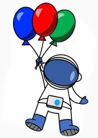
# QCD equation of state with improved precision from lattice simulations



**Paolo Parotto**, Pennsylvania State University

September 5, 2023

**Quark Matter 2023, Houston, TX** ♡

**Wuppertal-Budapest**:

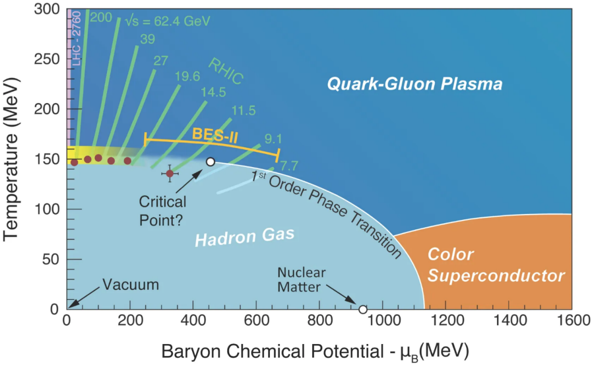
S. Borsányi, Z. Fodor, J. N. Guenther, R. Kara, A. Pásztor, C. Ratti, K. K. Szabó

**Note:** references in the talk are links to papers

# The equation of state of QCD

From a combination of approaches (experiment, models, **lattice QCD calculations**, ...), we have *some knowledge* of the QCD phase diagram

- Ordinary nuclear matter at $T \simeq 0$ and $\mu_B \simeq 922\,\mathrm{MeV}$
- Deconfinement transition at $\mu_B = 0$ is a smooth crossover at $T \simeq 155 - 160\,\mathrm{MeV}$
- Transition line at finite $\mu_B$ is known to some precision
- Expansion(s) up to $\mu_B/T \simeq 2 - 3.5$
- Critical point? Exotic phases?




The equation of state (EoS) of QCD is invaluable.

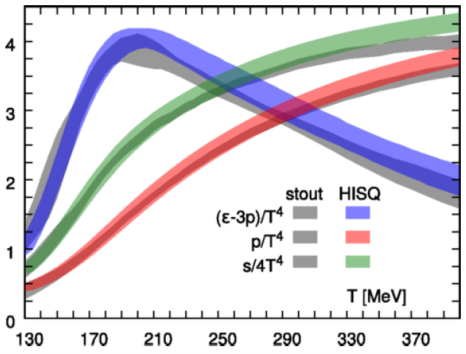
Knowing it would mean we can *really* draw the phase diagram of QCD.

# The equation of state of QCD

- Highly demanded in heavy-ion collisions community, e.g. for hydrodynamics
- **Lattice QCD is the most robust tool to determine QCD thermodynamics**
- Known at $\mu_B = 0$ to high precision for a few years now (continuum limit, physical quark masses) $\longrightarrow$ Agreement between different calculations (2013-2014)

From grancanonical partition function $\mathcal{Z}$

- **Pressure**: $p = -k_B T \frac{\partial \ln \mathcal{Z}}{\partial V}$
- **Entropy density**: $s = \left(\frac{\partial p}{\partial T}\right)_{\mu_i}$
- **Charge densities**: $n_i = \left(\frac{\partial p}{\partial \mu_i}\right)_{T,\mu_{j \neq i}}$
- **Energy density**: $\epsilon = Ts - p + \sum_i \mu_i n_i$
- More (**Fluctuations**, etc...)




**WB: Borsányi+ '13, HotQCD: Bazavov+ '14**

# Finite density: the sign/complex action problem

Euclidean path integrals on the lattice are calculated with MC methods using importance sampling, interpreting the factor $\det M[U]\, e^{-S_G[U]}$ as the Boltzmann weight for the configuration $U$

$$Z(V,T,\mu) = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-S_F(U,\psi,\bar{\psi})-S_G(U)}$$

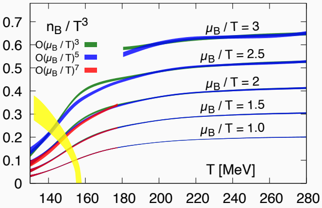
$$= \int \mathcal{D}U\; \det M(U) e^{-S_G(U)}$$

- If there is particle-antiparticle-symmetry ($\mu = 0$) $\det M(U)$ is real
- For real chemical potential ($\mu^2 > 0$) $\rightarrow$ $\det M(U)$ is complex (**complex action problem**) and has wildly oscillating phase (**sign problem**)

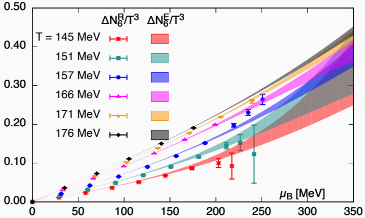
  $\Rightarrow$ It cannot serve as a statistical weight
- For *purely imaginary* chemical potential ($\mu^2 < 0$) $\rightarrow$ $\det M(U)$ is real again, simulations can be made!

# Alternatives: Taylor, analytic continuation, reweighting

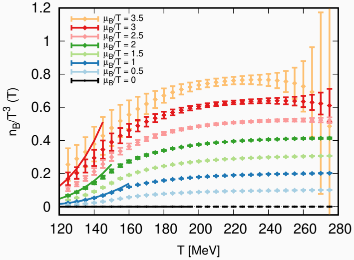
## Taylor expansion (1D) Bollweg+ '22



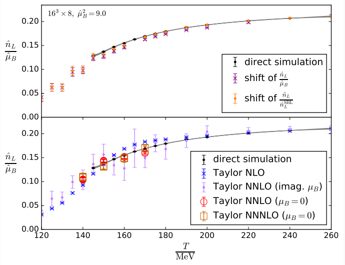

## Approximate reweighting Mondal+ '21




## Alternative expansion (2D) Borsanyi+ '21, '22




## Reweighting Borsanyi+ '22, C. H. Wong's talk



# Motivation: finite $\mu_B$ vs zero $\mu_B$ equation of state

As just seen, in recent years a great deal of attention has been given to means to determine the equation of state at finite chemical potential, with great progress (see **A. Pásztor's plenary**)

However:

**i.** New results for the equation of state at $\mu_B = 0$ have been produced, but focus is now on cosmology (**Borsányi+ '16**, **Bazavov+ '17**). Around $T_c$ no updates in $\sim 10$ years

**ii.** **No matter what method is used** to extrapolate or directly simulate at finite $\hat{\mu}_B$, the **equation of state at** $\mu_B = 0$ **is always needed**

**iii.** Except for large-ish $\mu_B/T \gtrsim 2 - 2.5$, the equation of state at $\mu_B = 0$ is (was) the main source of uncertainty

**iv.** LHC physics ($\mu_B \simeq 0$) is getting more and more precise

**Our goal:** dramatically reduce the uncertainty on the equation of state at $\mu_B = 0$

# Equation of state from the lattice

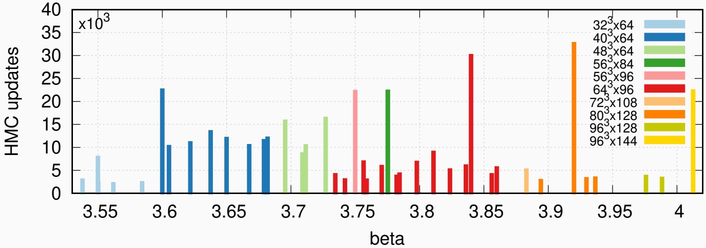
The pressure cannot be determined directly (not a derivative of $\ln Z$ wrt to a parameter), but via an integral of the trace anomaly $I(T)$:

$$\boxed{\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \frac{dT'}{T'} \frac{I(T')}{T'^4}}$$

where the **trace anomaly** $I(T)$ *can* be determined directly on the lattice:

$$\frac{I(T)}{T^4} = N_\tau^4 \left( \boxed{T > 0} - \boxed{T = 0} \right)$$

but **needs renormalization**, which means (a lot of) **simulations at $T = 0$ are needed**



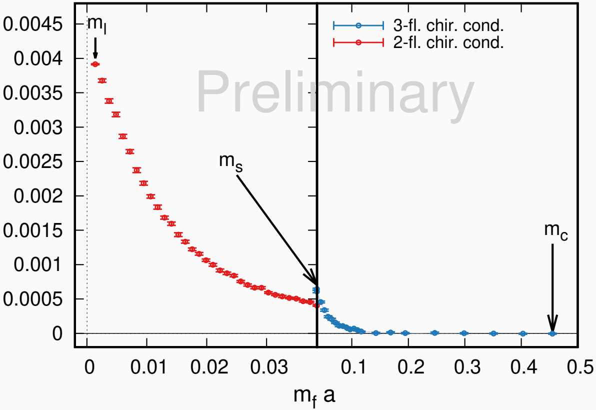
# I. Pressure constant

We calculate the integration constant $\frac{p(T_0)}{T_0^4}$ at a chosen $T_0 = 185\,\mathrm{MeV}$.

The pressure is determined as an integral in the quark masses down from infinity (where $p = 0$):

$$\frac{p(T_0)}{T_0^4} = \int_{m_s}^{m_l} dm_2 \left\langle \bar{\psi}\psi \right\rangle_{R,2}(m_2) + \int_{\infty}^{m_s} dm_3 \left\langle \bar{\psi}\psi \right\rangle_{R,3}(m_3)$$

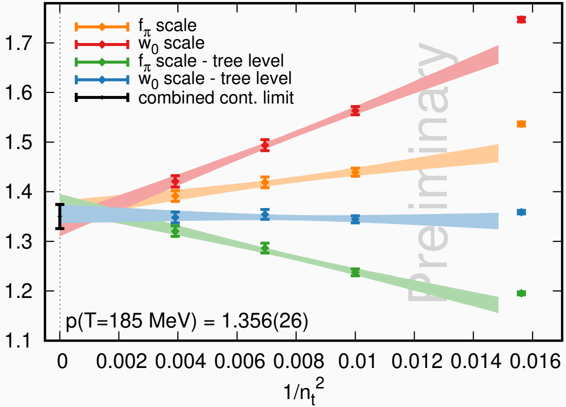
We first integrate in the two light flavours up to $m_s$, then the three flavours up to infinity by fitting an exponential.




**Note:** the chiral condensates $\left\langle \bar{\psi}\psi \right\rangle_{R,i}$ are the renormalized ones

# I. Pressure constant

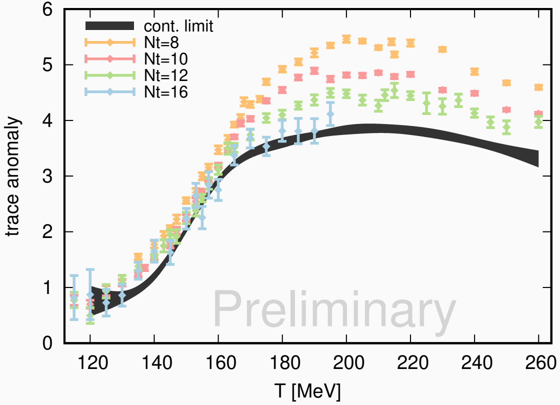
We calculate the integration constant with two settings of the scale, with and without applying the tree level improvement on the observables → 4x systematics




For the first time, we have results for up to $N_t = 16$ which allow us to discard $N_t = 8$ in the continuum extrapolation. ⇒ <u>$\sim$ 2x improvement in uncertainty</u>

# II. Trace anomaly

We determine on our lattices $32^3{\times}8$, $40^3{\times}10$, $48^3{\times}12$, $64^3{\times}16$ the trace anomaly:




then perform a global continuum extrapolation + spline fit in $T$.

# Equation of state at $\hat{\mu}_B = 0$

Now we have both ingredients to determine the equation of state at $\mu_B = 0$, as shown previously:

$$\boxed{\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \frac{dT'}{T'} \frac{I(T')}{T'^4}}$$

From the pressure, the other quantities follow. At $\mu_B = 0$, normalized quantities $\hat{O}(T)$:

$$\hat{s} = 4\hat{p}(T) + T\frac{d\hat{p}(T)}{dT}$$

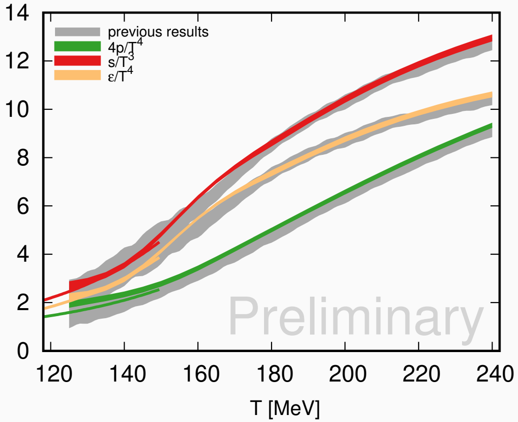
$$\hat{\epsilon}(T) = \hat{s}(T) - \hat{p}(T)$$

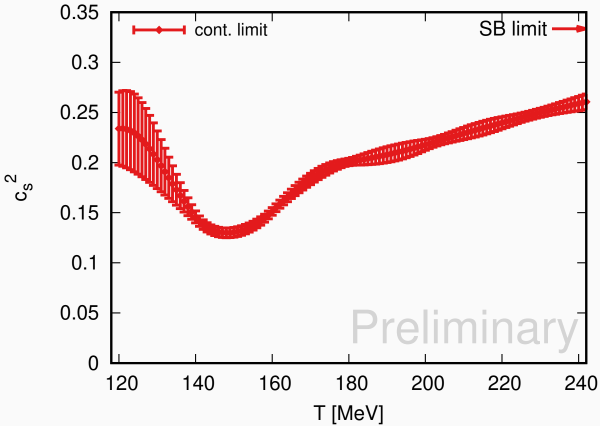
$$c_s^2(T) = \frac{\hat{I}(T) + 4\hat{p}(T)}{7\hat{I}(T) + 12\hat{p}(T) + T\frac{d\hat{I}(T)}{dT}}$$

# Equation of state at $\hat{\mu}_B = 0$

We can compare the resulting equation of state at $\mu_B = 0$ to our previous result from 2014

**Borsányi+ '14**

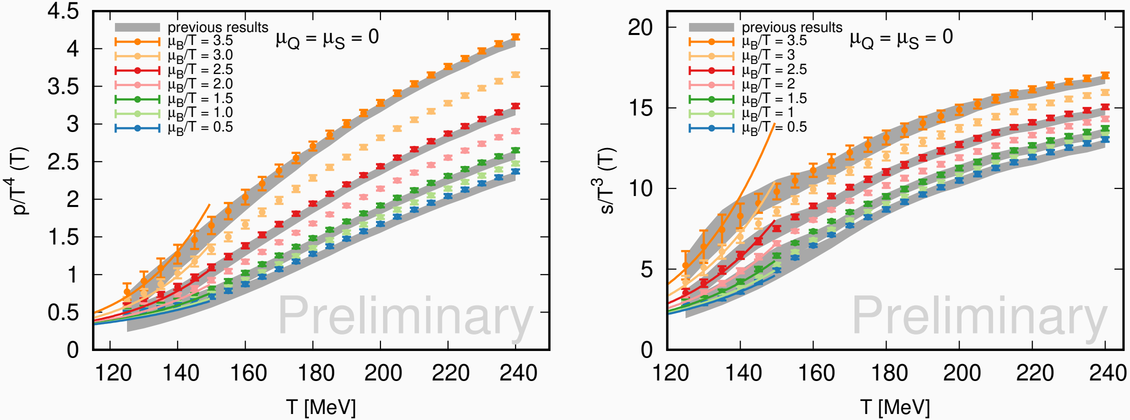



**Note:** full systematics analysis still in the making, more statistics coming at $N_\tau = 16$

# Equation of state at finite $\hat{\mu}_B$

We get the equation of state at finite $\mu_B$ with our expansion scheme **Borsanyi+ '21, '22**.
Here pressure and entropy density up to $\hat{\mu}_B = 3.5$ for $\mu_Q = \mu_S = 0$:




A substantial improvement in the error is seen up to $\mu_B/T \simeq 2 - 2.5$

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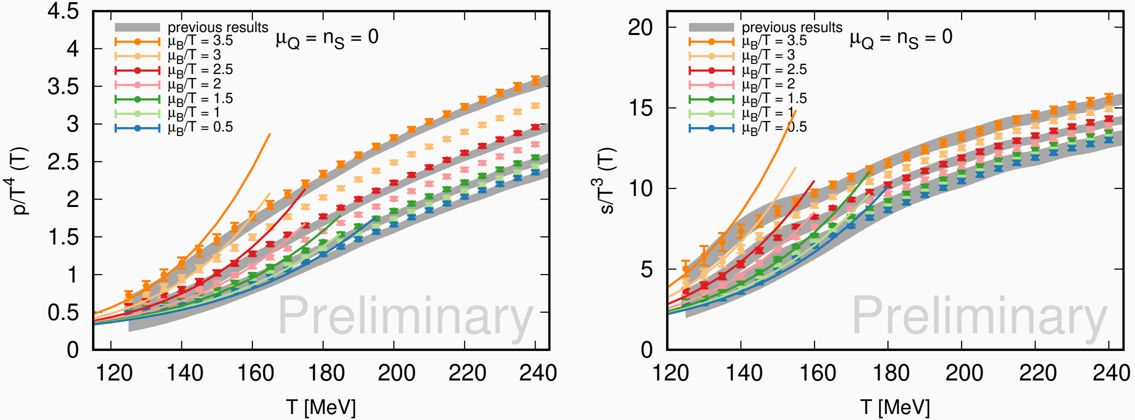


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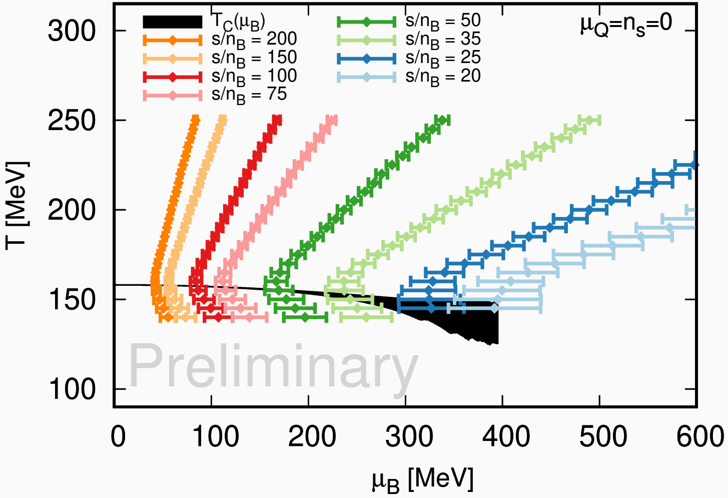
# Equation of state at finite $\hat{\mu}_B$: isentropes

We now have isentropic lines at very large $\mu_B$ with small errors $\Rightarrow$ **no critical lensing**



# Summary

- The QCD equation of state at zero and finite chemical potential is highly demanded in the heavy-ion collisions community, e.g. for hydrodynamic simulations
- **Regardless of the method** used to calculate the equation of state at finite $\mu_B$, its $\mu_B = 0$ determination is necessary and independent
- Thanks to our sizeable zero-T program and increased finite-T statistics, **we reached unprecedented precision in the zero-density equation of state**
- The precision is also increased at finite $\mu_B$, up to $\mu_B/T \simeq 2 - 2.5$

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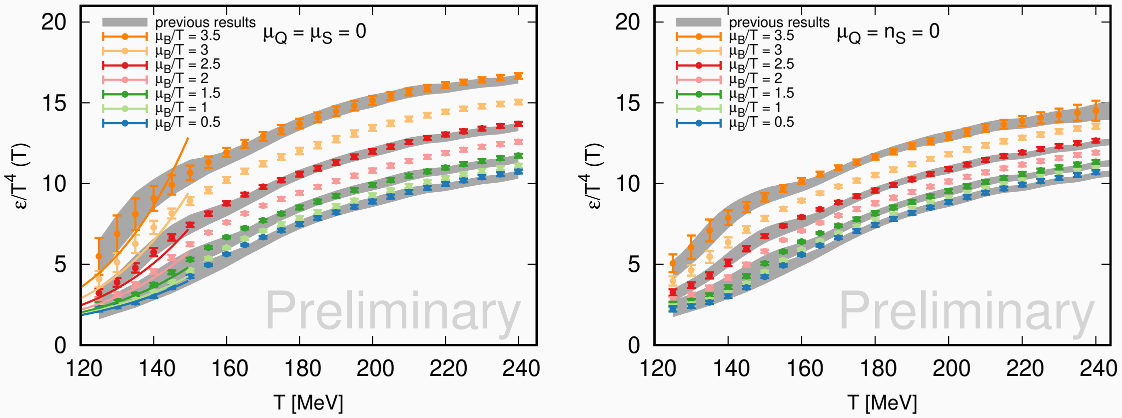
**THANK YOU!**

# BACKUP

# Equation of state at finite $\hat{\mu}_B$

Energy density up to $\hat{\mu}_B = 3.5$ with $\mu_S = 0$ and $n_S = 0$



# Equation of state from the lattice

Pressure is not determined directly:

$$\boxed{\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \frac{dT'}{T'} \frac{I(T')}{T'^4}}$$

Trace anomaly *is* determined directly on the lattice

$$\frac{I(T)}{T^4} \frac{dT}{T} = N_\tau^4 \left( d\beta \left\langle -s_G \right\rangle_R + \sum_f dm_f \left\langle \bar{\psi}_f \psi_f \right\rangle_R \right)$$

with gauge coupling $\beta = 6/g^2$ and fermion masses $m_f$.

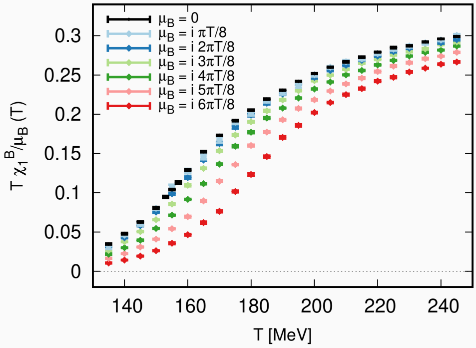
The **gauge action** $\langle -s_G \rangle$ and **chiral condensates** $\langle \bar{\psi}_f \psi_f \rangle$ require renormalization:

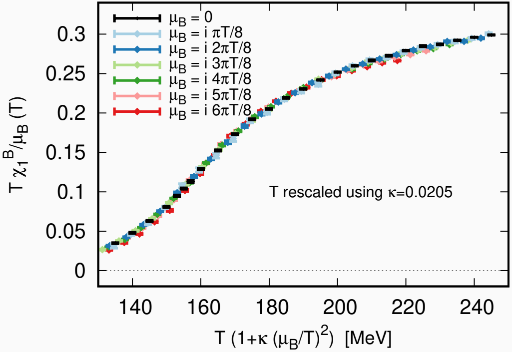
$$\langle -s_G \rangle_R = \langle -s_G \rangle_T - \langle -s_G \rangle_0$$
$$\left\langle \bar{\psi}_f \psi_f \right\rangle_R = \left\langle \bar{\psi}_f \psi_f \right\rangle_T - \left\langle \bar{\psi}_f \psi_f \right\rangle_0$$

# An alternative approach

From simulations at imaginary $\mu_B$ we *observe* that $\chi_1^B(T,\,\hat{\mu}_B)$ at (imaginary) $\hat{\mu}_B$ appears to be differing from $\chi_2^B(T,0)$ mostly by a rescaling of $T$:

$$\frac{\chi_1^B(T,\,\hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T',0)\ , \quad T' = T\left(1+\kappa\,\hat{\mu}_B^2\right)$$



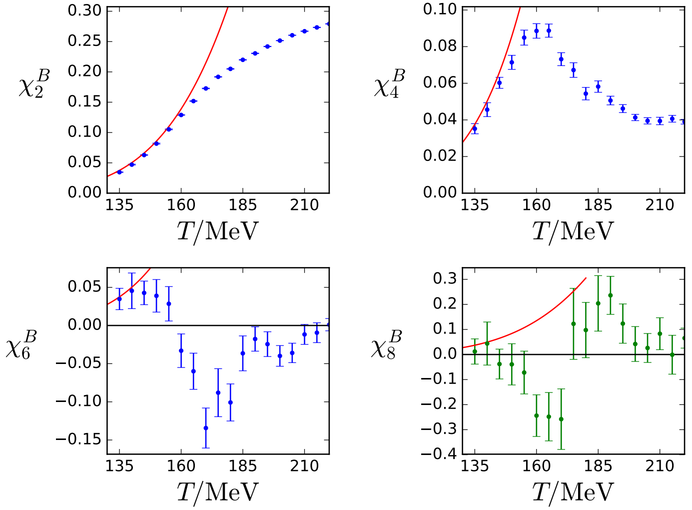



# Lattice QCD at finite $\mu_B$ - Taylor coefficients

- Fluctuations of baryon number are the Taylor expansion coefficients of the pressure

$$\chi^{BQS}_{ijk}(T) = \left.\frac{\partial^{i+j+k} p/T^4}{\partial\hat\mu_B^i \partial\hat\mu_Q^j \partial\hat\mu_S^k}\right|_{\vec\mu=0}$$

- Signal extraction is increasingly difficult with higher orders, especially in the transition region
- Higher order coefficients present a more complicated structure



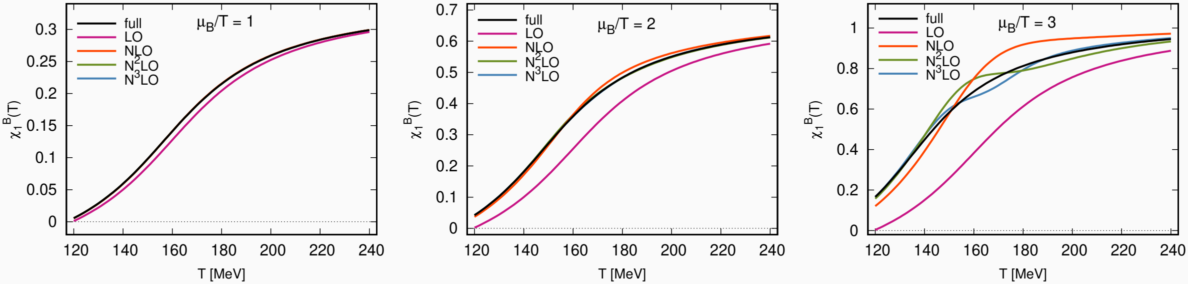

**WB: Borsányi *et al.* JHEP 10 (2018) 205;**
**(also e.g., HotQCD: Bazavov *et al.* PRD101 (2020), 074502)**

# Taylor expanding a (shifting) sigmoid

*Assume* we have a sigmoid function $f(T)$ which shifts with $\hat{\mu}$, with a simple $T$-independent shifting parameter $\kappa$. How does Taylor cope with it?

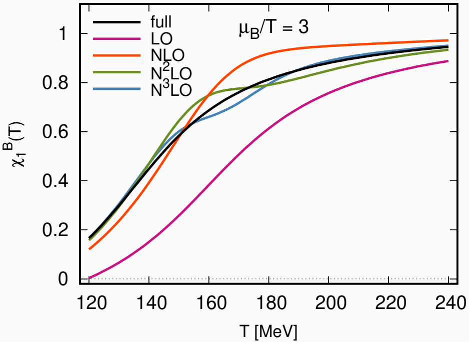
$$f(T,\,\hat{\mu}) = f(T',0)\ , \qquad\qquad T' = T(1+\kappa\,\hat{\mu}^2)\ ,$$

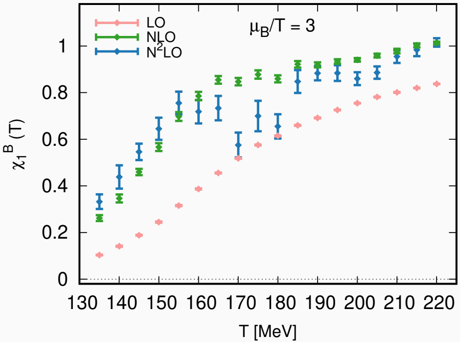
We fitted $f(T,0) = a + b\arctan(c(T-d))$ to $\chi_2^B(T,0)$ data for a $48 \times 12$ lattice



# Taylor expanding a (shifting) sigmoid

- The Taylor expansion seems to have problems reproducing the original function (left)
- Quite suggestive comparison with actual Taylor-expanded lattice data (right)




- Problems at $T$ slightly larger than $T_{pc}$ $\Rightarrow$ influence from structure in $\chi_6^B$ and $\chi_8^B$

# Determine $\kappa_n$

**I.** Directly determine $\kappa_2(T)$ at $\hat{\mu}_B = 0$ from the previous relation

**II.** From our imaginary-$\hat{\mu}_B$ simulations ($\hat{\mu}_Q = \hat{\mu}_S = 0$) we calculate:

$$\frac{T' - T}{T\,\hat{\mu}_B^2} = \kappa_2(T) + \kappa_4(T)\,\hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4) = \Pi(T)$$

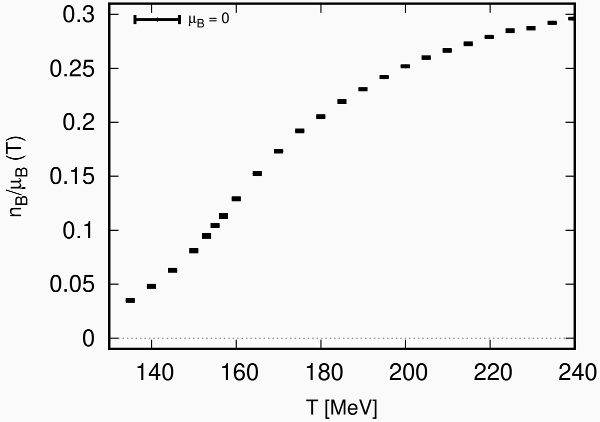
**III.** Calculate $\Pi(T, N_\tau, \hat{\mu}_B^2)$ for $\hat{\mu}_B = in\pi/8$ and $N_\tau = 10, 12, 16$

**IV.** Perform a combined fit of the $\hat{\mu}_B^2$ and $1/N_\tau^2$ dependence of $\Pi(T)$ at each temperature, yielding a continuum estimate for the coefficients

$\Rightarrow$ The $\mathcal{O}(1)$ and $\mathcal{O}(\hat{\mu}_B^2)$ coefficients of the fit are $\kappa_2(T)$ and $\kappa_4(T)$

# Determine $\kappa_n$

The procedure, visualized:




Spline fit both at $\mu_B = 0$ and $\mu_B \neq 0$, then determine $T - T'$ (horizontal segments)

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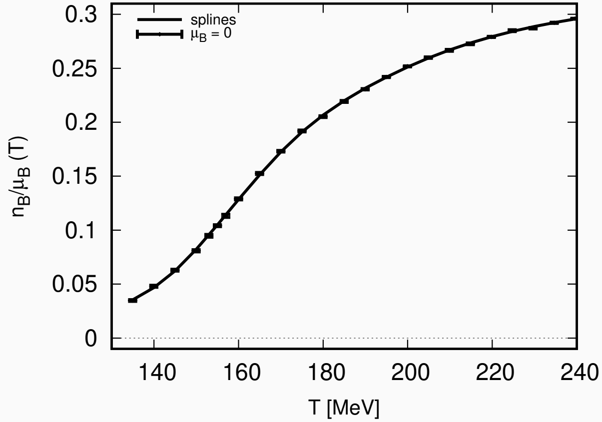


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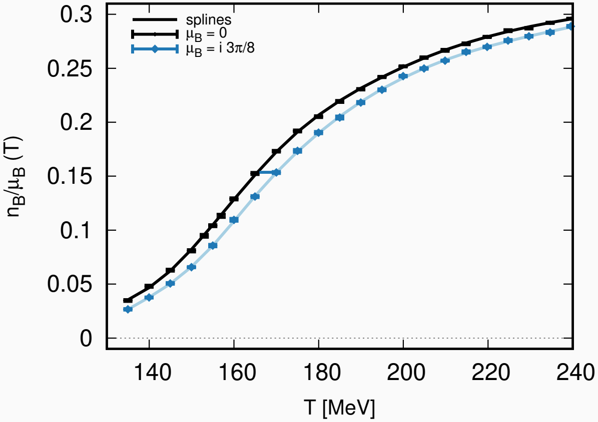


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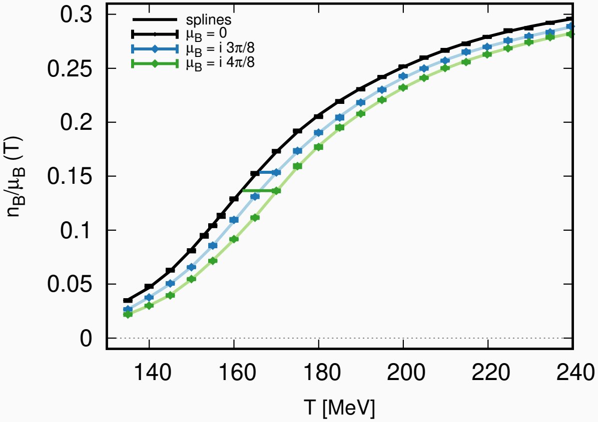


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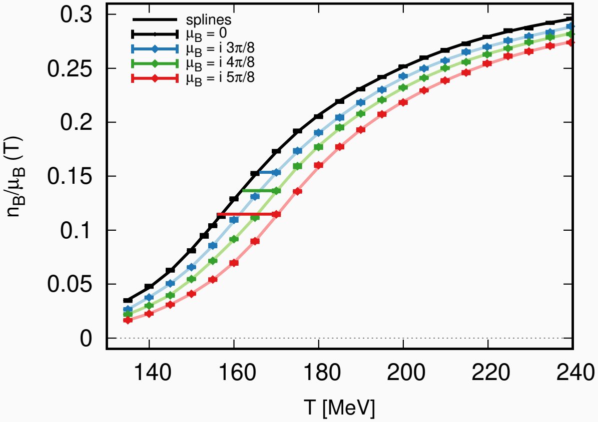


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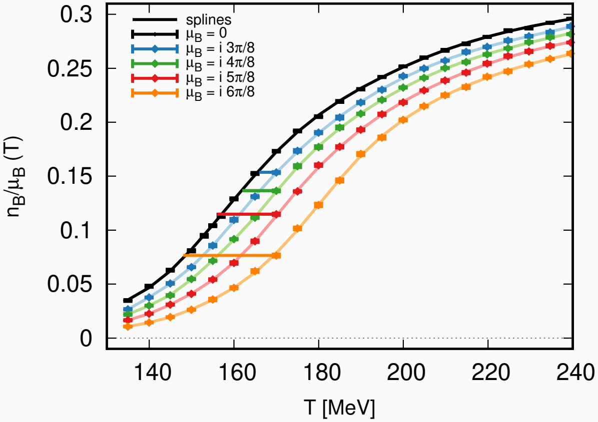


Spline fit both at $\hat{\mu}_B = 0$ and $\hat{\mu}_B \neq 0$, then determine $T - T'$ (horizontal segments)

# Thermodynamics at finite (real) $\mu_B$

- We also check the results without the inclusion of $\kappa_4(T)$ (darker shades)
- Including $\kappa_4(T)$ only results in added error, but does not "move" the results

$\longrightarrow$ Good convergence