

Testing Eigenstate Thermalization Hypothesis for Non-Abelian Gauge Theories

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XY, PRD 108, L031504 (2023)

B.Müller, XY, arXiv: 2307.00045

L.Ebner, B.Müller, A.Schäfer, C.Seidl, XY, arXiv: 2308.16202



InQubator for Quantum Simulation

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2023



Motivation: Quantum Understanding of Hydrodynamization and Thermalization

- Current understanding of rapid hydrodynamization mainly uses semiclassical approximations and may neglect important quantum effects
- System of two fast moving nuclei is **pure** state, evolves unitarily, how approximate thermalization happens?
- Time evolution of local operator expectation value

$$\langle O \rangle(t) = \text{Tr}[O\rho(t)] = \sum_{n,m} \langle n|O|m\rangle \langle m|\rho(0)|n\rangle e^{i(E_n - E_m)t}$$

↓ After some time?

$$\langle O \rangle_{\text{mc}}(E) \qquad \qquad E = \text{Tr}(H\rho)$$

Microcanonical ensemble average

Eigenstate Thermalization Hypothesis (ETH)

- For most non-integrable systems, matrix elements of local operators for typical eigenstates

$$\langle n|O|m\rangle = \langle O \rangle_{\text{mc}}(E)\delta_{nm} + e^{-S(E)/2}f(E, \omega)R_{nm}$$

Diagonal part close to
microcanonical
ensemble average

$$E = (E_n + E_m)/2$$

Correction suppressed
exponentially by
entropy (system size)

$$\omega = E_n - E_m$$

Gaussian
random
matrix

Spectral function decays with ω

Deutsch, Phys. Rev. A 43, 2046 (1991)

Srednicki, Phys. Rev. E 50, 888 (1994)

Rigol, Dunjko, Olshanii, Nature 452, 854 (2008)

Thermalization from ETH

- For large system and initial state with small energy variation, ETH leads to
 - (1) Long time average $\bar{O} \approx$ thermal expectation value $\langle O \rangle_T \rightarrow$ ergodic
 - (2) Fluctuation of $\langle O \rangle(t)$ around \bar{O} is exponentially small in system size
 - (3) Quantum fluctuation \approx thermal fluctuation
 - (4) Time correlation function

$$\langle n|O(t)O(0)|n\rangle - \langle n|O(t)|n\rangle\langle n|O(0)|n\rangle \approx \int d\omega e^{-i\omega t} e^{\beta\omega/2} |f(E, \omega)|^2$$

$f(E, \omega)$ is related to the spectral function

- The system, when observed through O , **behaves like a thermal state**

Goal: Demonstrate ETH for SU(N) Gauge Theory

$$\langle n|O|m\rangle = \langle O \rangle_{\text{mc}}(E)\delta_{nm} + e^{-S(E)/2}f(E, \omega)R_{nm}$$

- Show diagonal part: exponentially close to mc average
- Show off-diagonal: Gaussian random matrix, spectral function decays in ω
- QFT contains infinite states, how to demonstrate? \rightarrow **lattice**
 - (1) Continuum limit: demonstrate for several $g(a)$, extrapolate $a \rightarrow 0$ (RG)
 - (2) Infinite volume limit: fixed $g(a)$, demonstrate for several system sizes
 \leftarrow ETH more manifest with larger system sizes
 - (3) Consider “physical” operators: multiplicatively renormalizable, gauge invariant, local or sufficiently averaged

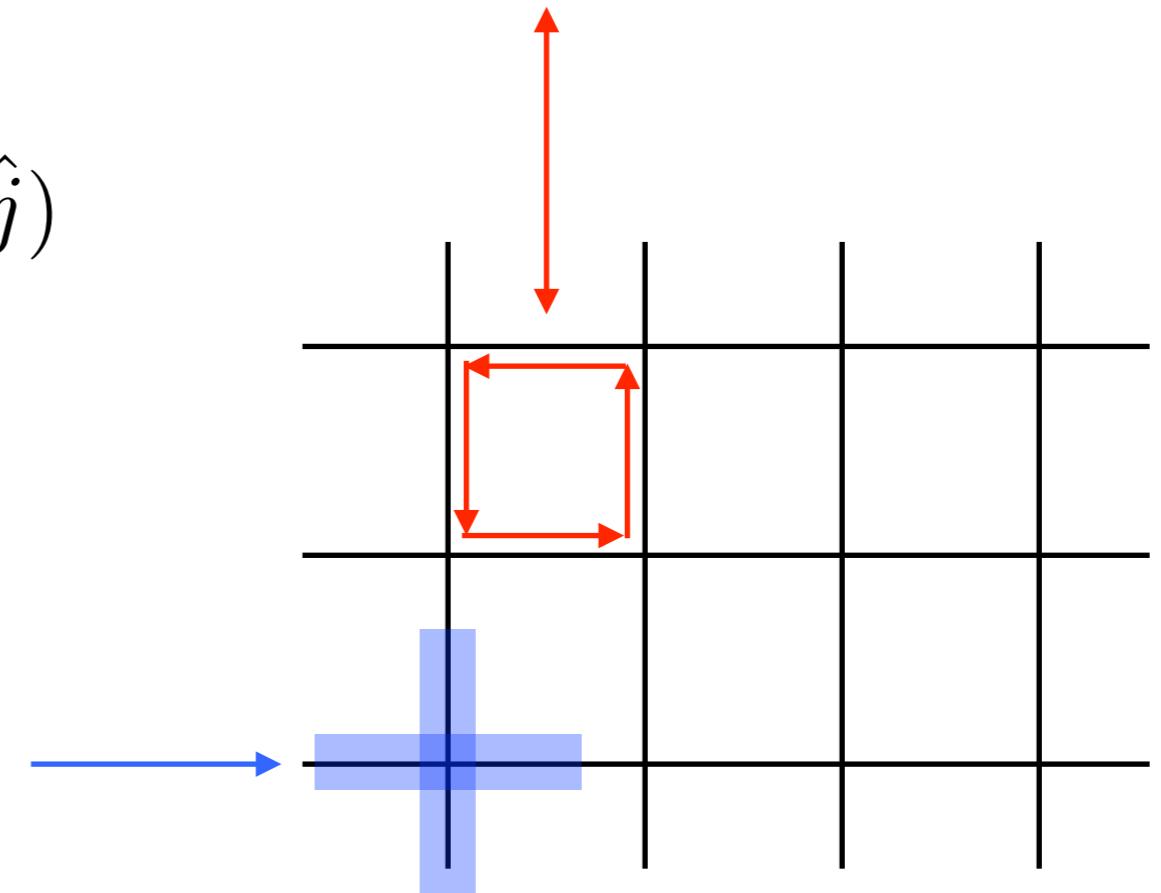
2+1D SU(2) Lattice Gauge Theory

- Kogut-Susskind Hamiltonian: $H = \frac{g^2}{2} \sum_{\text{links}} (E_i^a)^2 - \frac{2}{a^2 g^2} \sum_{\text{plaquettes}} \square(\mathbf{n})$

$$\square(\mathbf{n}) = \text{Tr}[U^\dagger(\mathbf{n}, \hat{y}) U^\dagger(\mathbf{n} + \hat{y}, \hat{x}) U(\mathbf{n} + \hat{x}, \hat{y}) U(\mathbf{n}, \hat{x})]$$

$$[E_i^a, U(\mathbf{n}, \hat{j})] = -\delta_{ij} T^a U(\mathbf{n}, \hat{j})$$

$$[E_i^a, E_i^b] = i f^{abc} E_i^c$$



- Gauss's law: every vertex transforms as a **singlet** for physical state

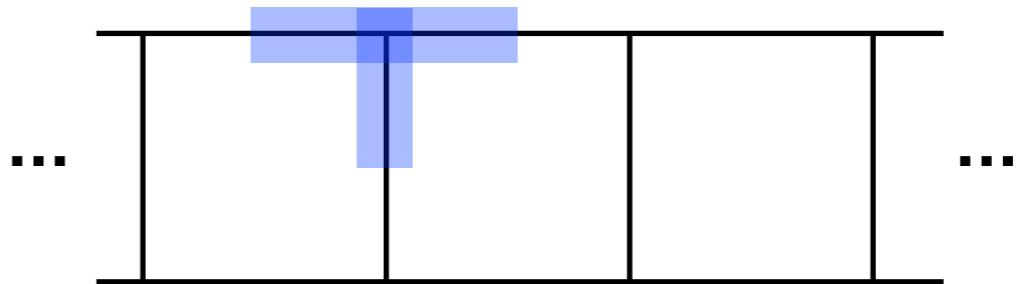
- Electric basis on links: $|j m_L m_R\rangle$

$$E^2 |j m_L m_R\rangle = j(j+1) |j m_L m_R\rangle$$

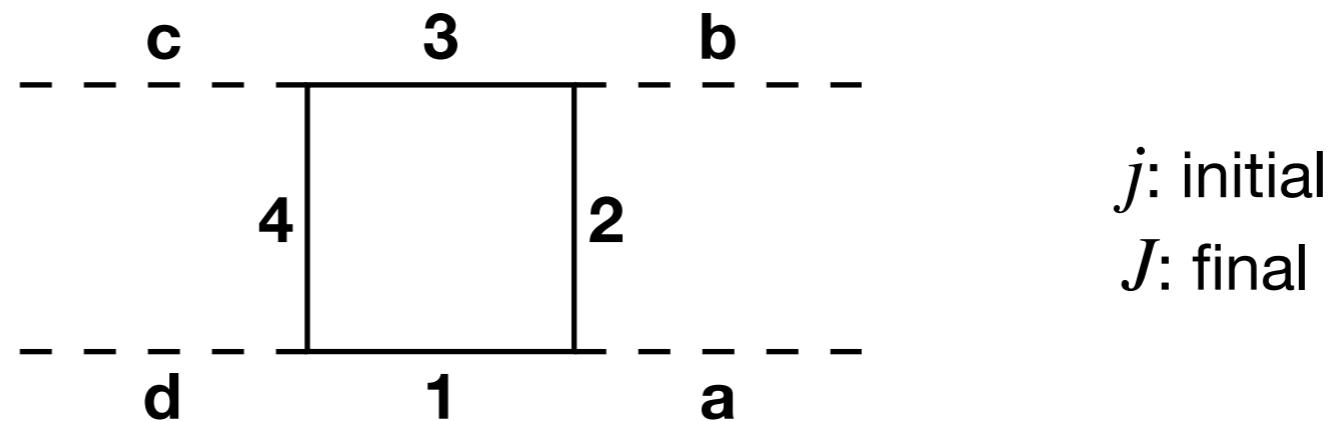
Byrnes, Yamamoto, quant-ph/0510027

2+1D SU(2) on Periodic Plaquette Chain

- Each vertex has three links:
singlet is **uniquely** defined by
the j values on the three links



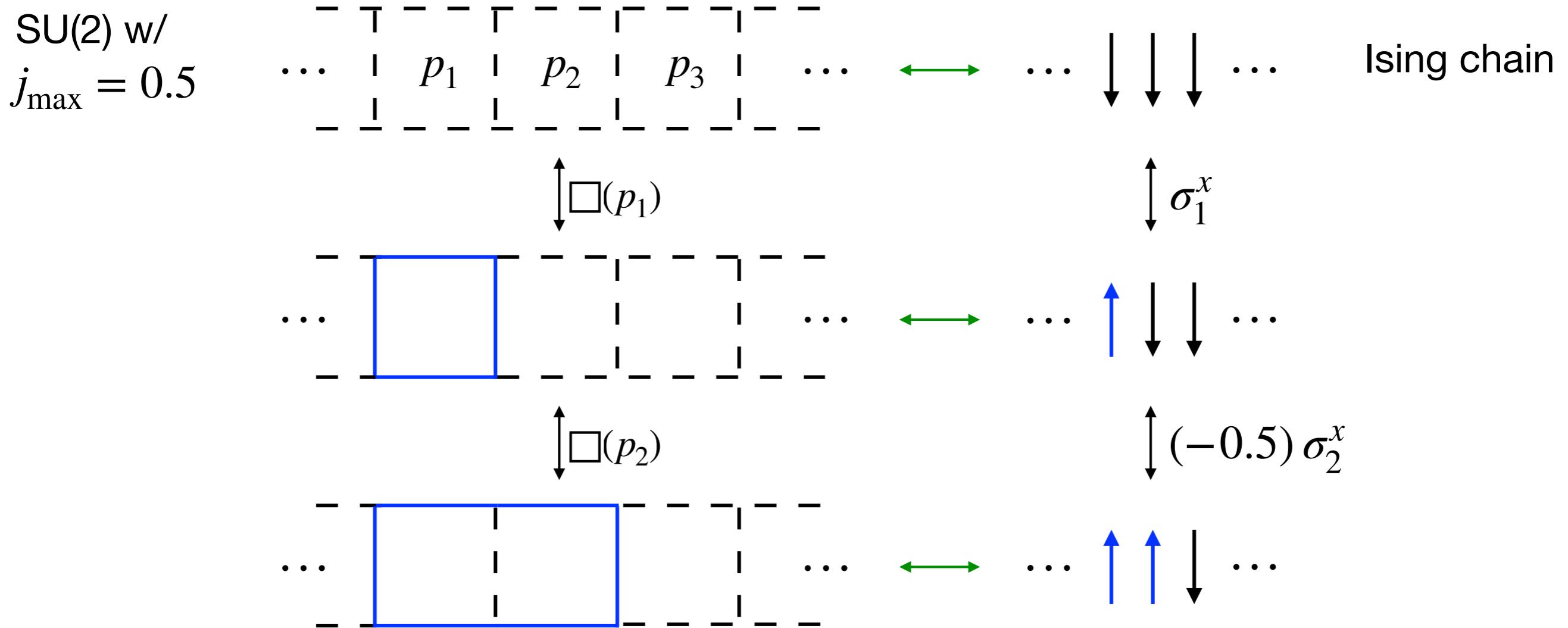
- Matrix elements between physical states (singlets)



$$\langle J_1 J_2 J_3 J_4 | \square | j_1 j_2 j_3 j_4 \rangle = \prod_{\alpha=a,b,c,d} (-1)^{j_\alpha} \prod_{\alpha=1,2,3,4} \left[(-1)^{j_\alpha + J_\alpha} \sqrt{(2j_\alpha + 1)(2J_\alpha + 1)} \right]$$

$$\left\{ \begin{array}{ccc} j_a & j_1 & j_2 \\ 1/2 & J_2 & J_1 \end{array} \right\} \left\{ \begin{array}{ccc} j_b & j_2 & j_3 \\ 1/2 & J_3 & J_2 \end{array} \right\} \left\{ \begin{array}{ccc} j_c & j_3 & j_4 \\ 1/2 & J_4 & J_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_d & j_4 & j_1 \\ 1/2 & J_1 & J_4 \end{array} \right\}$$

Simplify Hamiltonian with $j_{\max} = 0.5$



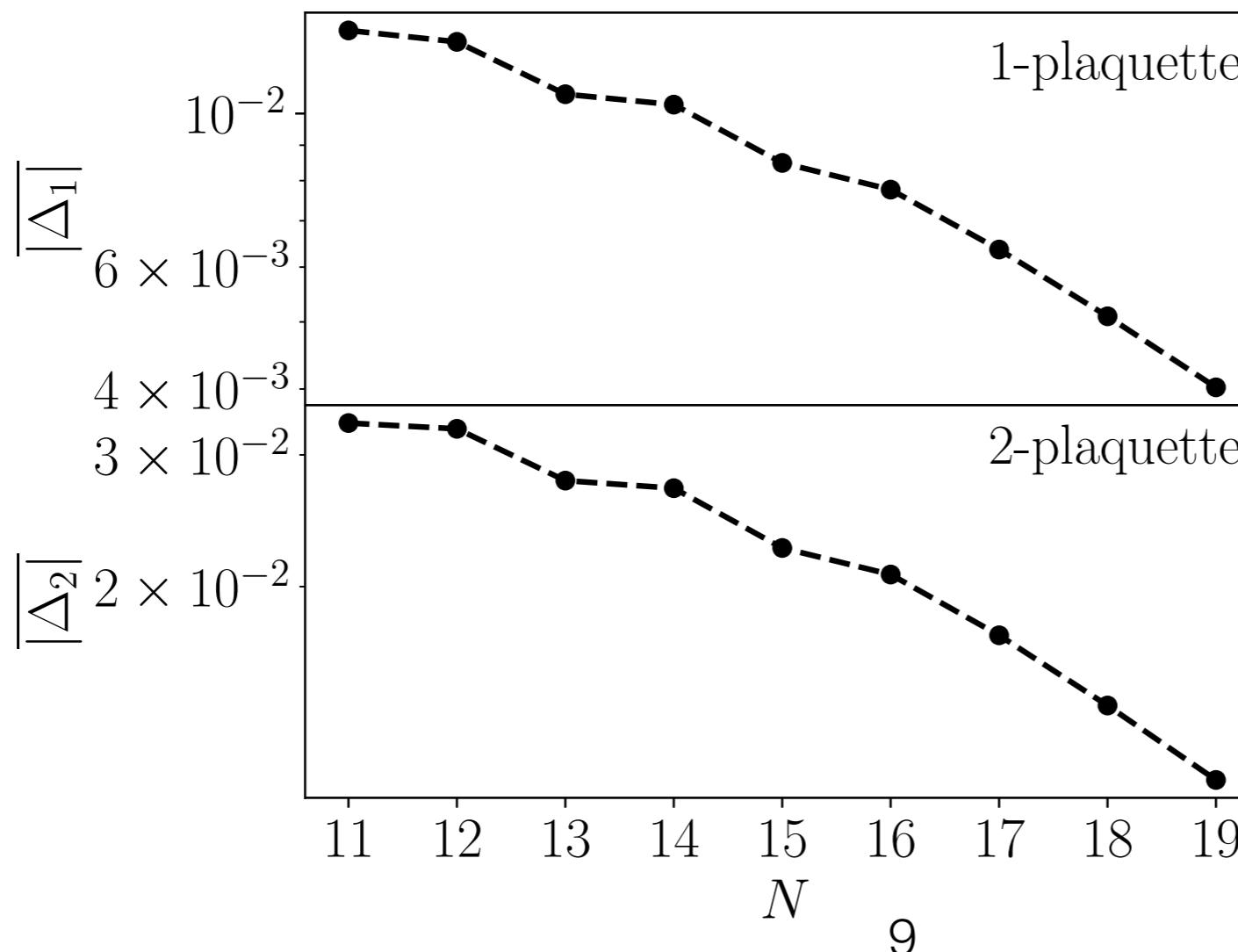
$$aH = J \sum_{i=0}^{N-1} \sigma_i^z \sigma_{i+1}^z + h_z \sum_{i=0}^{N-1} \sigma_i^z + h_x \sum_{i=0}^{N-1} (-0.5)^{(\sigma_{i-1}^z + \sigma_{i+1}^z)/2 + 1} \sigma_i^x$$

$$J = -3ag^2/16, \quad h_z = 3ag^2/8, \quad h_x = -2/(ag^2)$$

Diagonal Part Test on Chain with $j_{\max} = 0.5$

- Consider 1-plaquette and 2-plaquette operators with $ag^2 = 1.2$
- Proxy for microcanonical ensemble:

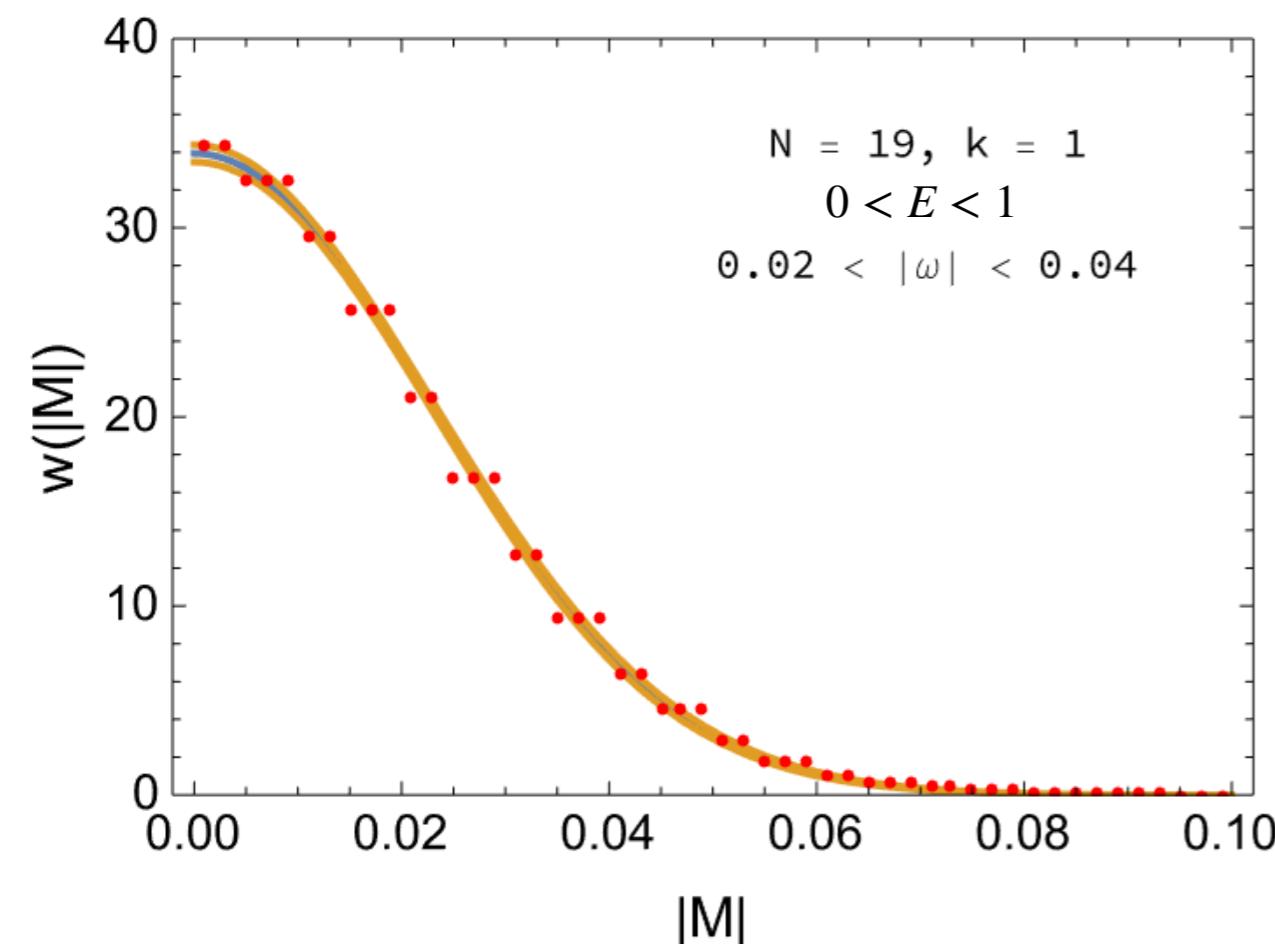
$$\Delta_i(n) = \langle n | O_i | n \rangle - \frac{1}{21} \sum_{m=n-10}^{n+10} \langle m | O_i | m \rangle$$



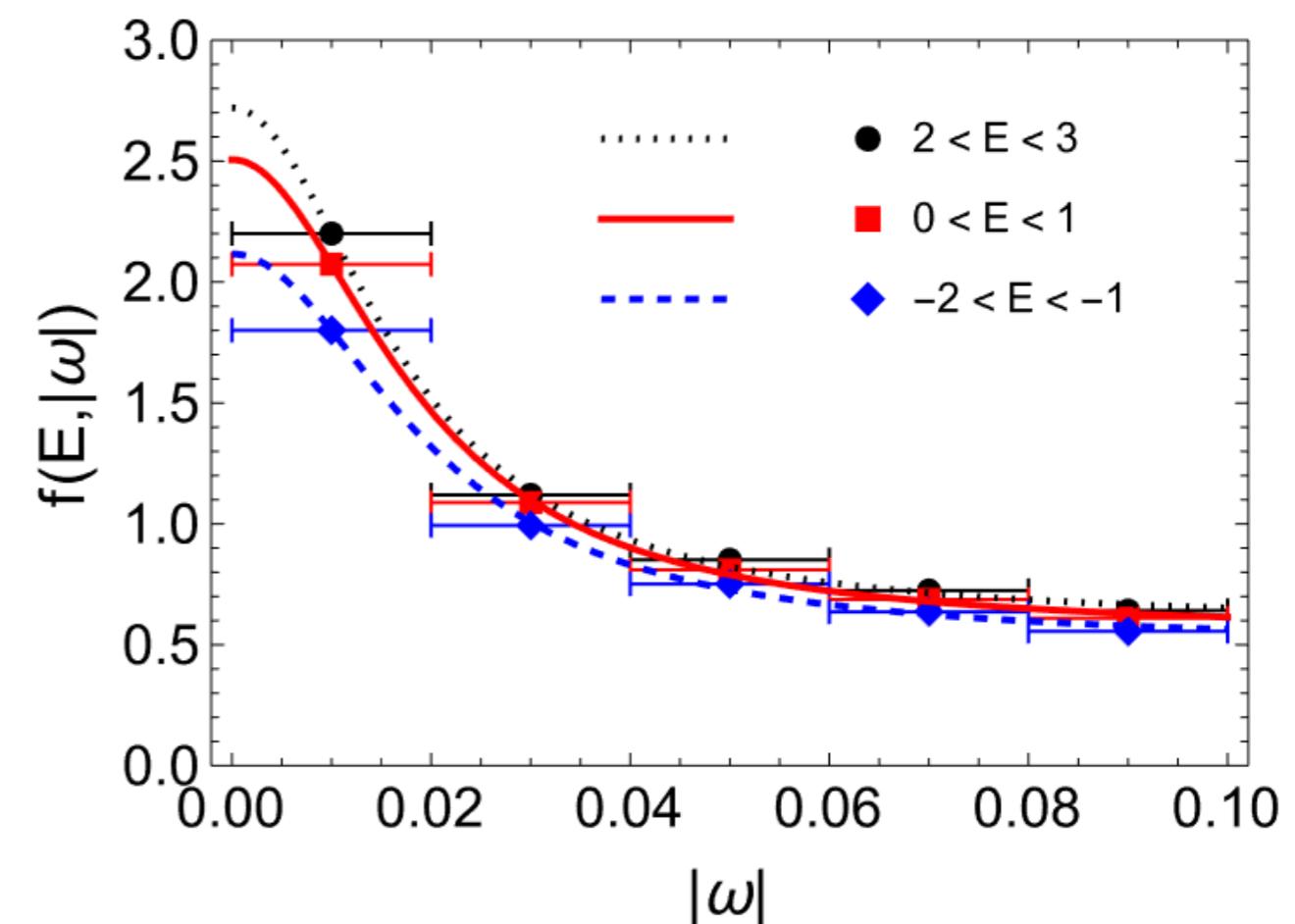
Exponential decay
with system size
when it is large

Off-Diagonal Part Test on Chain with $j_{\max} = 0.5$

- Consider off-diagonal part $M_{mn} \equiv \langle m | H_{\text{el}} | n \rangle$



Well described
by Gaussian

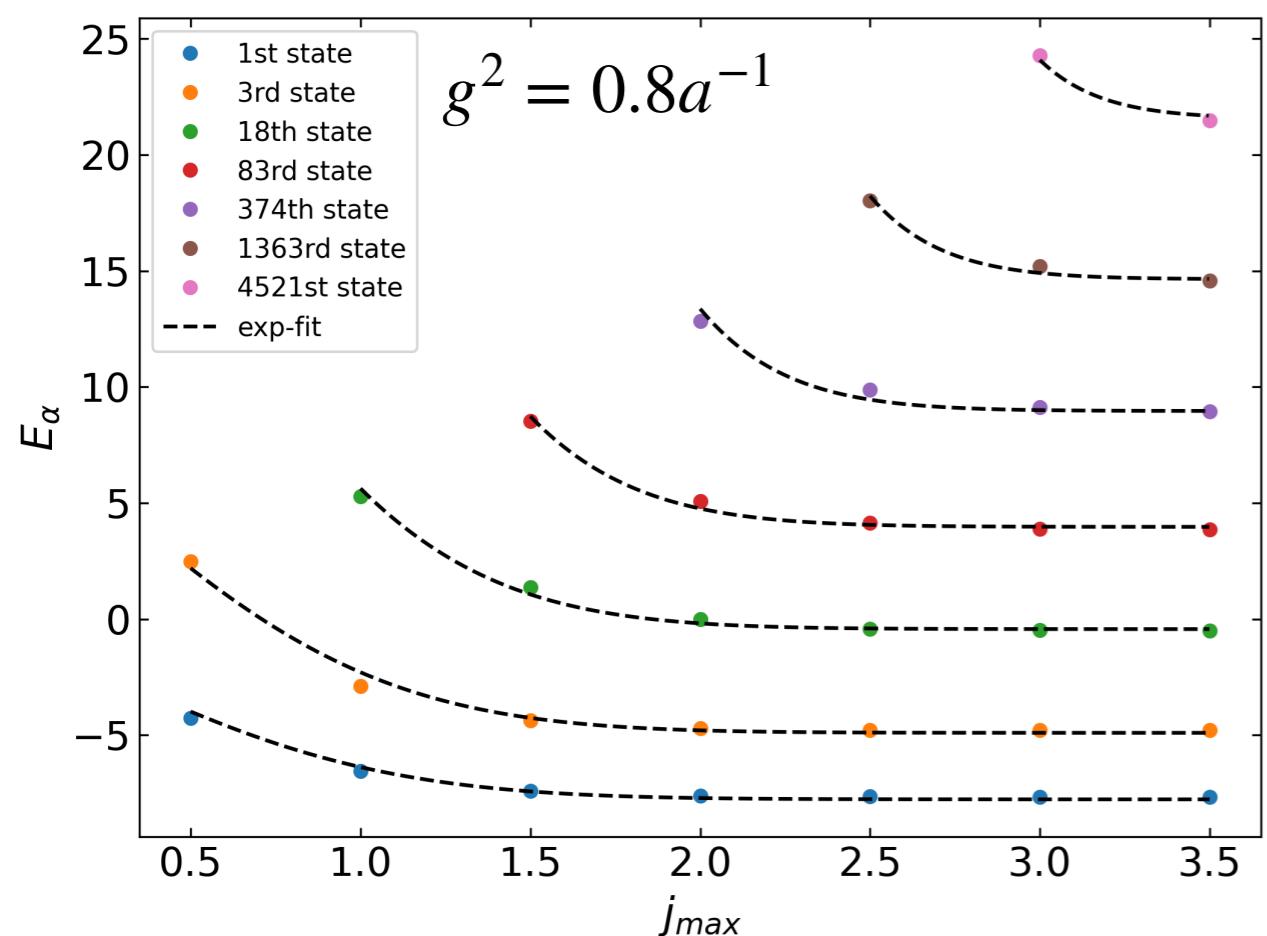


Spectral function at small $|\omega|$
well fitted by

$$f(E, \omega) = \frac{a}{\omega^2 + b^2} + c$$

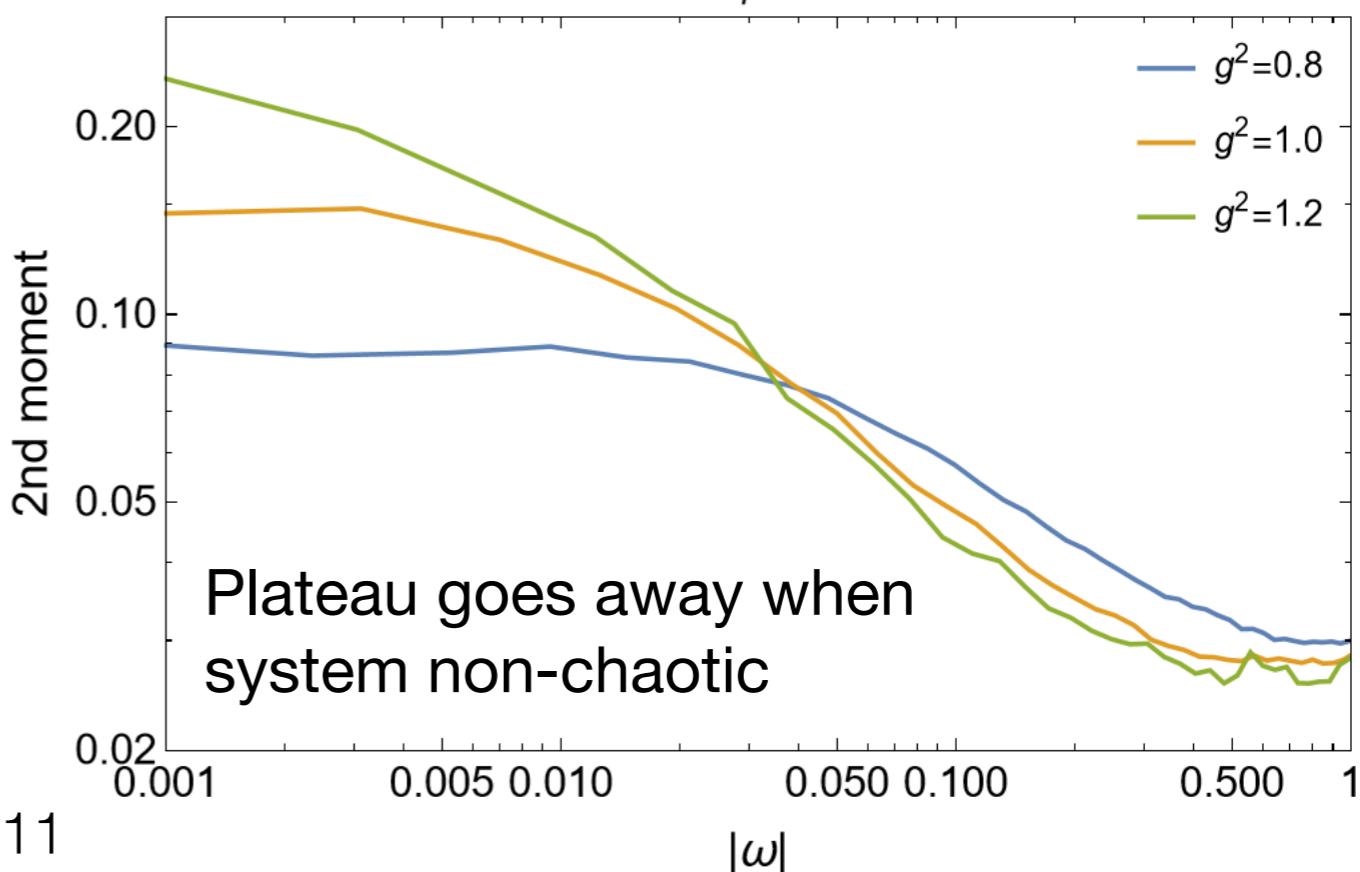
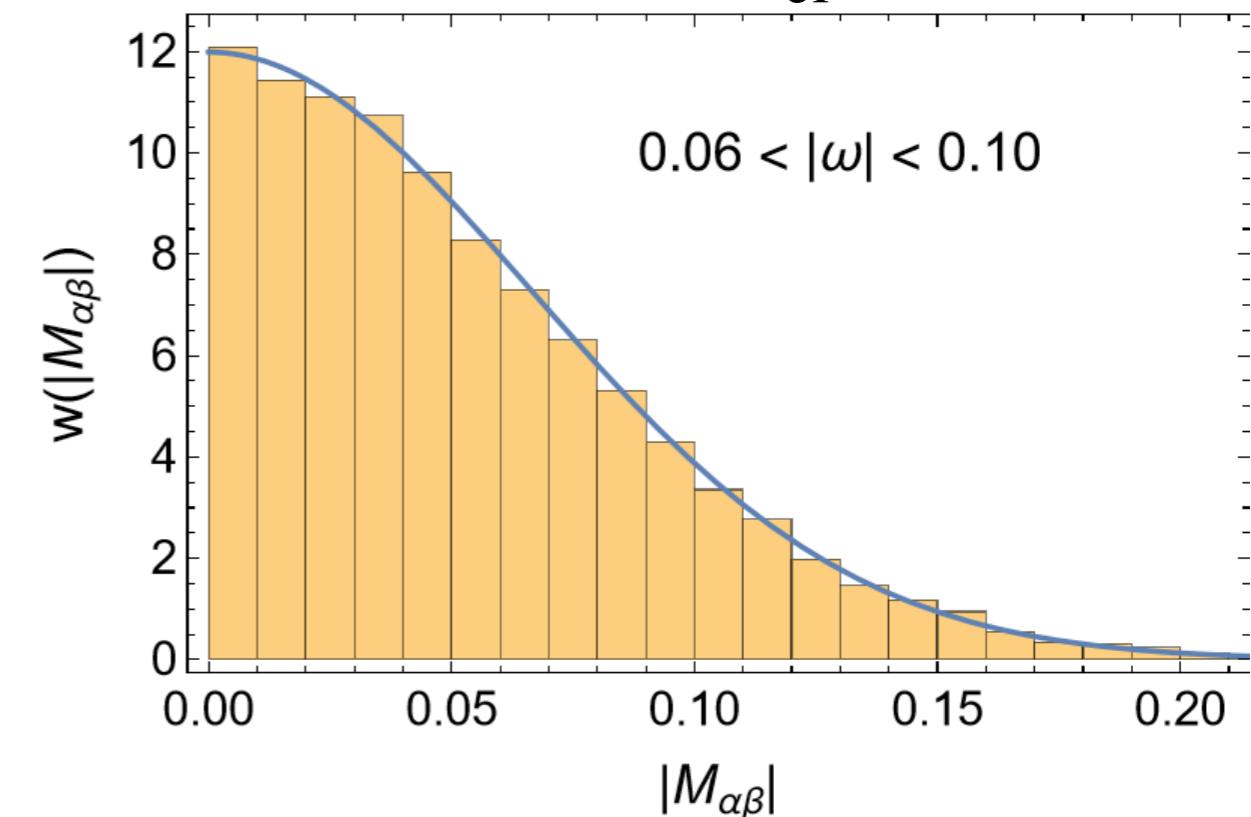
j_{\max} Cutoff Dependence and Convergence

Eigenenergy on N=3 chain v.s. j_{\max}



Take $j_{\max} = 3.5$, only use states within 5% error from asymptotic eigenenergy values

Off-diagonal of H_{el} is Gaussian



Test Random Matrix Theory Behavior

- Take band matrix by dropping elements less important at later time

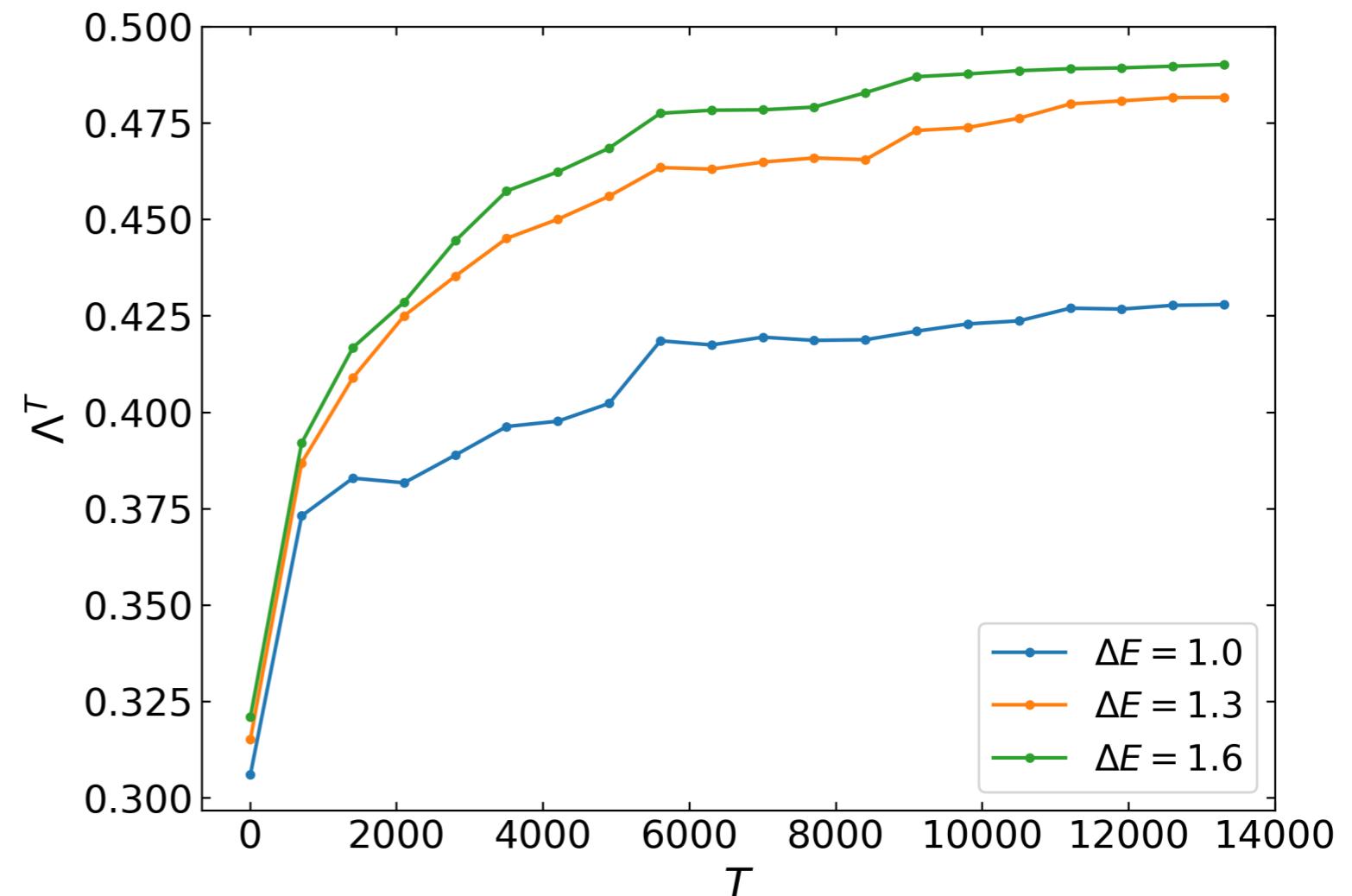
$$O_{mn}^T = \begin{cases} \langle m | O | n \rangle, & |E_m - E_n| \leq \frac{2\pi}{T} \\ 0, & |E_m - E_n| > \frac{2\pi}{T} \end{cases}$$

RMT measure

$$O_c^T = O^T - \text{Tr}[O^T]/d$$

$$\Lambda^T = \frac{(\text{Tr}[(O_c^T)^2])^2}{d (\text{Tr}[(O_c^T)^4])}$$

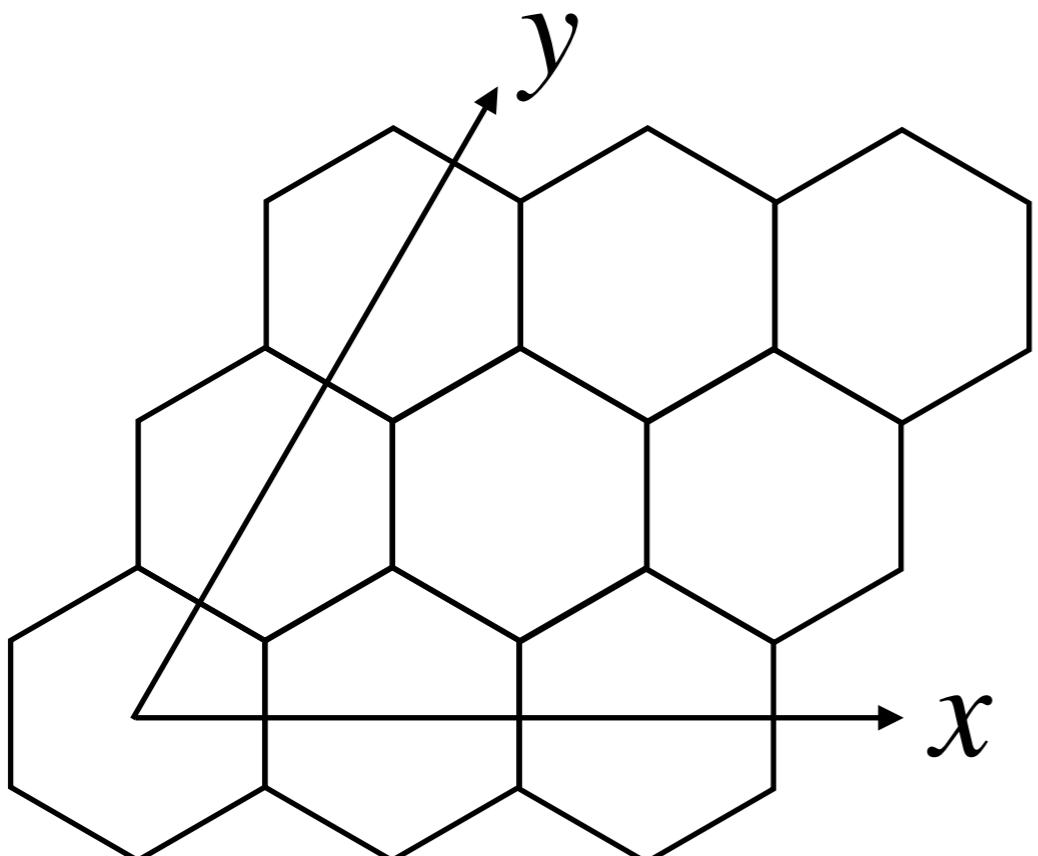
For Gaussian Orthogonal Ensemble (GOE), $\Lambda^T = 0.5$



We see GOE behavior in small ω window when statistics is good

2+1D SU(2) on Plaquette Plane

- Problem: on square lattice each vertex has four links and singlet is **not uniquely** defined by four j values
- **Solution: use honeycomb lattice**



$$H_{\text{el}} = \frac{g^2}{2} \frac{3\sqrt{3}}{2} \sum_n \sum_{i=1}^3 E_i^2(n)$$

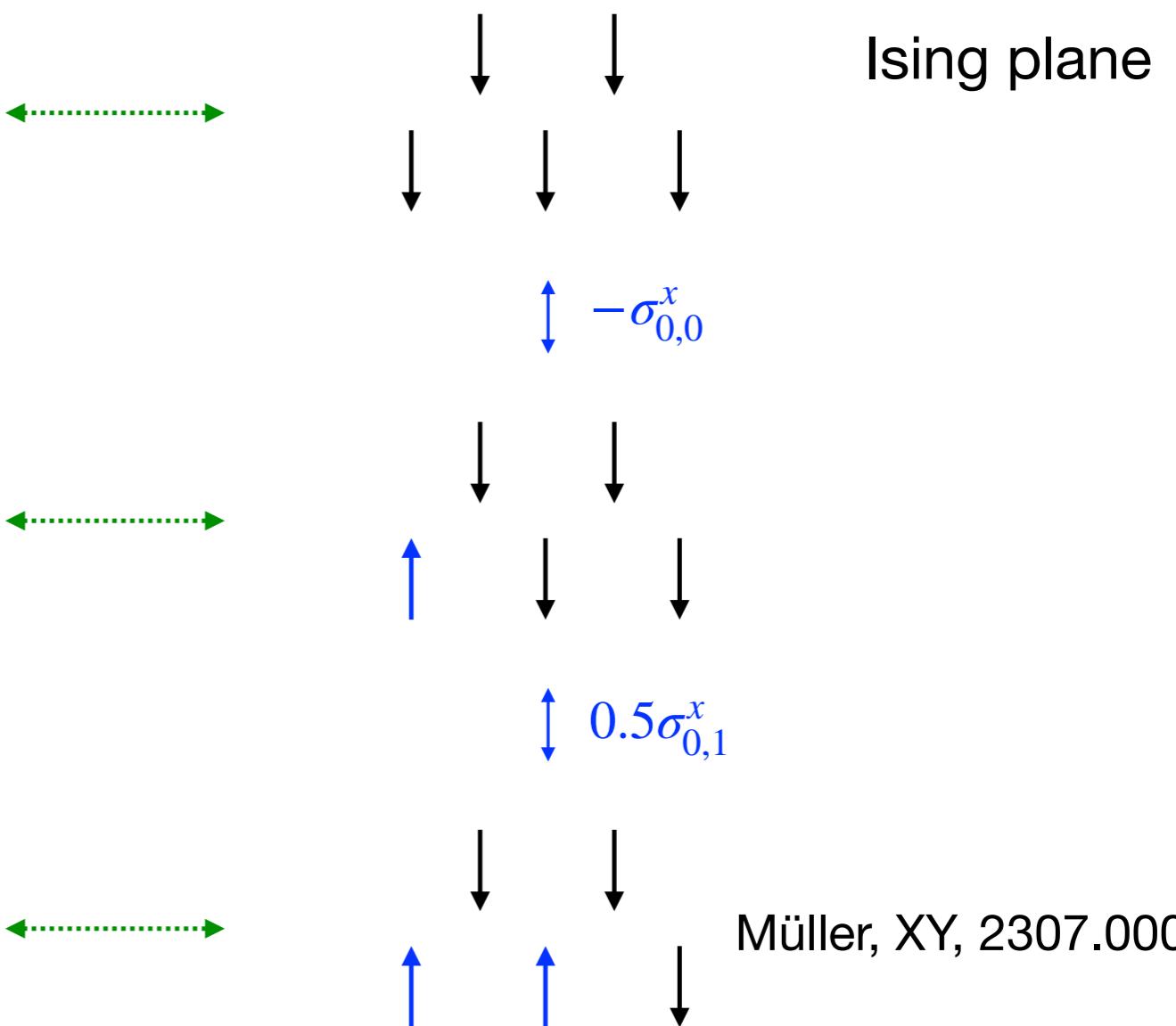
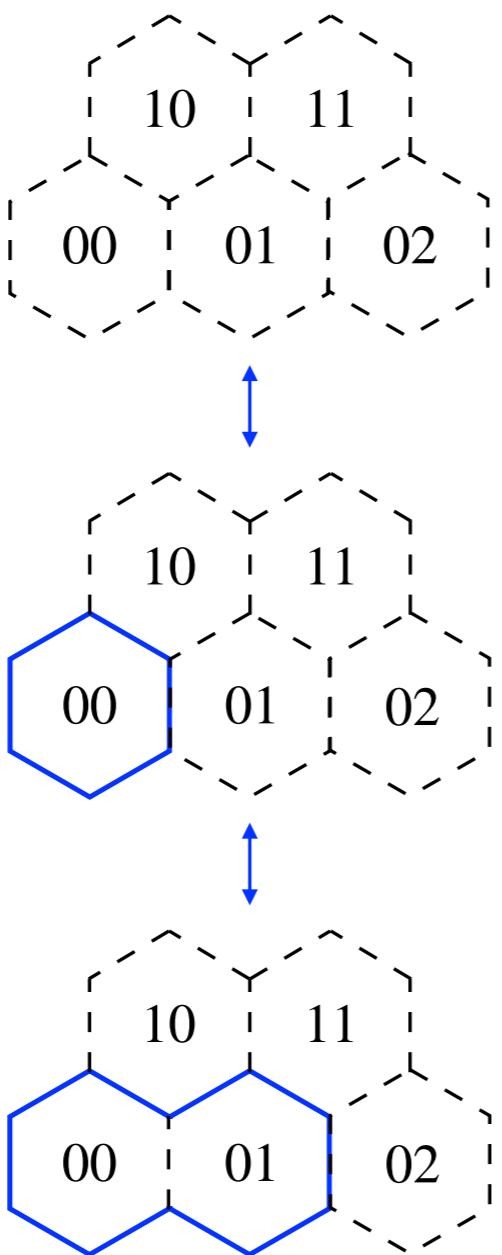
$$H_{\text{mag}} = -\frac{4\sqrt{3}}{9a^2 g^2} \sum_n \langle \text{hexagon}(n) \rangle$$

$\langle J_i | \text{hexagon} | j_i \rangle$ between physical states

Müller, XY, 2307.00045

Simplify Hamiltonian with $j_{\max} = 0.5$

$SU(2)$ w/
 $j_{\max} = 0.5$



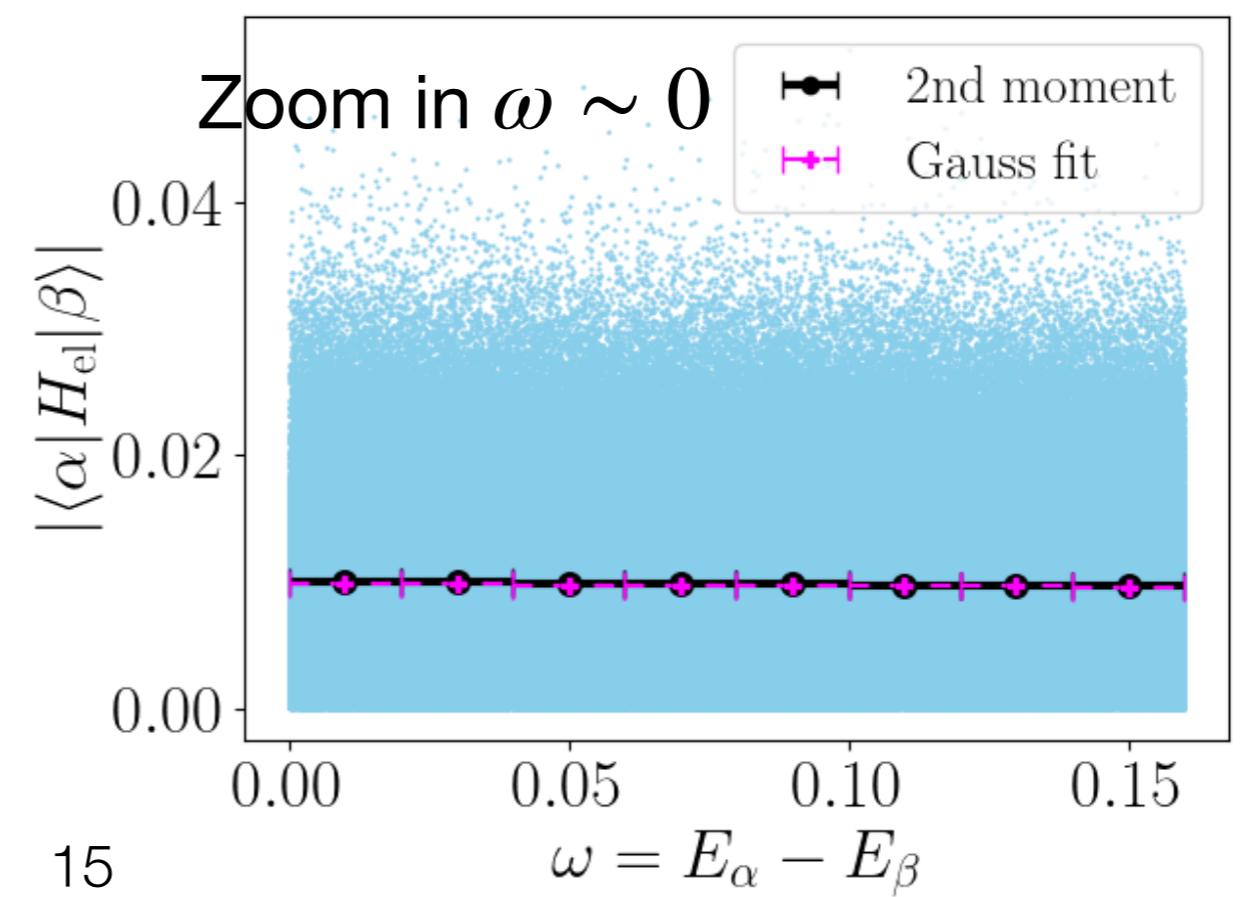
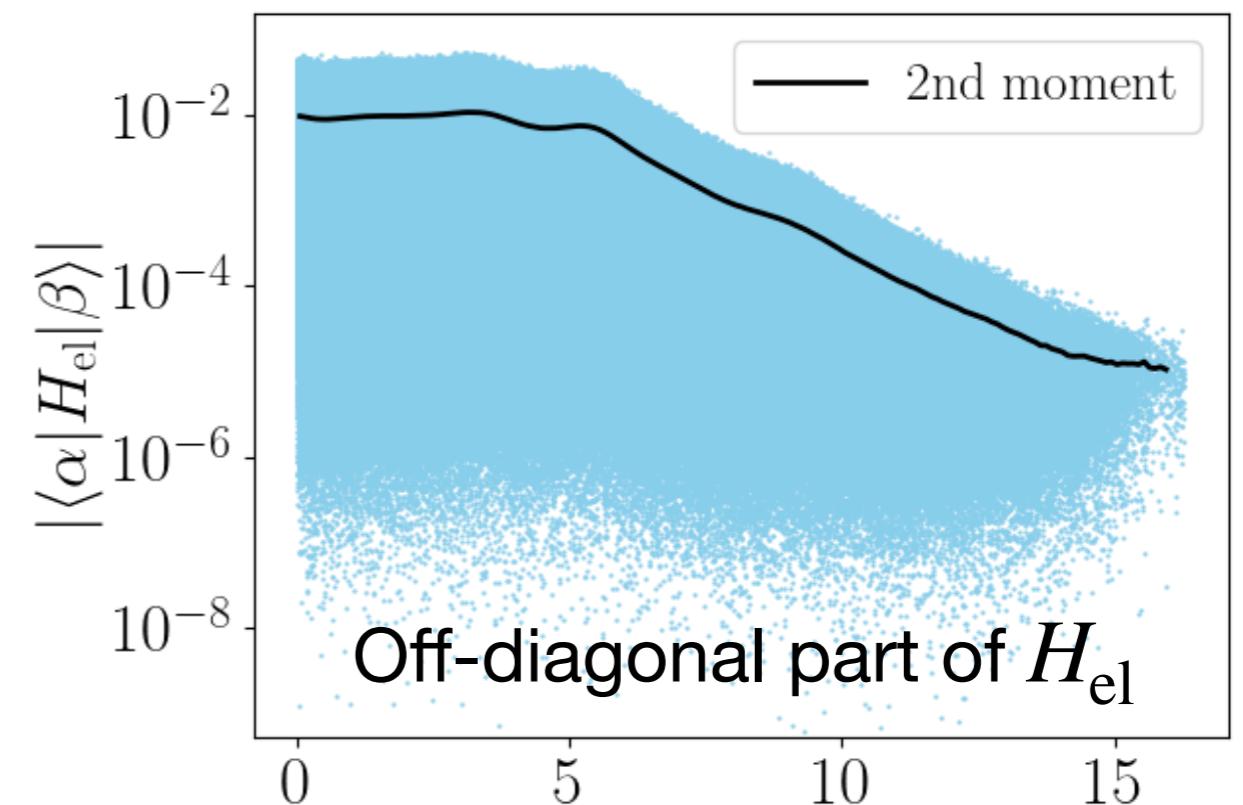
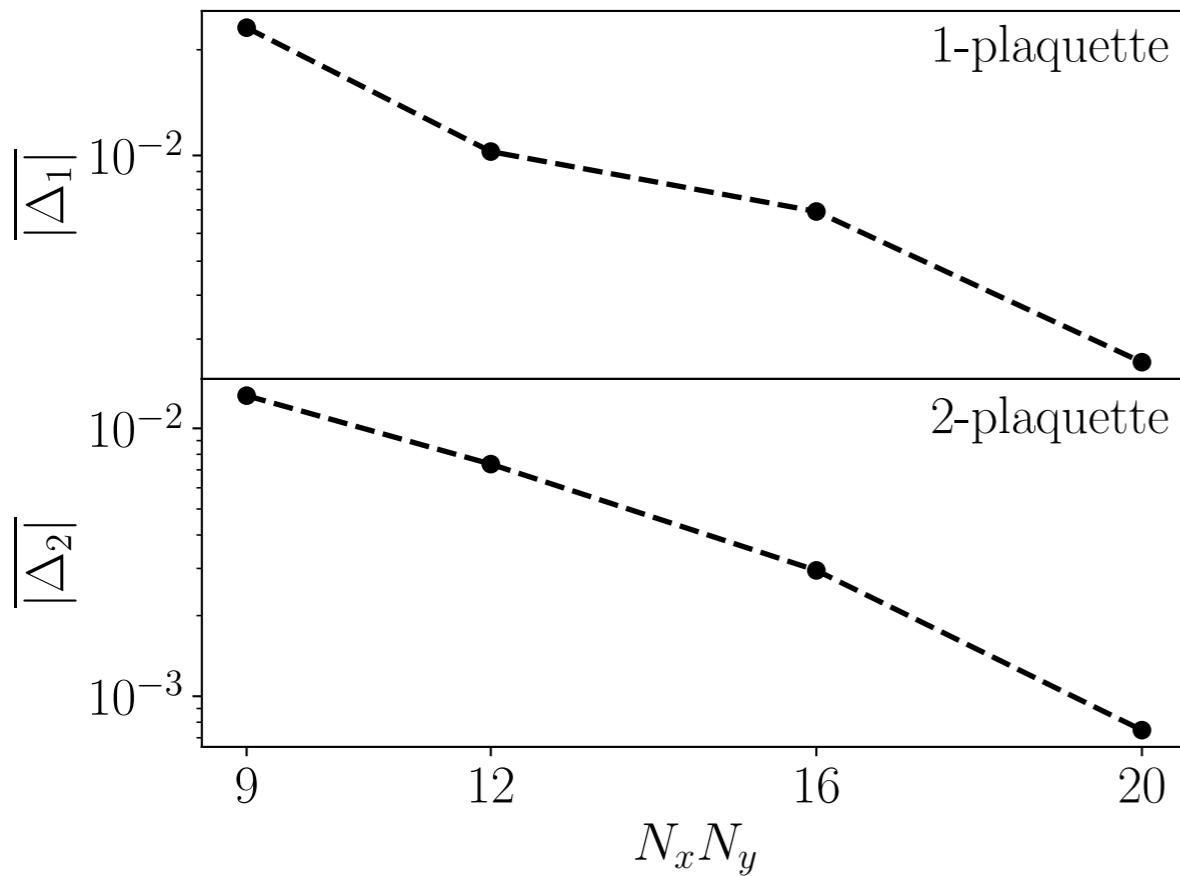
$$aH = h_+ \sum_{(i,j)} \Pi_{i,j}^+ - h_{++} \sum_{(i,j)} \Pi_{i,j}^+ (\Pi_{i+1,j}^+ + \Pi_{i,j+1}^+ + \Pi_{i+1,j-1}^+) + h_x \sum_{(i,j)} (-0.5)^{c_{i,j}} \sigma_{i,j}^x$$

$$\Pi_{i,j}^+ = (1 + \sigma_{i,j}^z)/2 \quad h_+ = \frac{27\sqrt{3}}{8} ag^2, \quad h_{++} = \frac{9\sqrt{3}}{8} ag^2, \quad h_x = \frac{4\sqrt{3}}{9ag^2}$$

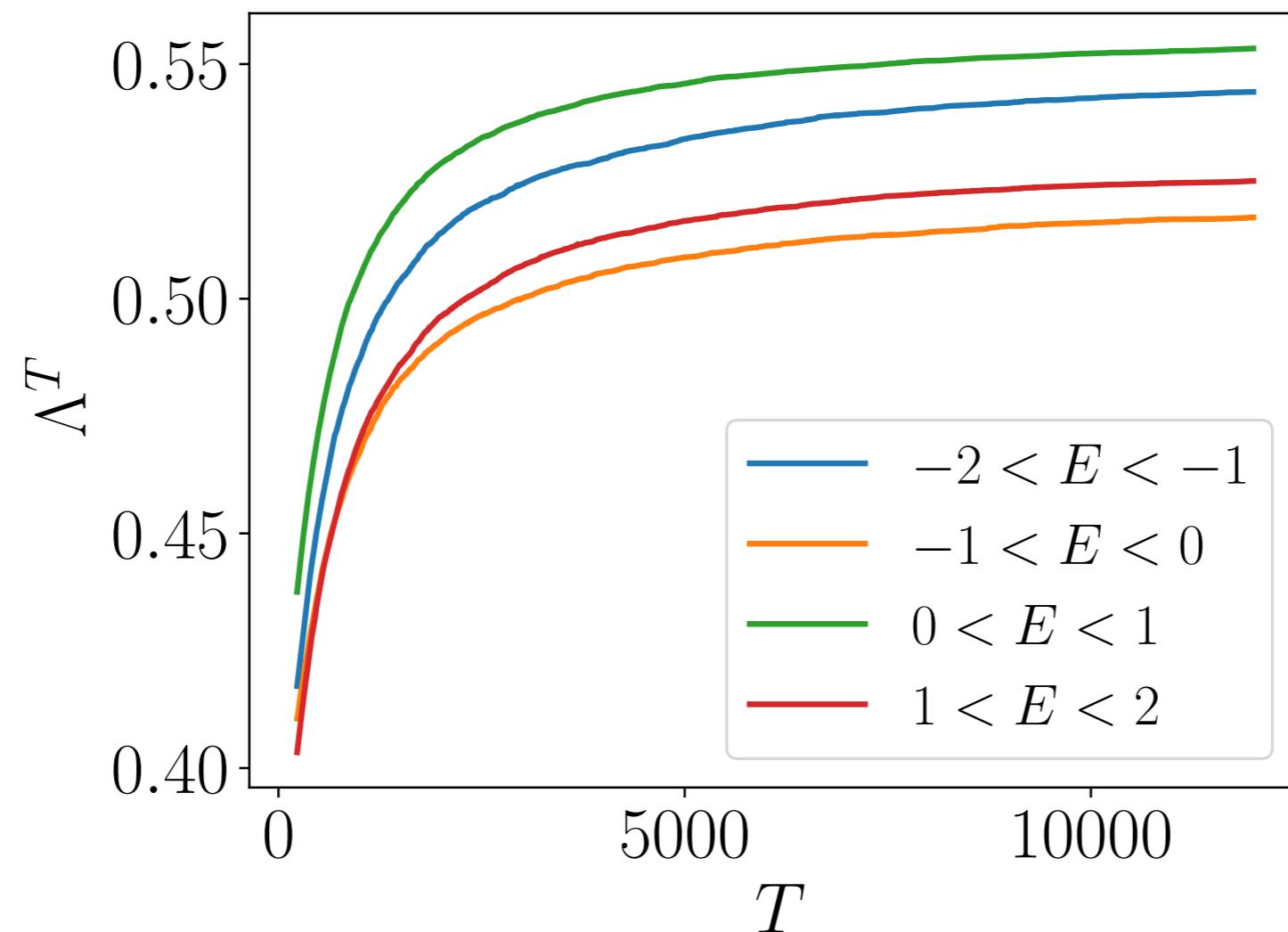
ETH Test on Honeycomb Lattice with $j_{\max} = 0.5$

- Consider $ag^2 = 0.8$

Diagonal part test for operators made up of a few plaquettes



Test RMT Behavior on Honeycomb Lattice



Within 10% of the RMT value 0.5 for several energy windows

Off-diagonal magnitude is well described by Gaussian random variables, but orthogonality is not perfect (sign correlation)

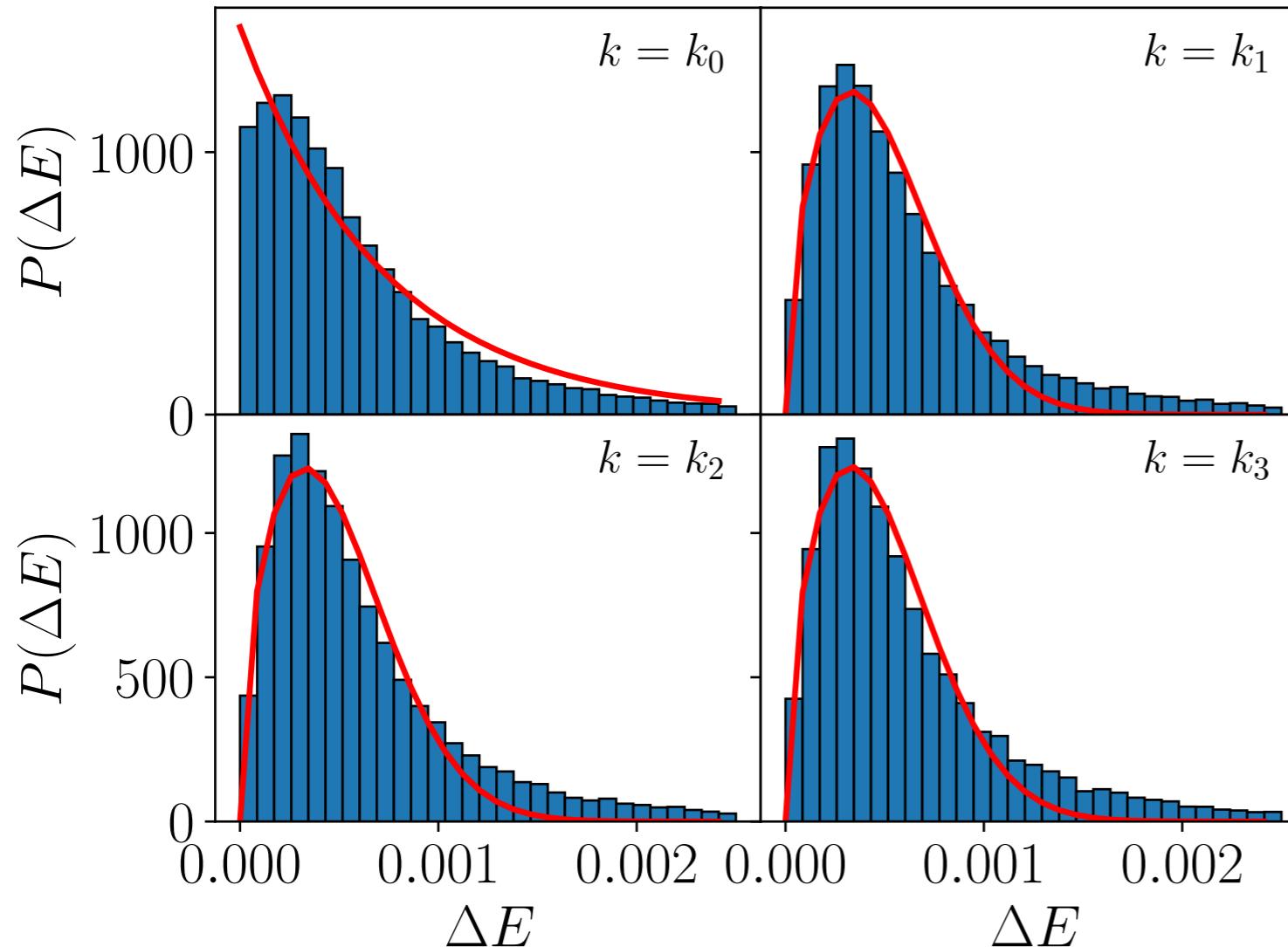
Summary and Future Plans

- Provided numerical evidence of ETH for 2+1D SU(2) lattice gauge theory
 - (1) long chain with $j_{\max} = 0.5 \checkmark$
 - (2) short chain with $j_{\max} = 3.5$ and converged spectrum \checkmark
 - (3) 2D honeycomb with $j_{\max} = 0.5 \checkmark$
- Future plan: (1) 2D honeycomb with higher j_{\max}
 - (2) 3+1D SU(2)
 - (3) SU(3) and include fermions
 - (4) Implementation on a **quantum computer**
- Final remark: ETH in terms of eigenstates, satisfied by quantum system whose classical counterpart is **chaotic**

Another quantum chaos indicator = Bohigas-Giannoni-Schmit conjecture:
level separation statistics described by GOE —> Wigner-Dyson statistics

Backup: Level Separation Statistics

- SU(2) plaquette chain with $N = 19$, look at neighboring eigenenergy gaps

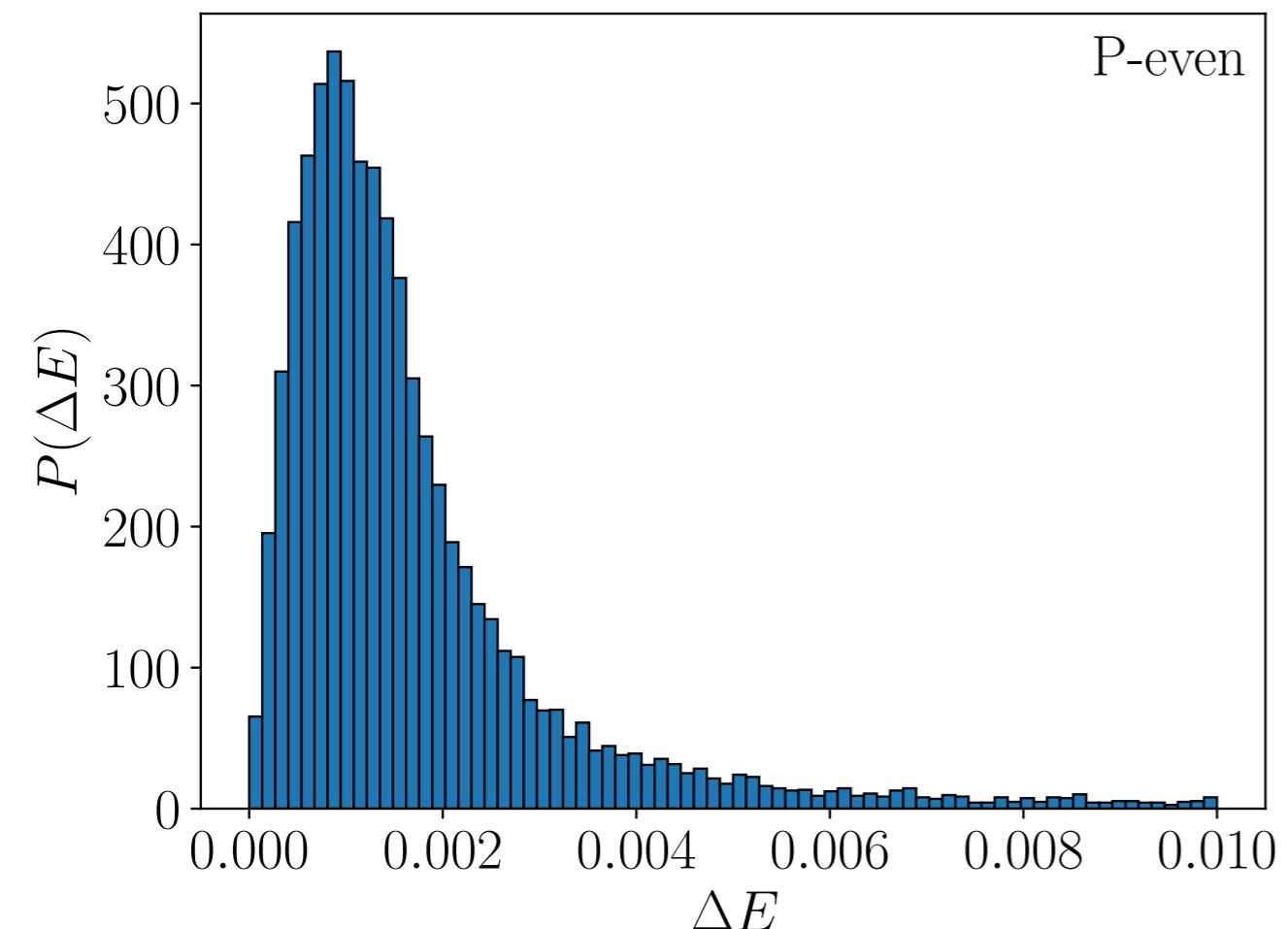


Different momentum sector $k_j = 2\pi j/N$

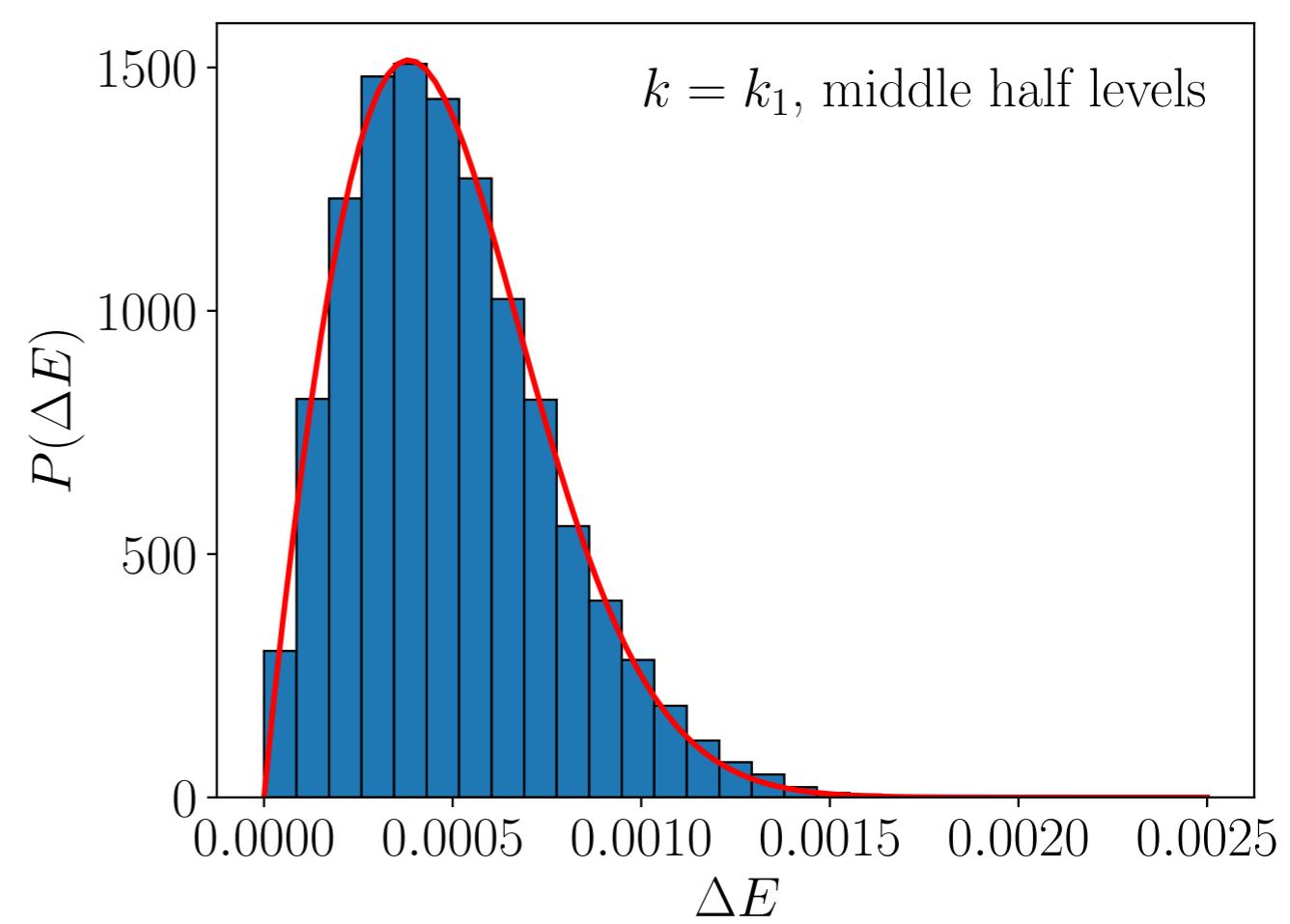
k=0 sector has remaining discrete symmetry (parity), so level statistics are more Poisson-like. But ETH still works fine.

$$P_{\text{ws}}(\Delta E) = a(\Delta E)^b \exp[-c(\Delta E)^2] \longrightarrow \text{Level repulsion in Wigner-Dyson statistics}$$

Backup: Level Separation Statistics



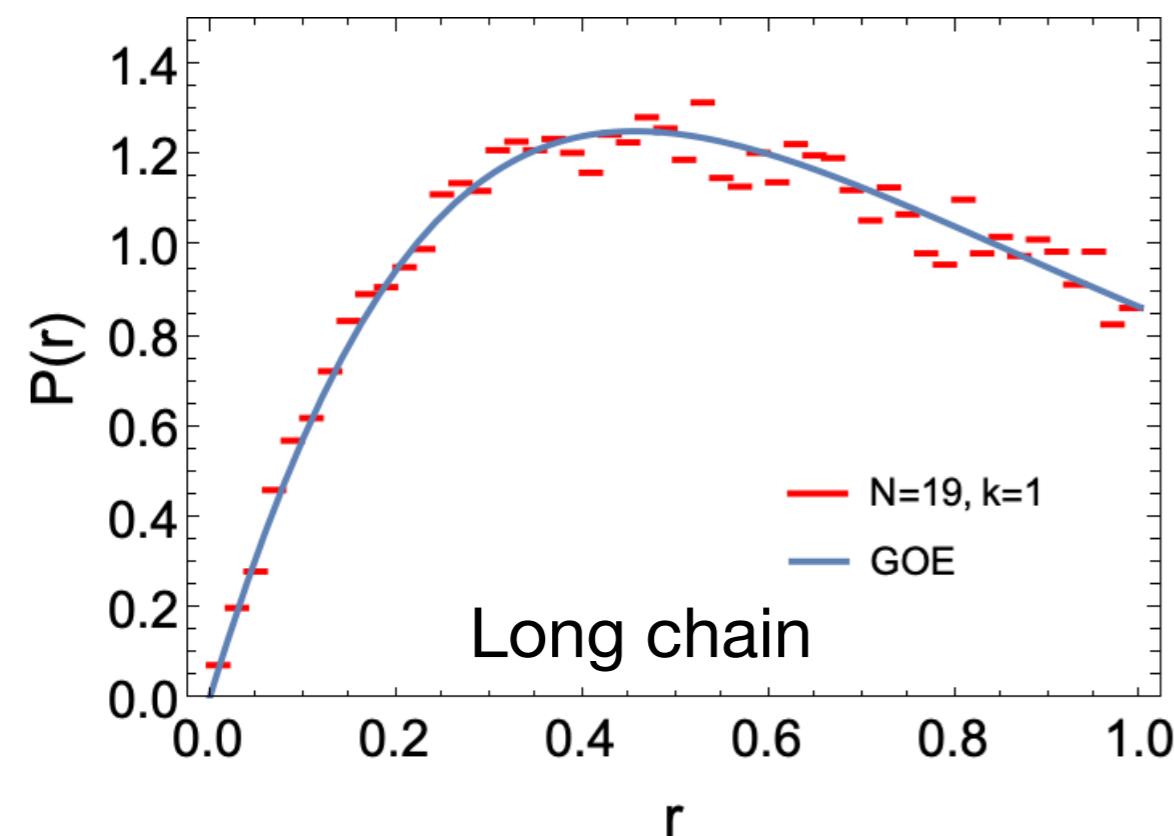
Parity-even sector in $k=0$



Middle half of the spectrum

Backup: restricted level gap ratio

$$0 < r_\alpha = \frac{\min[\delta_\alpha, \delta_{\alpha-1}]}{\max[\delta_\alpha, \delta_{\alpha-1}]} \leq 1$$

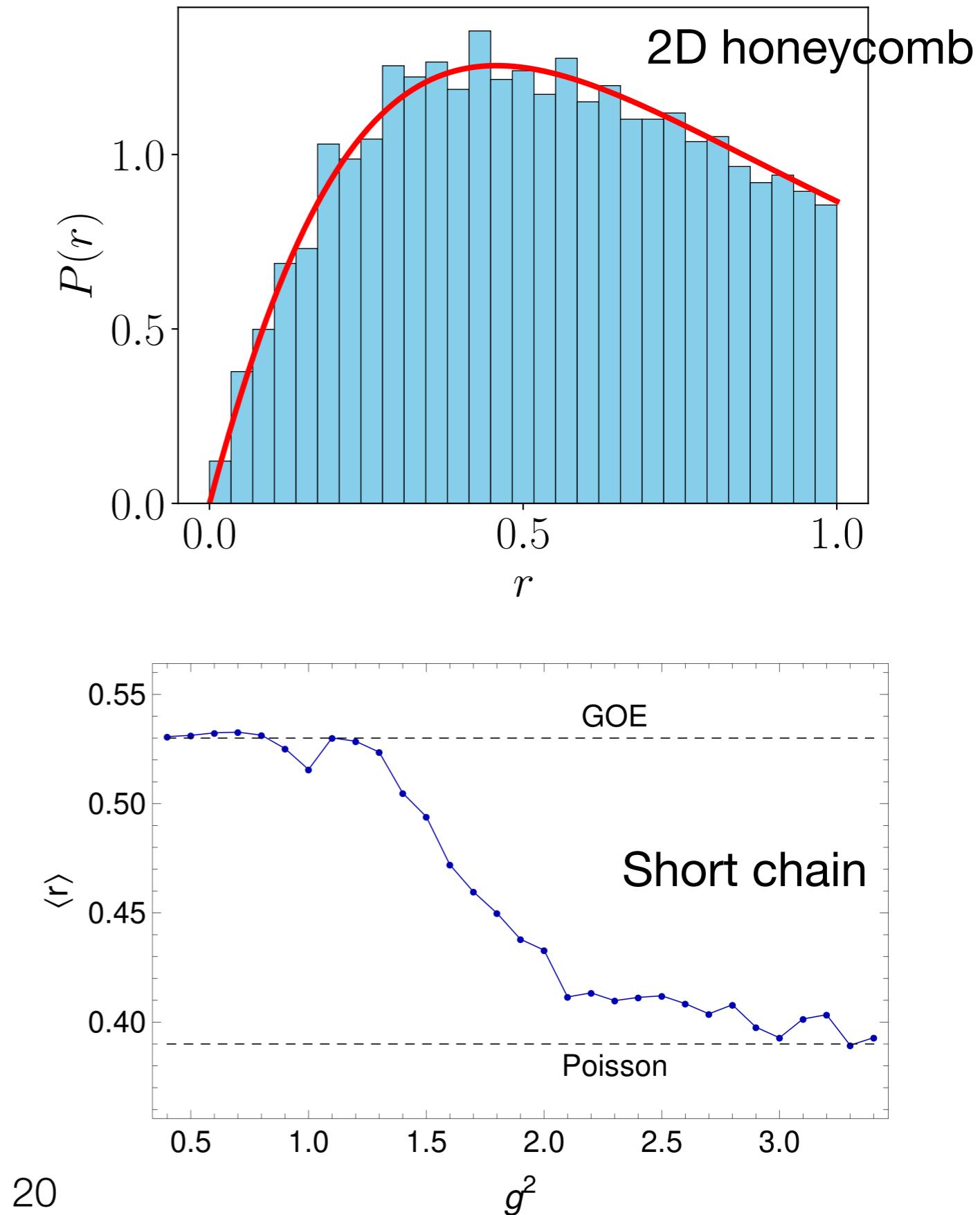


Long chain

$$\langle r \rangle_{\text{GOE}} = 0.5307$$

$$\langle r \rangle_{\text{long chain}} = 0.5340$$

$$\langle r \rangle_{\text{honeycomb}} = 0.5312$$



2D honeycomb

GOE

Short chain

Poisson