INSTITUTO DE FÍSICA
Universidade Federal Fluminense

Causal and Stable Relativistic Magnetohydrodynamics

(work in preparation...)

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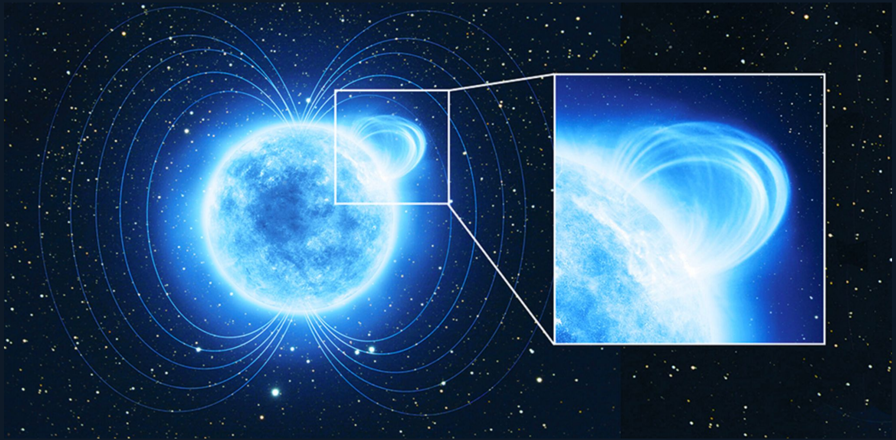
International conference on
ultra-relativistic nucleus nucleus collision

CNPq
Conselho Nacional de Desenvolvimento Científico e Tecnológico

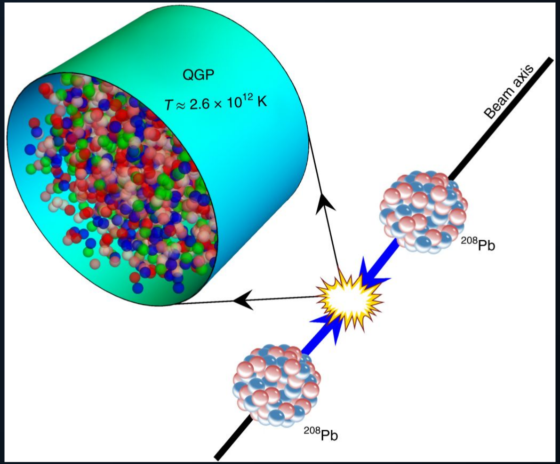
Outline

1. Motivation
2. Relativistic Magnetohydrodynamics from Boltzmann equation
3. Linear Regime
4. Bjorken flow
5. Conclusion

Why Relativistic Magnetohydrodynamics?



Magnetic fields in universe:
Magnetars! (~ 10^{15} gauss)



Heavy Ion Collisions (~ 10^{18} - 10^{19} gauss)

Magnetic field + Relativistic Fluids

Relativistic Magnetohydrodynamics

Relativistic hydrodynamics framework (B = 0)

Relativistic Dissipative Hydrodynamics

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu},$$
$$N^\mu = n u^\mu + V^\mu.$$

Conservation Equations $\partial_\mu T^{\mu\nu} = 0\ ; \quad \partial_\mu N^\mu = 0$

Israel-Stewart Equations

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = \frac{-\zeta\theta}{\tau_\Pi} + \mathcal{O}(2),$$
$$\dot{V}^{\langle\mu\rangle} + \frac{V^{\langle\mu\rangle}}{\tau_V} = \frac{\kappa\nabla^\mu\alpha}{\tau_V} + \mathcal{O}(2),$$
$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\langle\mu\nu\rangle}}{\tau_\pi} = \frac{2\eta\sigma^{\mu\nu}}{\tau_\pi} + \mathcal{O}(2),$$

Relativistic hydrodynamics framework (B = 0)

Relativistic Dissipative Hydrodynamics

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu},$$
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Relativistic hydrodynamics framework ($B \neq 0$)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu},$$
$$N^\mu = n u^\mu + V^\mu.$$

Conservation Equations $\partial_\mu T^{\mu\nu} = -F^{\mu\lambda} J_\lambda$

1st order hydro: Hattori, Hongo, Huang (Symmetry 14 (2022) 9, 1851, Symmetry 14 (2022) 1851)

BDNK: Armas, Camilioni(JCAP 10 (2022) 039, JCAP 10 (2022) 039)

Israel-Stewart Equations

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = \frac{-\zeta\theta}{\tau_\Pi} + \mathcal{O}(2),$$
$$\dot{V}^{\langle\mu\rangle} + \frac{V^{\langle\mu\rangle}}{\tau_V} = \frac{\kappa\nabla^\mu\alpha}{\tau_V} + \mathcal{O}(2),$$
$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\langle\mu\nu\rangle}}{\tau_\pi} = \frac{2\eta\sigma^{\mu\nu}}{\tau_\pi} + \mathcal{O}(2),$$

What are the transient equations at finite B?

For Dilute gases

Possibly Causal Theory

arXiv:1804.05210v2

Resistive dissipative magnetohydrodynamics from the Boltzmann-Vlasov equation

Gabriel S. Denicol,[1] Etele Molnár,[2,3] Harri Niemi,[2,4,5] and Dirk H. Rischke[2,6]

IS theory in presence of magnetic field.

Boltzmann Vlasov Equations

$$k^\mu \partial_\mu f + q k_\nu F^{\mu\nu} \frac{\partial}{\partial k^\mu} f = C[f]$$

Particles undergo binary elastic collisions only.

$$F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha B_\beta$$

e.g., the equation for shear stress tensor

$$\begin{aligned}
\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \\
&- \tau_{\pi V} V_f^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi V}\nabla^{\langle\mu}V_f^{\nu\rangle} + \lambda_{\pi V} V_f^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 \\
&- \delta_{\pi B}\, \mathfrak{q} B b^{\alpha\beta}\Delta^{\mu\nu}_{\alpha\kappa} g_{\lambda\beta}\pi^{\kappa\lambda} \,.
\end{aligned}$$

For Dilute gases

Possibly Causal Theory

Very simple single component system

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Particles undergo binary elastic collisions only.

$$\mathbf{F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha B_\beta}$$

e.g., the equation for shear stress tensor

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} - \delta_{\pi\pi} \pi^{\mu\nu}\theta - \tau_{\pi\pi} \pi^{\lambda\langle\mu} \sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi} \Pi\sigma^{\mu\nu}$$
$$- \tau_{\pi V} V_f^{\langle\mu} \dot{u}^{\nu\rangle} + \ell_{\pi V} \nabla^{\langle\mu} V_f^{\nu\rangle} + \lambda_{\pi V} V_f^{\langle\mu} \nabla^{\nu\rangle} \alpha_0$$
$$- \delta_{\pi B}\, \mathfrak{q} B b^{\alpha\beta} \Delta^{\mu\nu}_{\alpha\kappa} g_{\lambda\beta} \pi^{\kappa\lambda} .$$

Will this hold for more complicated system as well?

Relativistic Magnetohydrodynamics Framework

and, the system in consideration

2 Species-Particle fluid

The system in consideration

- 2 massless particle species with opposite charges.
- Vanishing dipole moment & chemical potential.
- Ultra relativistic limit and constant cross section.
- Vanishing magnetization.
- Using Method of Moments to solve the equation of motion. Gabriel S. Denicol, Dirk H. Rischke (2021) *Microscopic foundations of relativistic fluid dynamics*

Equations of motion for the Shear Stress Tensor

Boltzmann Equations

$$k^\mu \partial_\mu f^+ + q^+ k_\nu F^{\mu\nu} \frac{\partial}{\partial k^\mu} f^+ = C[f^+]$$

$$k^\mu \partial_\mu f^- + q^- k_\nu F^{\mu\nu} \frac{\partial}{\partial k^\mu} f^- = C[f^-]$$

Positive $\longrightarrow \pi_+^{\mu\nu}$

Negative $\longrightarrow \pi_-^{\mu\nu}$

$+$: $\pi^{\mu\nu} = \pi_+^{\mu\nu} + \pi_-^{\mu\nu}$

$-$: $\delta\pi^{\mu\nu} = \pi_+^{\mu\nu} - \pi_-^{\mu\nu}$

Method of moments leads to Coupled Equations !

$$\Sigma = \frac{3P}{5T}\left(\sigma_T^{+-} + \sigma_T\right)$$

$$\Sigma' = \frac{P}{5T}\left(5\sigma_T^{+-} + 3\sigma_T\right)$$

$$\Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} + \Sigma\pi^{\mu\nu} + \boxed{\frac{2qB}{5T} qb^{\lambda\langle\mu}\delta\pi_\lambda^{\nu\rangle}} = \frac{8}{15}\varepsilon\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\sigma^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} - 2\omega^{\lambda\langle\nu}\pi_\lambda^{\mu\rangle},$$

$$\Delta^{\mu\nu}_{\alpha\beta}\delta\dot{\pi}^{\alpha\beta} + \Sigma'\delta\pi^{\mu\nu} + \boxed{\frac{2qB}{5T} qb^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle}} = -\frac{4}{3}\delta\pi^{\mu\nu}\theta - \frac{10}{7}\sigma^{\lambda\langle\mu}\delta\pi_\lambda^{\nu\rangle} - 2\omega^{\lambda\langle\nu}\delta\pi_\lambda^{\mu\rangle}.$$

$$b^{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta}u_\alpha \frac{B_\beta}{B}$$

Projected equation for shear tensor

➢ *Decomposition with respect to the magnetic field*

$$\pi^{\mu\nu} = \left(b^\mu b^\nu + \frac{1}{2}\Xi^{\mu\nu}\right)\pi_{||} - b^\mu \pi_\perp^\nu - b^\nu \pi_\perp^\mu + \pi_\perp^{\mu\nu}$$

$$b^\mu \equiv \frac{B^\mu}{B}$$

- → $\pi_{||}$ (longitudinal),
- → $\pi_\perp^\mu$ (semi - transverse),
- → $\pi_\perp^{\mu\nu}$ (transverse).

3 components of shear viscosity.

Each component relaxes differently.

➢ *New basis*

Individual components in new basis for simplicity. Taking projection with respect to $\ell^\mu_\pm$.

$$\pi_\perp^\mu = \pi_\perp^+ \ell_+^\mu + \pi_\perp^- \ell_-^\mu$$

$$\pi_\perp^{\mu\nu} = \pi_\perp^{++} \ell_+^\mu \ell_+^\nu + \pi_\perp^{--} \ell_-^\mu \ell_-^\nu$$

where

$$\ell_\pm^\mu = \frac{1}{\sqrt{2}}(x^\mu \pm i y^\mu)$$

Final equations for shear stress tensor

Iterating order by order in gradient & truncating upto 2nd order in gradient.

2-particle species

Scalar component

$$\dot{\pi}_{\parallel} + \pi_{\perp}^{\mu}\dot{b}_{\mu} + \Sigma\pi_{\parallel} = \frac{8}{15}\varepsilon\sigma_{\parallel} - \frac{4}{3}\pi_{\parallel}\theta - \frac{10}{7}\left(-\frac{1}{2}\pi_{\parallel}\sigma_{\parallel} + \frac{1}{3}\sigma_{\perp}^{\mu}\pi_{\perp\mu} + \frac{1}{3}\pi_{\perp\alpha\beta}\sigma_{\perp}^{\alpha\beta}\right) - \frac{2}{3}\left(\omega_{\perp}^{\mu}\pi_{\perp\mu} + \omega_{\perp}^{\alpha\beta}\pi_{\perp\alpha\beta}\right)$$

Vector component

$$\begin{aligned}(1-\varphi^2)\dot{\pi}_{\perp}^{\mp} + \left(\Sigma + \varphi^2\Sigma'\right)\pi_{\perp}^{\mp} &= \frac{8}{15}\varepsilon\sigma_{\perp}^{\mp} - \left[(1-\varphi^2)\left(\ell_{\mp}^{\nu}\dot{\ell}_{\nu}^{\pm} + \frac{4\theta}{3} - \frac{5}{14}\sigma_{\parallel}\right) + \varphi\dot{\varphi}\right]\pi_{\perp}^{\mp} + (1+\varphi^2)\left[\omega_{\perp}^{\mp\mp} + \frac{5}{7}\sigma_{\perp}^{\mp\mp}\right]\pi_{\perp}^{\pm} \\ &+ (1+2\varphi^2)\left[-\ell_{\mp}^{\nu}\dot{b}_{\nu} + \omega_{\perp}^{\pm} + \frac{5}{7}\sigma_{\perp}^{\pm}\right]\pi_{\perp}^{\mp\mp} + \left(\frac{3}{2}\ell_{\pm}^{\nu}\dot{b}_{\nu} + \frac{\omega_{\perp}^{\mp}}{2} + \frac{5}{14}\sigma_{\perp}^{\mp}\right)\pi_{\parallel}\end{aligned}$$

Tensor component

$$\begin{aligned}(1-4\varphi^2)\dot{\pi}_{\perp}^{\mp\mp} + \left(\Sigma + 4\Sigma'\varphi^2\right)\pi_{\perp}^{\mp\mp} &= \frac{8}{15}\varepsilon\sigma_{\perp}^{\mp\mp} - \left[(1-4\varphi^2)\left(2\ell_{\mp}^{\beta}\dot{\ell}_{\beta}^{\pm} + \frac{4}{3}\theta + \frac{5}{7}\sigma_{\parallel}\right) - 4\varphi\dot{\varphi}\right]\pi_{\perp}^{\mp\mp} \\ &+ (1+2\varphi^2)\left(2\ell_{\beta}^{\pm}\dot{b}^{\beta} + \frac{10}{7}\sigma_{\perp}^{\mp} + 2\omega_{\perp}^{\mp}\right)\pi_{\perp}^{\mp} - \left(\frac{5}{7}\sigma_{\perp}^{\mp\mp} + \omega_{\perp}^{\mp\mp}\right)\pi_{\parallel}\end{aligned}$$

1-particle species

$$\begin{aligned}&\tau_{\pi}\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} \\ &= 2\eta\sigma^{\mu\nu} + 2\tau_{\pi}\pi_{\lambda}^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\,\pi^{\mu\nu}\theta \\ &- \delta_{\pi B}\,\mathfrak{q}Bb^{\alpha\beta}\Delta_{\alpha\kappa}^{\mu\nu}g_{\lambda\beta}\pi^{\kappa\lambda} - \tau_{\pi\pi}\,\pi^{\lambda\langle\mu}\sigma_{\lambda}^{\nu\rangle}\end{aligned}$$

$$\boxed{\tau_{\perp\perp} = \frac{1-\phi^2}{\Sigma + 4\Sigma'\phi^2}}$$

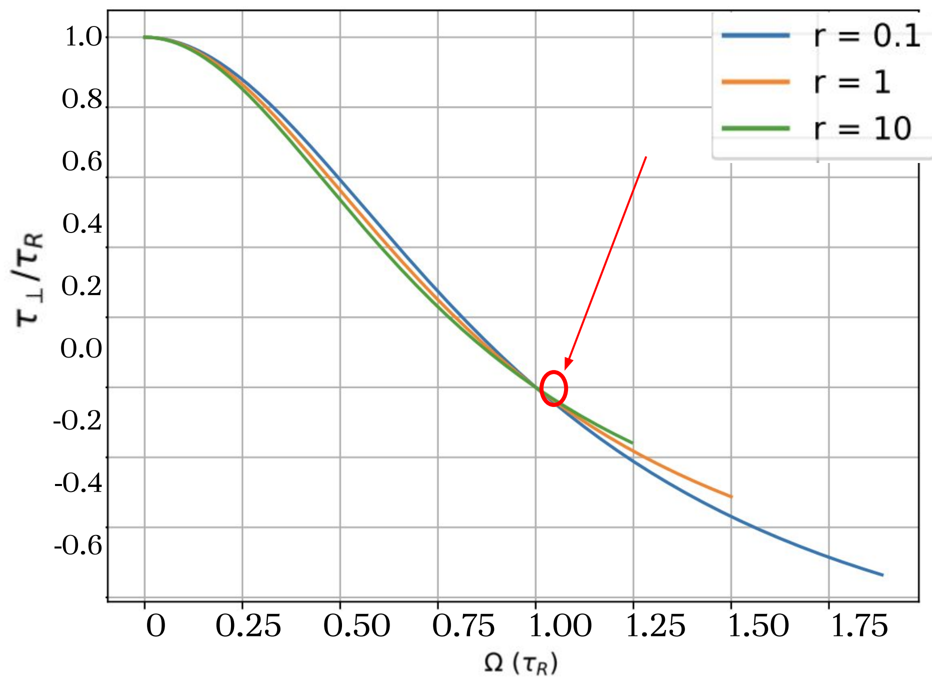
$$\boxed{\varphi = \frac{qB}{P}\frac{1}{\left(5\sigma_T^{+-} + 3\sigma_T^{++}\right)}}$$

Relaxation time as function of magnetic field

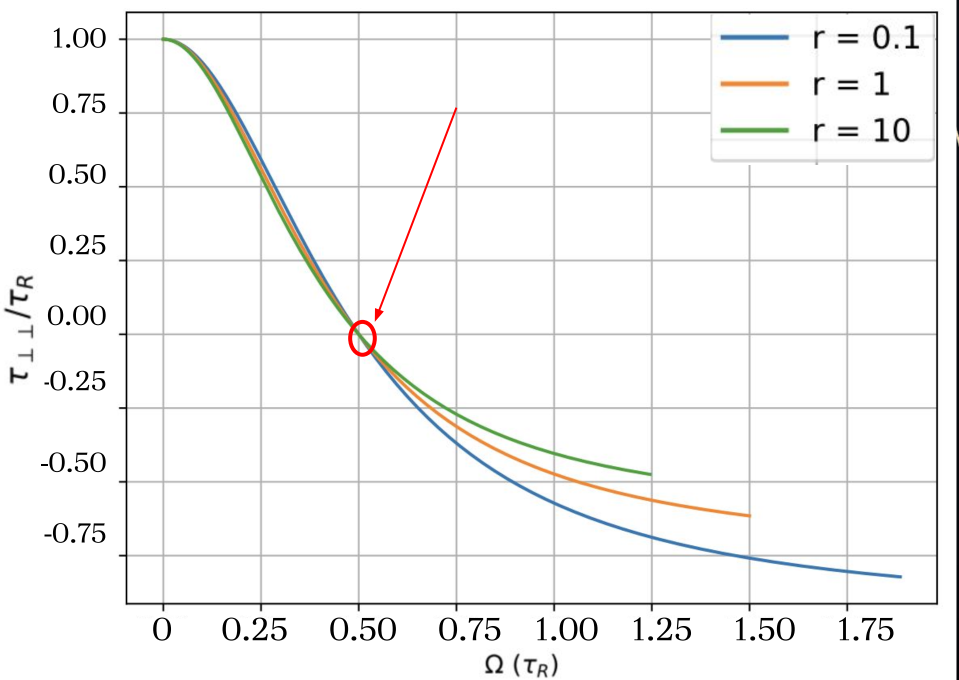
$$\Omega = \frac{qB}{5T} \qquad r = \frac{\sigma_T^{+-}}{\sigma_T^{++}}$$

- ➔ *Different component of shear relaxes differently.*
- ➔ *Theory fails at large B: Negative relaxation times.*

Semi transverse component



Fully transverse component



Shear Stress Tensor in Linear Regime: Oscillatory dynamics ?

→ *Linearising equations around equilibrium and in constant magnetic field.*

$$\underbrace{\begin{aligned}\ddot{\pi}_{\perp}^{\mp}+\left(\Sigma+\Sigma'\right)\dot{\pi}_{\perp}^{\mp}+\left(\Sigma\Sigma'+\Omega^2\right)\pi_{\perp}^{\mp}&=\frac{8}{15}\varepsilon\Sigma'\sigma_{\perp}^{\mp}+\frac{8}{15}\varepsilon\dot{\sigma}_{\perp}^{\mp},\\ \ddot{\pi}_{\perp}^{\mp\mp}+\left(\Sigma+\Sigma'\right)\dot{\pi}_{\perp}^{\mp\mp}+\left(\Sigma\Sigma'+4\Omega^2\right)\pi_{\perp}^{\mp\mp}&=\frac{8}{15}\varepsilon\Sigma'\sigma_{\perp}^{\mp\mp}+\frac{8}{15}\varepsilon\dot{\sigma}_{\perp}^{\mp\mp}.\end{aligned}}_{\text{Harmonic Oscillator type equation!}}$$

$$\Sigma=\frac{3P}{5T}\left(\sigma_T^{+-}+\sigma_T\right)$$
$$\Sigma'=\frac{P}{5T}\left(5\sigma_T^{+-}+3\sigma_T\right)$$

$$\Omega=\frac{qB}{5T}$$

→ *Mode for the transverse component*

$$\omega=\frac{i}{2}\left[\Sigma+\Sigma'\pm\sqrt{\left(\Sigma-\Sigma'\right)^2-4\Omega^2}\right]$$

$$\Omega<\frac{\Sigma'-\Sigma}{4}\implies\frac{2qB}{T}<n_0\sigma_T^{+-}$$

Overdamped

Relaxing to equilibrium

B = 0

Critically damped

$\omega=i\Sigma$ ***and*** $\omega=i\Sigma'$

Shear Stress Tensor in Linear Regime: Oscillatory dynamics ?

→ *Linearising equations around equilibrium and in constant magnetic field.*

$$\ddot{\pi}_{\perp}^{\mp} + \left(\Sigma + \Sigma'\right)\dot{\pi}_{\perp}^{\mp} + \left(\Sigma\Sigma' + \Omega^2\right)\pi_{\perp}^{\mp} = \frac{8}{15}\varepsilon\Sigma'\sigma_{\perp}^{\mp} + \frac{8}{15}\varepsilon\dot{\sigma}_{\perp}^{\mp},$$
$$\ddot{\pi}_{\perp}^{\mp\mp} + \left(\Sigma + \Sigma'\right)\dot{\pi}_{\perp}^{\mp\mp} + \left(\Sigma\Sigma' + 4\Omega^2\right)\pi_{\perp}^{\mp\mp} = \frac{8}{15}\varepsilon\Sigma'\sigma_{\perp}^{\mp\mp} + \frac{8}{15}\varepsilon\dot{\sigma}_{\perp}^{\mp\mp}.$$

$$\Sigma = \frac{3P}{5T}\left(\sigma_T^{+-} + \sigma_T\right)$$
$$\Sigma' = \frac{P}{5T}\left(5\sigma_T^{+-} + 3\sigma_T\right)$$

Harmonic Oscillator type equation!

$$\Omega = \frac{qB}{5T}$$

→ *Mode for the transverse component*

$$\omega = \frac{i}{2}\left[\Sigma + \Sigma' \pm \sqrt{\left(\Sigma - \Sigma'\right)^2 - 4\Omega^2}\right]$$

$$\Omega > \frac{\Sigma' - \Sigma}{4} \Longrightarrow \frac{2qB}{T} > n_0\sigma_T^{+-}$$

Oscillatory dynamics

B = 0

Critically damped

$\omega = i\,\Sigma$ ***and*** $\omega = i\,\Sigma'$

Can't be described by IS theory!

Bjorken Flow

A simple test of the theory

Our System in Bjorken flow

The equations of motion for different components of shear stress tensor

$$\frac{d\varepsilon}{d\tau} = \frac{\pi_{\|}}{2\tau} + \frac{\pi_{\perp}}{2\tau} - \frac{4\varepsilon}{3\tau},$$

Energy density

$$\dot{\pi}_{\|} + \Sigma\pi_{\|} = \frac{8}{45\tau}\varepsilon - \frac{23}{21\tau}\pi_{\|} - \frac{5}{21\tau}\left(\pi_{\perp}^{--} + \pi_{\perp}^{++}\right)$$

Scalar component

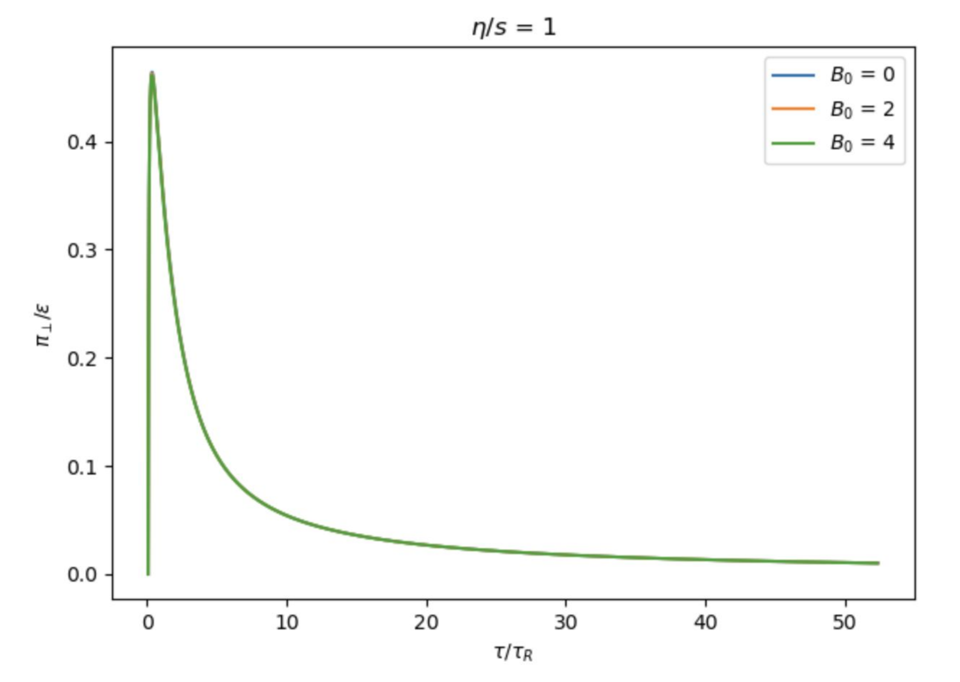
$$\dot{\pi}_{\perp}^{\mp\mp} + \Sigma\pi_{\perp}^{\mp\mp} \pm i\frac{2Bq}{5T}\delta\pi_{\perp}^{\mp\mp} = \frac{4}{15\tau}\varepsilon - \frac{5}{14\tau}\pi_{\|} - \frac{11}{7\tau}\pi_{\perp}^{\mp\mp}$$

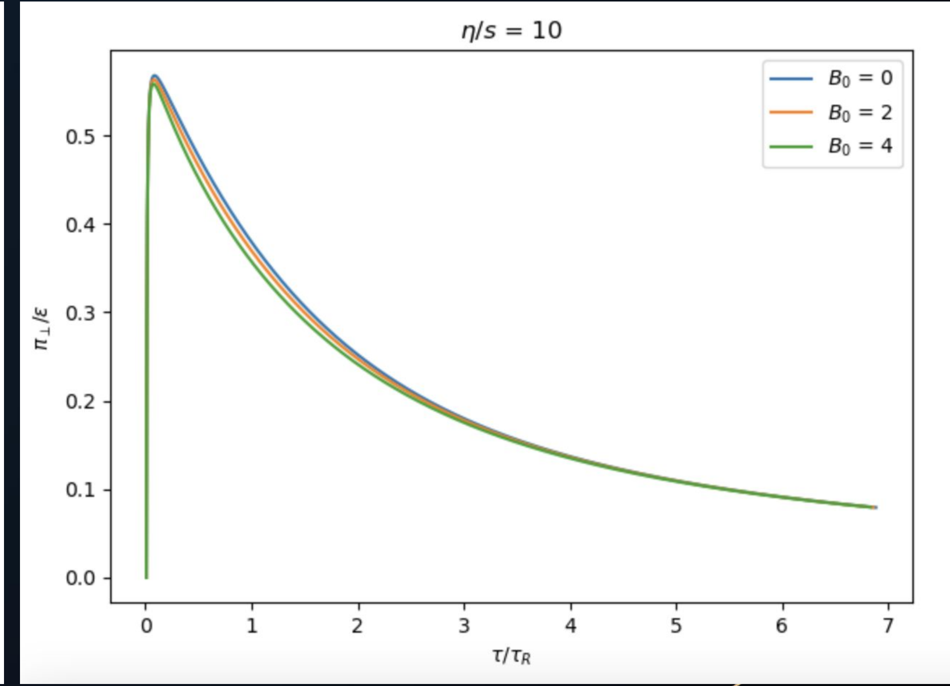
Tensor component

Our System in Bjorken flow

$$B \sim \left(\frac{\tau_0}{t}\right)$$

No oscillations!

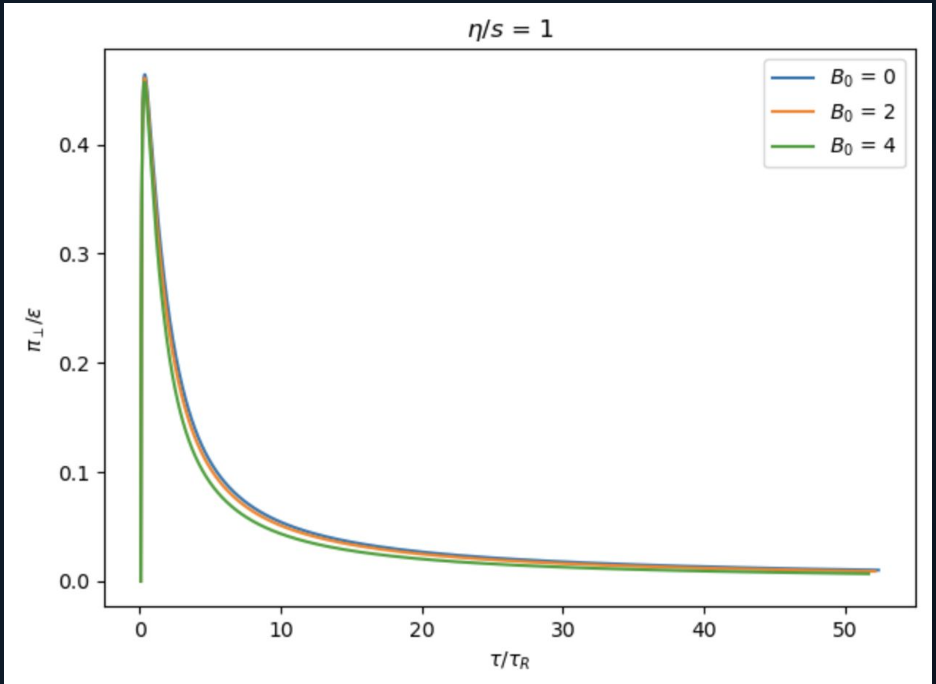


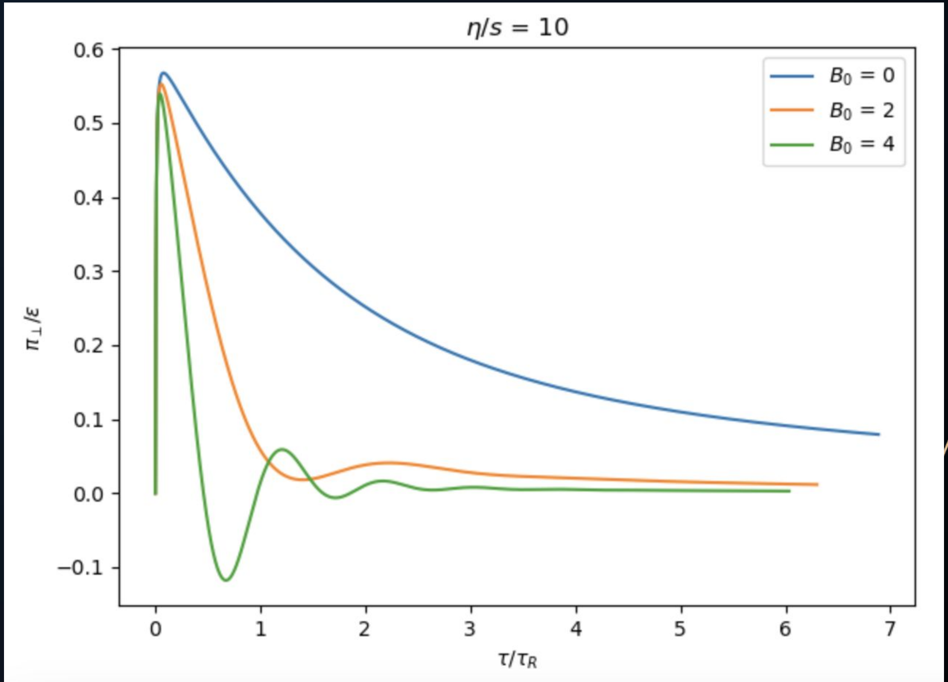


Our System in Bjorken flow

$$B \sim \left(\frac{\tau_0}{t}\right)^{1/2}$$

We see oscillations for large viscosity!

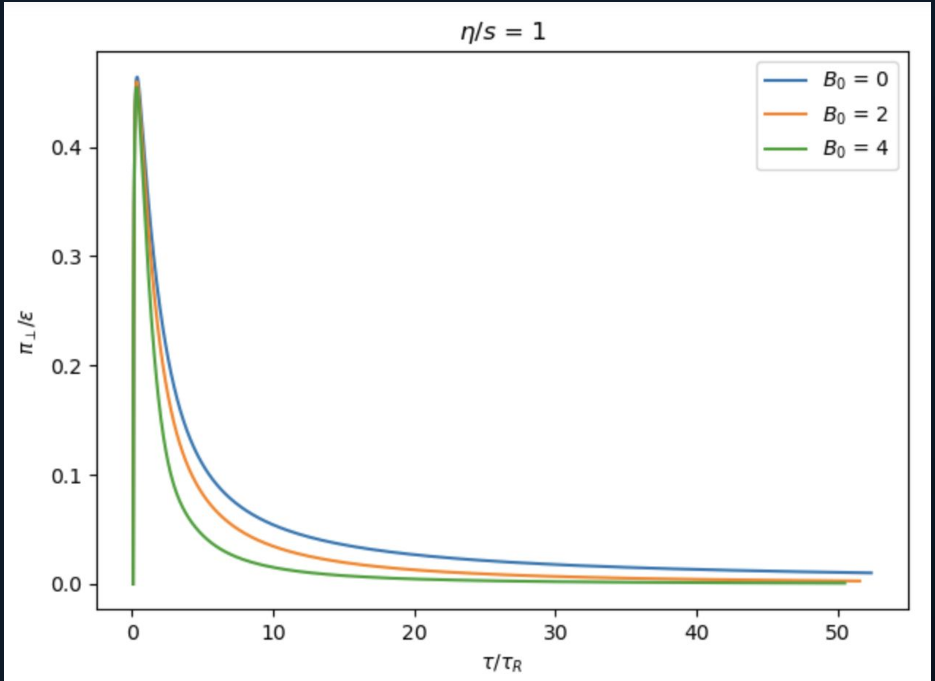


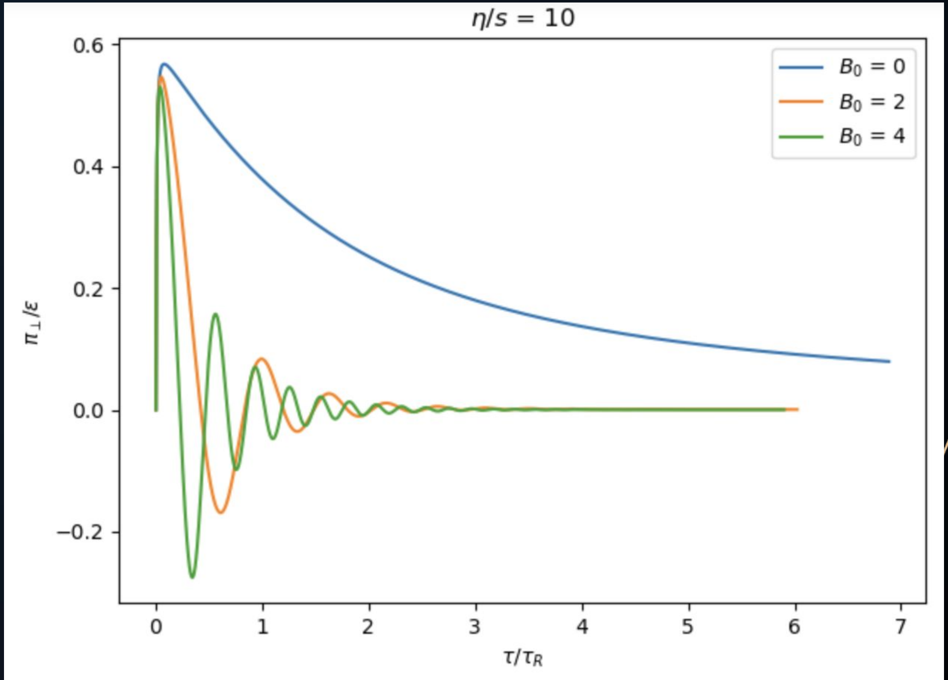


Our System in Bjorken flow

$$B \sim \left(\frac{\tau_0}{t}\right)^{1/3}$$

Oscillations for even smaller magnetic fields.





Conclusion

- A ***causal and stable theory*** for relativistic magnetohydrodynamics has ***not been fully understood yet.***

- We derived the equations of motion of relativistic second-order dissipative magnetohydrodynamics from the Boltzmann equation using the method of moments for a 2-species particle fluid.

- Different components of the shear stress tensor, with respect to the magnetic field obey different dynamical equations.

- At large magnetic fields, IS theory no longer works: Oscillatory dynamics appears.

- The oscillations are apparent only when B decays slowly in time.

–Thankyou!

Backup

Order of magnitude

Second Order Equations of motion

Iterating order by order in gradient & truncated upto 2$^{\text{nd}}$ order in gradient.

$$\delta\pi_\perp^{\mp\mp} = \pm 2i\varphi\pi_\perp^{\mp\mp} + \mathscr{O}(2),$$

$$\Sigma'\delta\pi_\perp^{\mp\mp} = \pm\frac{4}{3}2i\varphi\pi_\perp^{\mp\mp}\theta \pm \frac{5}{7}2i\varphi\pi_\perp^{\mp\mp}\sigma_\parallel \pm \frac{10}{7}i\varphi\pi_\perp^{\mp}\sigma_\perp^{\mp} \mp 2i\Omega\pi_\perp^{\mp\mp} \pm 2\varphi i\dot{\pi}_\perp^{\mp\mp} \pm 2i\pi_\perp^{\mp\mp}\dot{\varphi}$$
$$\pm 4i\varphi\pi_\perp^{\mp\mp}\ell_\mp^\beta\dot{\ell}_\beta^\pm \pm 2i\varphi\pi_\perp^{\mp}\ell_\beta^\pm\dot{b}^\beta \pm 2i\varphi\pi_\perp^{\mp}\omega_\perp^{\mp} + \mathscr{O}(3).$$

$$\dot{\pi}_\perp^{\mp\mp} \pm i\frac{2Bq}{5T}\delta\pi_\perp^{\mp\mp} = (\cdots)\pi_\perp^{\mp\mp} + \text{other } \pi_\perp^{\mp\mp}\text{terms}(\cdots)$$

Substituting the third order equation of relative shear in the equation of total shear stress tensor.

Second Order Equation of Motion for components of $\pi^{\mu\nu}$