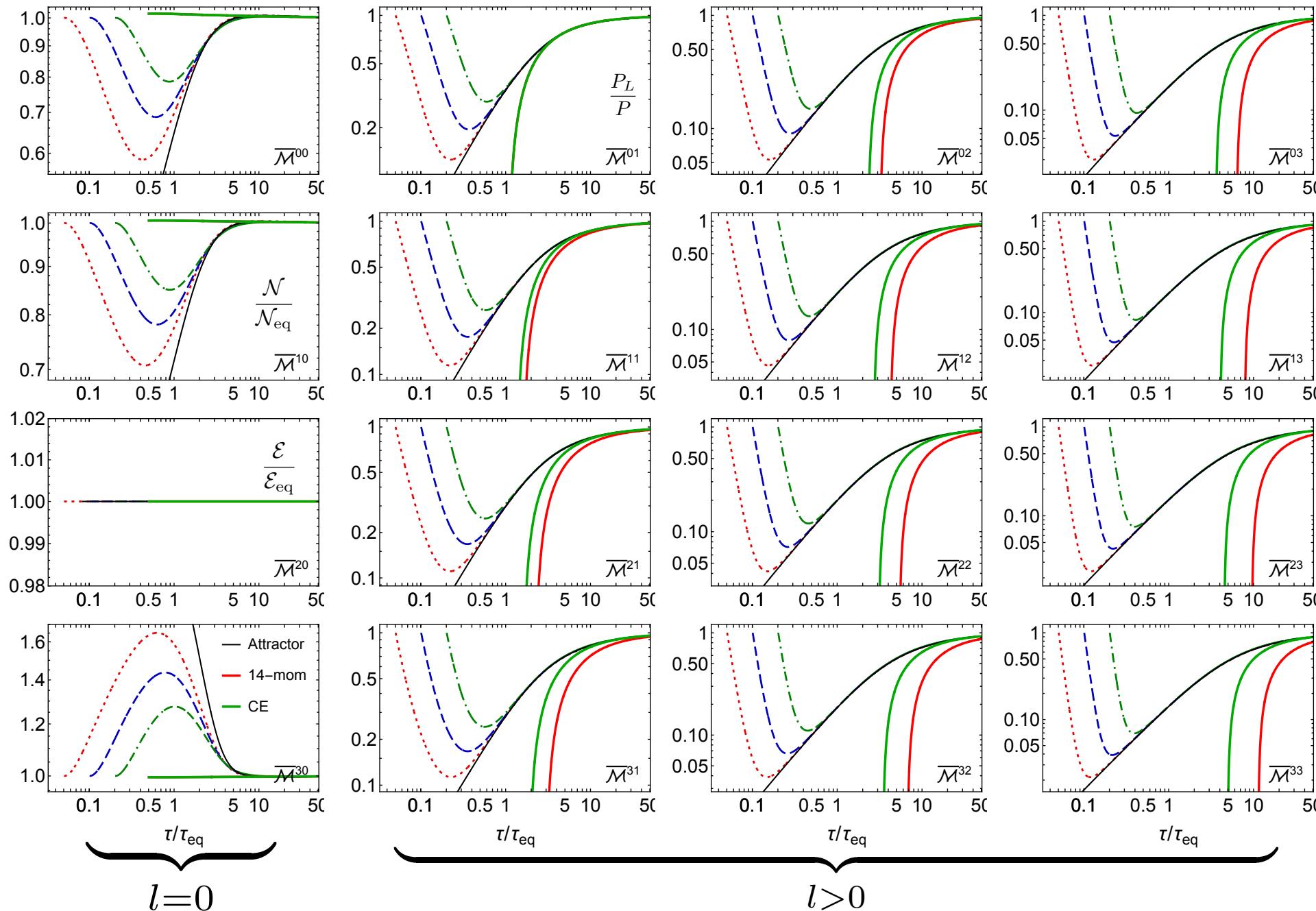
$$\tau_0 \in \{0.01, 0.02, 0.04\} \text{ fm/c}$$

} **Early-time attractor**

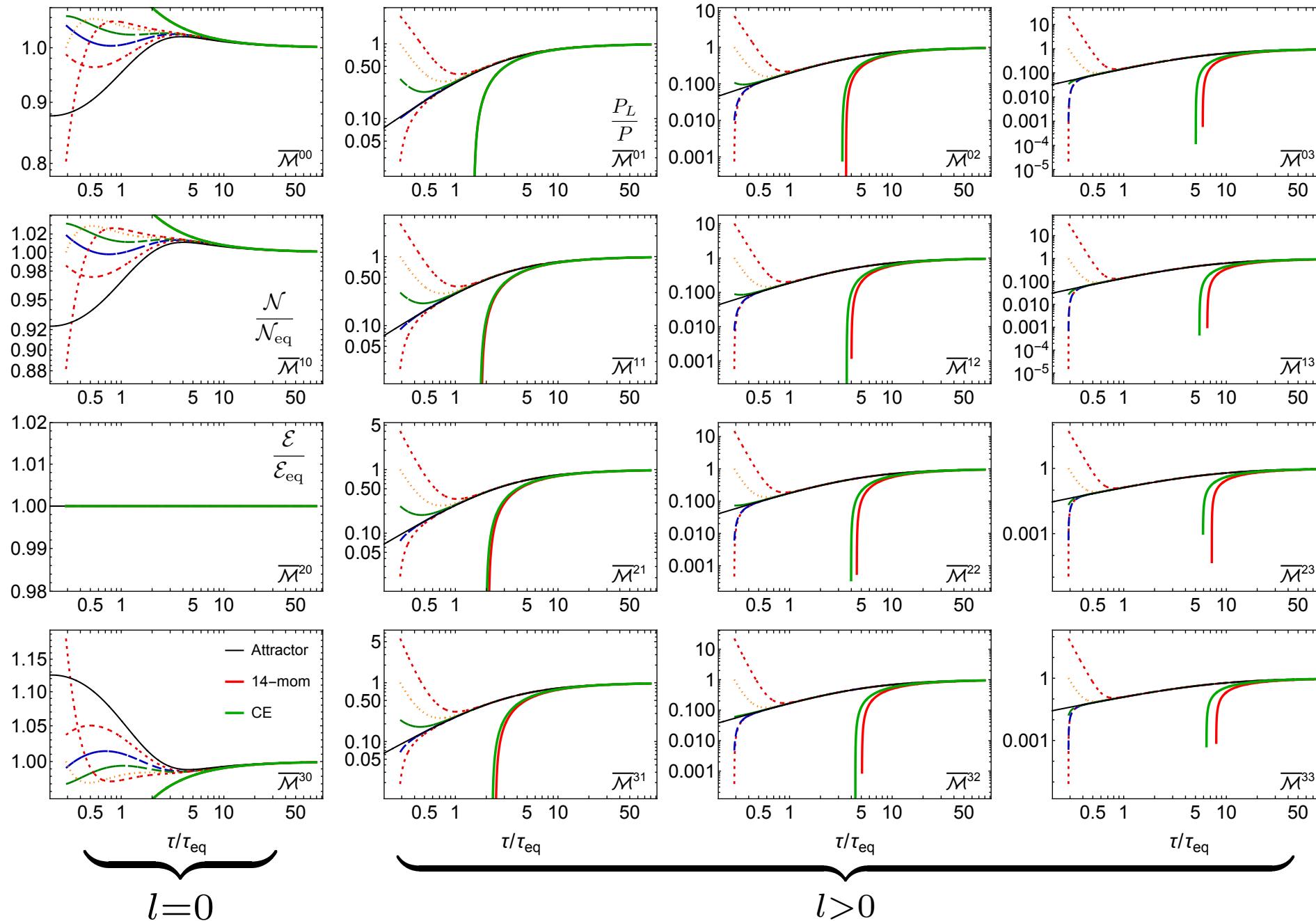
Results: General moments → case (1) Late-time attractor, $m = 0.2$ GeV



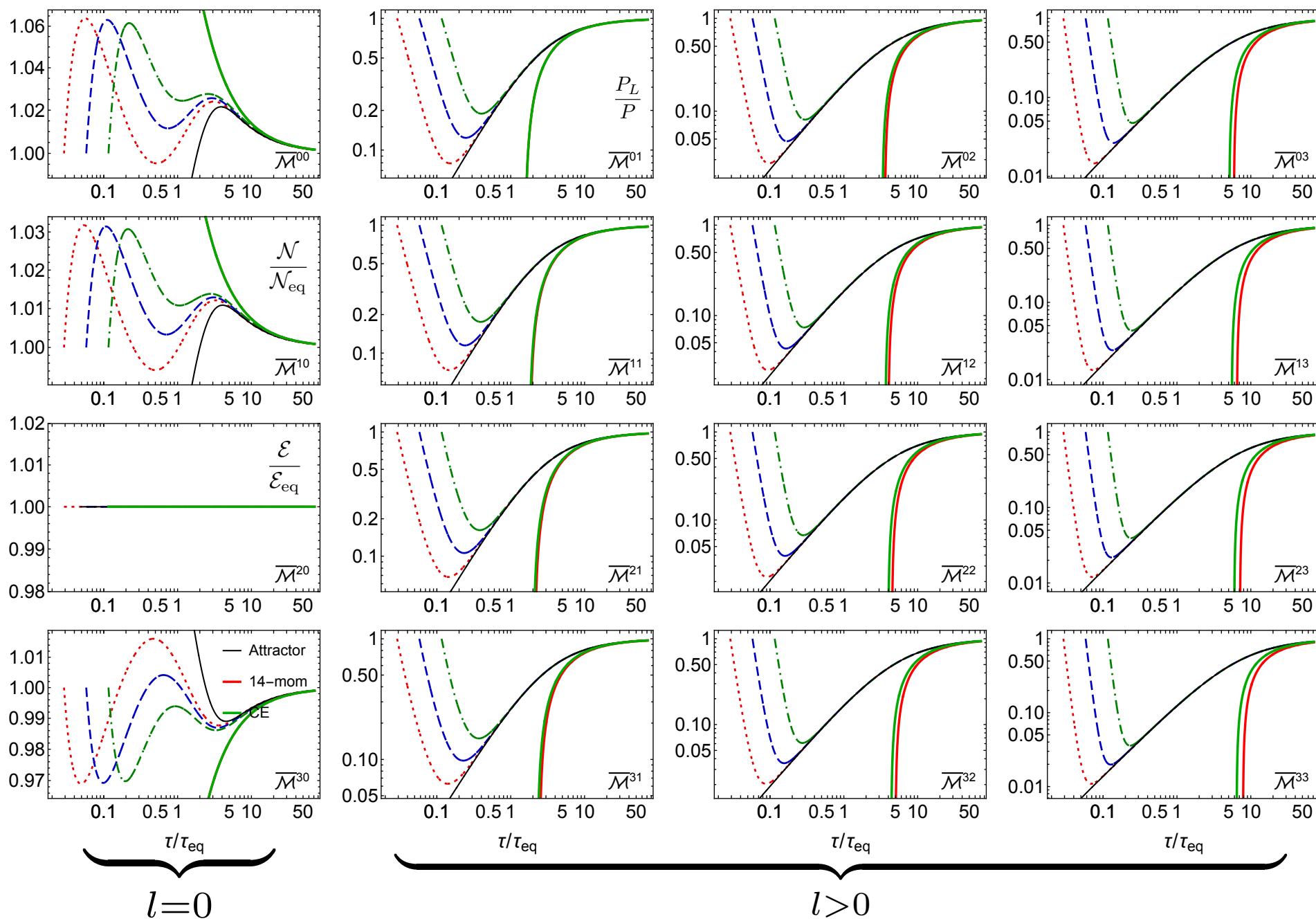
Results: General moments → case (2) Early-time attractor , $m = 0.2$ GeV



Results: General moments → case (1) Late-time attractor $m = 5 \text{ GeV}$



Results: General moments → case (2) Early-time attractor , $m = 5 \text{ GeV}$



Viscous corrections expressed in terms of moments

- ▶ Shear and bulk viscous corrections can be expressed in terms of the moments:

- ▶ Shear viscous correction:

$$\pi^{\mu\nu} = \Delta_{\alpha\beta}^{\mu\nu} \int dP p^\alpha p^\beta (f - f_{\text{eq}})$$

Scaling by equilibrium pressure:

$$\tilde{\pi} \equiv \frac{\pi}{P} = 1 - \bar{\mathcal{M}}^{01} + \tilde{\Pi}$$

- ▶ Bulk viscous correction:

$$\Pi = -\frac{1}{3} \Delta_{\mu\nu} \int dP p^\mu p^\nu (f - f_{\text{eq}}) = -\frac{1}{3} m^2 [\mathcal{M}^{00} - \mathcal{M}_{\text{eq}}^{00}]$$

Scaling by equilibrium pressure:

$$\tilde{\Pi} \equiv \frac{\Pi}{P} = -\frac{m^2 (\mathcal{M}^{00} - \mathcal{M}_{\text{eq}}^{00})}{\mathcal{M}_{\text{eq}}^{20} - m^2 \mathcal{M}_{\text{eq}}^{00}}$$

- ▶ Equilibrium pressure:

$$P = -\frac{1}{3} \Delta_{\mu\nu} \int dP p^\mu p^\nu f_{\text{eq}} = \frac{1}{3} [\mathcal{M}_{\text{eq}}^{20} - m^2 \mathcal{M}_{\text{eq}}^{00}]$$

Results

- ▶ Shear and bulk viscous corrections

- 1) holding the initial energy density and initialization time fixed, while varying the initial momentum anisotropy.

$$\tau_f = 100 \text{ fm/c}$$

$$m = 0.2, 5 \text{ GeV}$$

$$T_0 = 1 \text{ GeV}$$

$$\tau_0 = 0.1 \text{ fm/c}$$

$$\alpha_0 = 1/\sqrt{1 + \xi_0} \in \{0.12, 0.25, 0.5, 1, 2\}$$

} **Late-time attractor**

- 2) holding the initial energy density and the initial momentum anisotropy fixed, while varying the initialization time.

$$\tau_f = 100 \text{ fm/c}$$

$$m = 0.2, 5 \text{ GeV}$$

$$T_0 = 1 \text{ GeV}$$

$$\xi_0 = 0$$

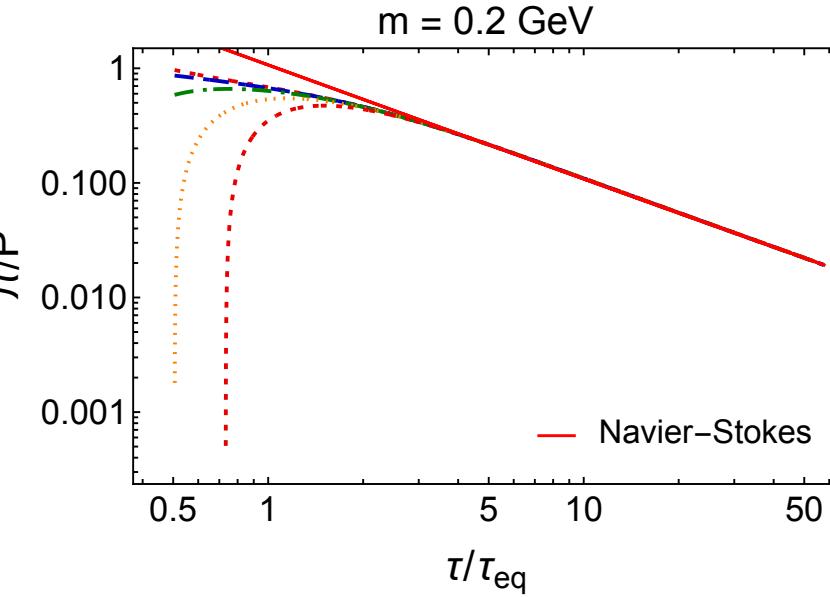
$$\tau_0 \in \{0.01, 0.02, 0.04\} \text{ fm/c}$$

} **Early-time attractor**

Results: shear viscous corrections

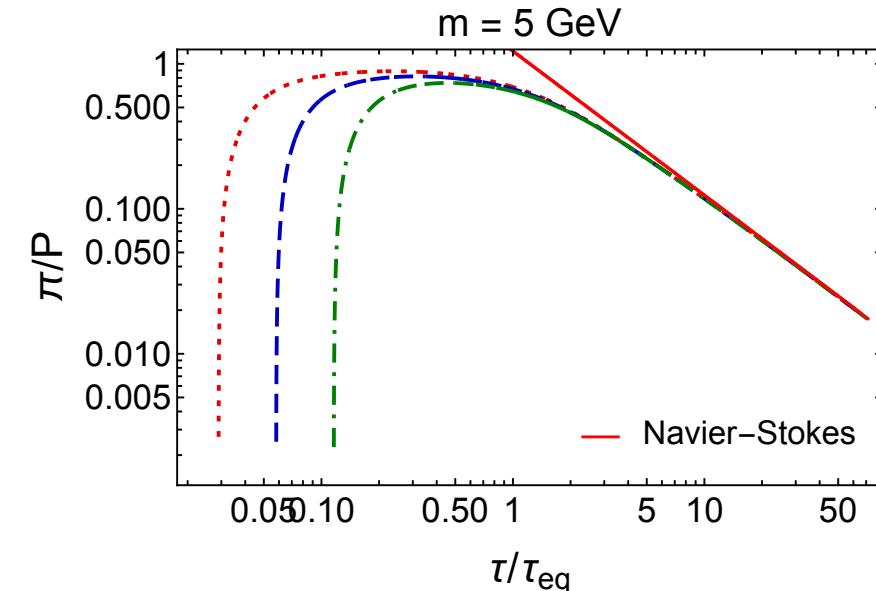
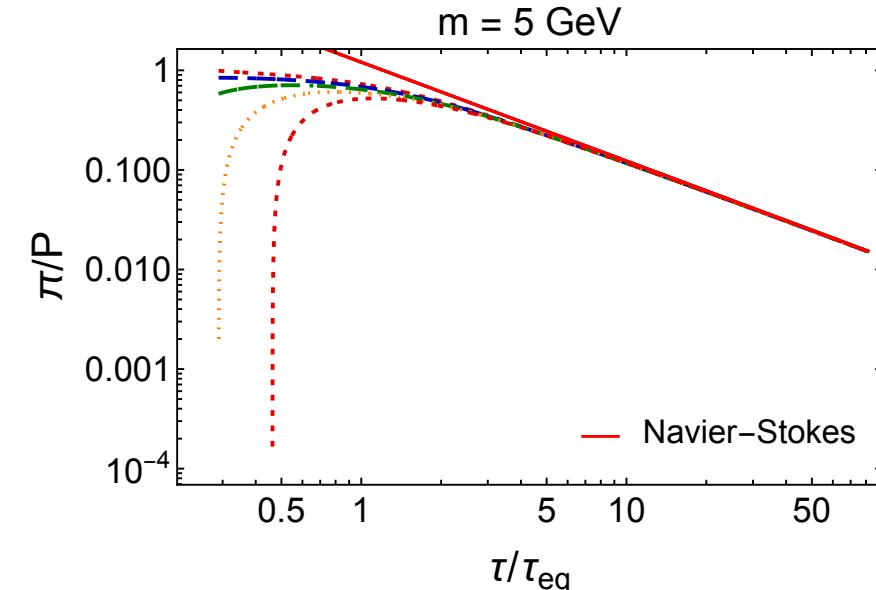
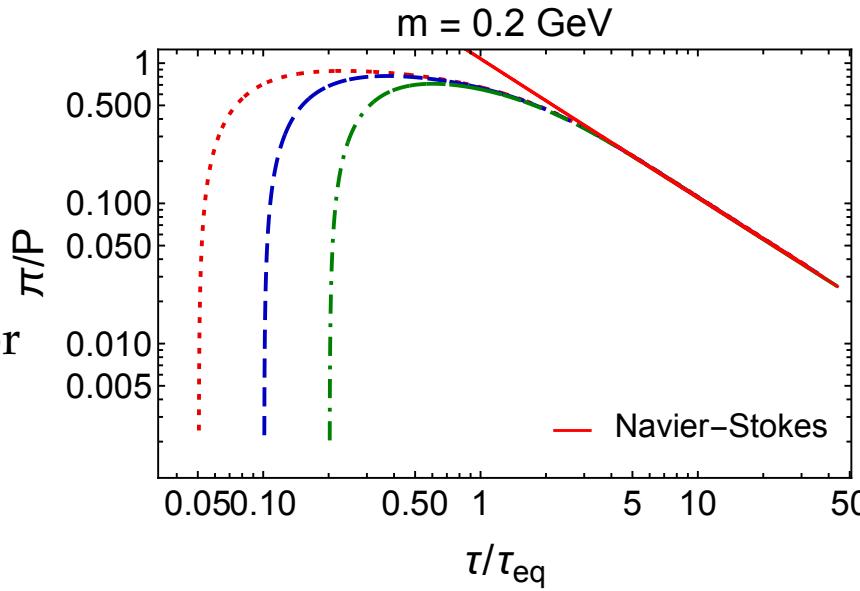
► Case (1) →

Late-time attractor



► Case (2) →

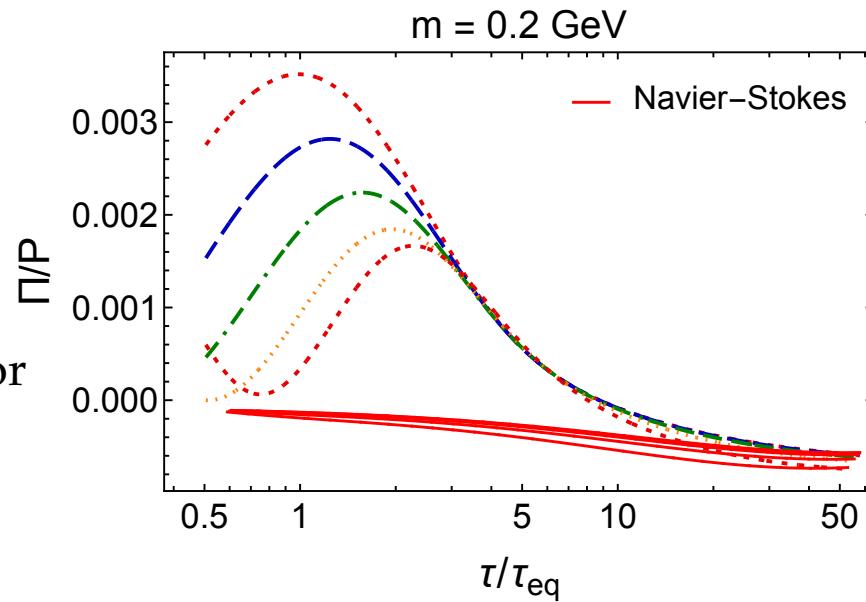
Early-time attractor



Results: bulk viscous corrections

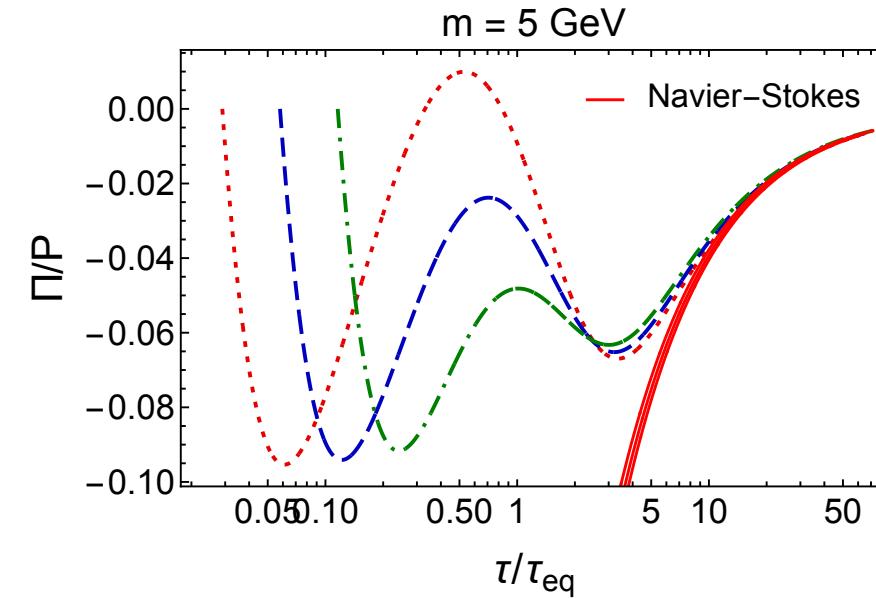
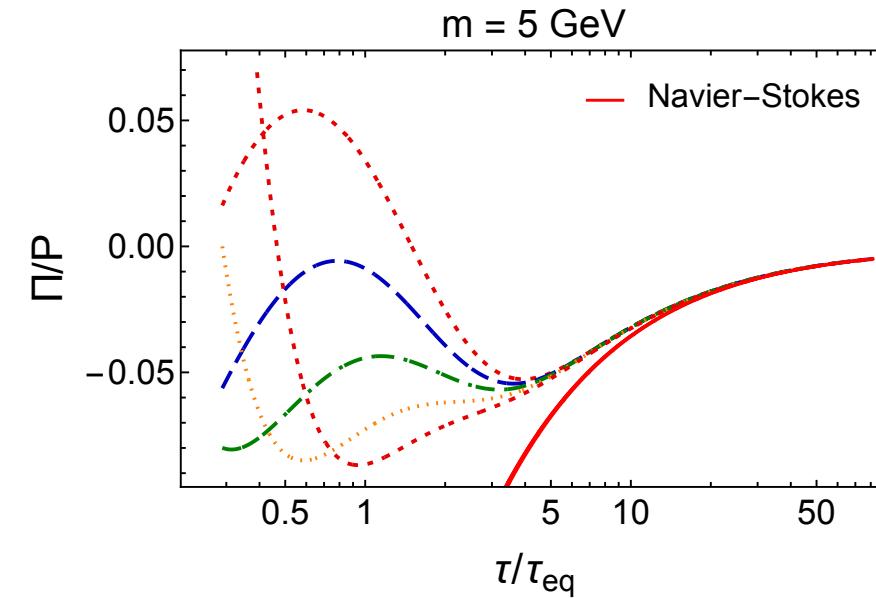
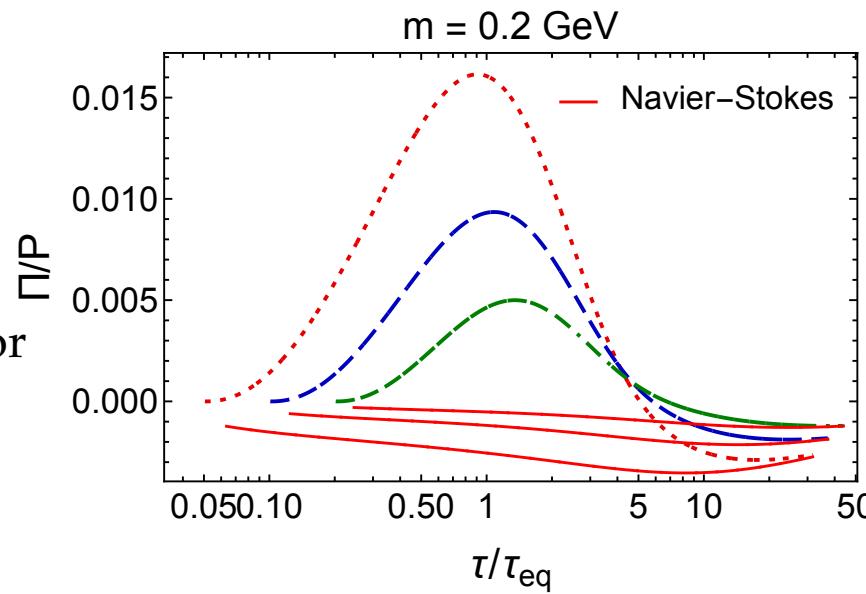
► Case (1) →

Late-time attractor



► Case (2) →

Early-time attractor



Conclusion

- ▶ Studied existence of attractors in non-conformal kinetic theory using an exact solution of the boost-invariant Boltzmann equation.
- ▶ Found both late-time and early-time attractors for scaled longitudinal pressure, indicating universal behavior.
- ▶ No separate attractors observed for shear and bulk viscous corrections.
- ▶ Moments with $l = 0$ did not exhibit early-time attractors.
- ▶ Bulk viscous correction at first-order in gradients did not collapse in the late-time limit.
- ▶ Implications for heavy-ion phenomenology: Existence of longitudinal pressure attractor still holds, but additional uncertainty for $l = 0$ moments.
- ▶ Forward attractor remains universal for moments with $l \neq 0$, semi-universal for $l = 0$ with phenomenologically relevant initialization times.

THANKS FOR YOUR ATTENTION

Evaluation of the moments to first order in hydrodynamic gradients

- ▶ The expressions for the shear and bulk viscosity corrected distribution functions

$$f = f_{\text{eq}} + \delta f_{\text{shear}} + \delta f_{\text{bulk}}$$

- **14-moment approximation:**

D. Teaney, arXiv: [nucl-th/0301099].

P. Bozek, arXiv: 0911.2397

J.-B. Rose, J.-F. Paquet, G. S. Denicol, M. Luzum, B. Schenke, S. Jeon et al., arXiv: 1408.0024

M. Alqahtani, M. Nopoush and M. Strickland, arXiv: 1605.02101

- ▶ the viscous corrections to the distribution function for a single component massive gas obeying classical statistics :

$$\delta f_{14-\text{moment}} = f_{\text{eq}} \left\{ 1 + \frac{p_\mu p_\nu \pi^{\mu\nu}}{2(\varepsilon + P)} T^2 - \frac{\beta}{\beta_\Pi} \left[\frac{m^2}{3 p \cdot u} - \left(\frac{1}{3} - c_s^2 \right) p \cdot u \right] \Pi \right\}$$

$$\text{with } \beta = 1/T, \quad \beta_\Pi = \frac{5}{3} \beta I_{42}^{(1)} - (\varepsilon + P) c_s^2$$

$$\text{and } I_{42}^{(1)} = \frac{T^5 \hat{m}^5}{30\pi^2} \left[\frac{1}{16} \left(K_5(\hat{m}) - 7K_3(\hat{m}) + 22K_1(\hat{m}) \right) - K_{i,1}(\hat{m}) \right]$$

A. Jaiswal, R. Ryblewski and M. Strickland, arXiv: 1407.7231

- ▶ the scaled moments in the Navier-Stokes (NS) limit

$$\begin{aligned} \bar{\mathcal{M}}_{14-\text{moment}}^{nl, \text{NS}} &= 1 + \frac{1}{15 \bar{\tau} T^2 \gamma(\hat{m})} \frac{[\mathcal{M}_{\text{eq}}^{n+2,l} - 3\mathcal{M}_{\text{eq}}^{n,l+1} - m^2 \mathcal{M}_{\text{eq}}^{n,l}]}{\mathcal{M}_{\text{eq}}^{n,l}} \\ &\quad + \frac{1}{3 \bar{\tau} T} \frac{[m^2 \mathcal{M}_{\text{eq}}^{n-1,l} - (1 - 3c_s^2) \mathcal{M}_{\text{eq}}^{n+1,l}]}{\mathcal{M}_{\text{eq}}^{n,l}} \end{aligned}$$

Evaluation of the moments to first order in hydrodynamic gradients

- ▶ The expressions for the shear and bulk viscosity corrected distribution functions

$$f = f_{\text{eq}} + \delta f_{\text{shear}} + \delta f_{\text{bulk}}$$

- **Chapman-Enskog approximation:**

- ▶ the viscous corrections to the distribution function for a single component massive gas obeying classical statistics :

$$\delta f_{\text{CE}} = f_{\text{eq}} \left\{ 1 + \frac{p^\mu p^\nu \pi_{\mu\nu}}{2(u \cdot p)\beta_\pi} \beta - \frac{\beta}{\beta_\Pi} \left[\frac{m^2}{3p \cdot u} - \left(\frac{1}{3} - c_s^2 \right) p \cdot u \right] \Pi \right\}$$

$$\text{with } \beta = 1/T, \quad \beta_\Pi = \frac{5}{3}\beta I_{42}^{(1)} - (\varepsilon + P)c_s^2, \quad \beta_\pi = \beta I_{42}^{(1)}$$

$$\text{and } I_{42}^{(1)} = \frac{T^5 \hat{m}^5}{30\pi^2} \left[\frac{1}{16} \left(K_5(\hat{m}) - 7K_3(\hat{m}) + 22K_1(\hat{m}) \right) - K_{i,1}(\hat{m}) \right]$$

A. Jaiswal, R. Ryblewski and M. Strickland, arXiv: 1407.7231

- ▶ the scaled moments in the Navier-Stokes (NS) limit

$$\begin{aligned} \bar{\mathcal{M}}_{\text{CE}}^{nl, \text{NS}} &= 1 + \frac{\varepsilon + P}{15 \bar{\tau} \gamma(\hat{m}) I_{42}^{(1)}} \frac{[\mathcal{M}_{\text{eq}}^{n+1,l} - 3\mathcal{M}_{\text{eq}}^{n-1,l+1} - m^2 \mathcal{M}_{\text{eq}}^{n-1,l}]}{\mathcal{M}_{\text{eq}}^{n,l}} \\ &\quad + \frac{1}{3 \bar{\tau} T} \frac{[m^2 \mathcal{M}_{\text{eq}}^{n-1,l} - (1 - 3c_s^2) \mathcal{M}_{\text{eq}}^{n+1,l}]}{\mathcal{M}_{\text{eq}}^{n,l}} \end{aligned}$$