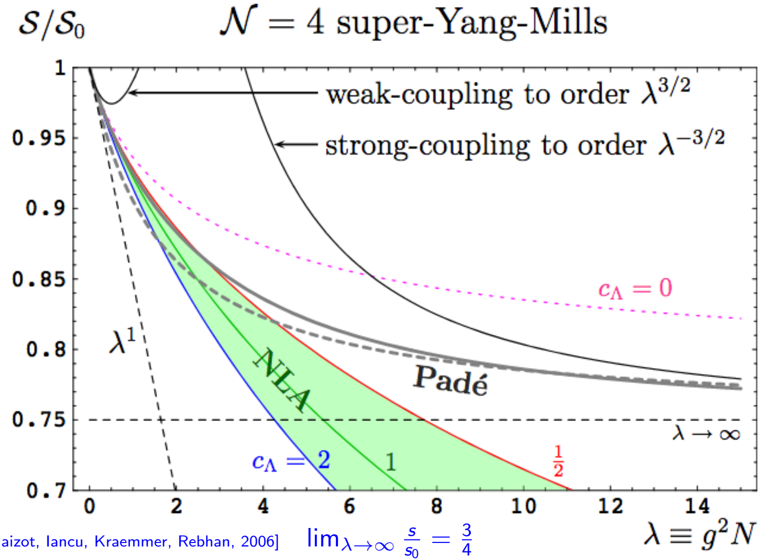
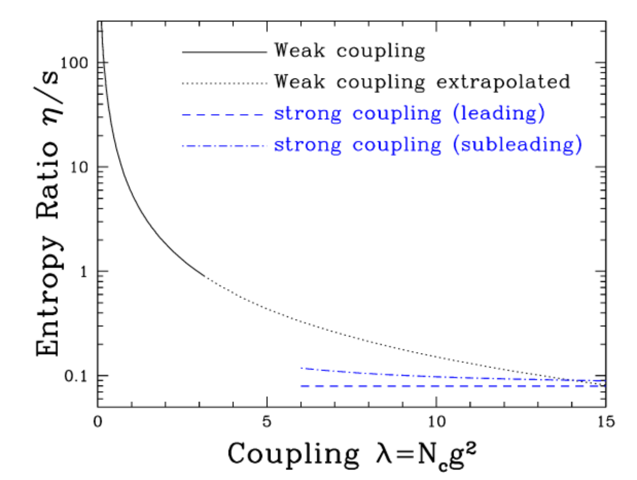
What if ϕ^4 theory in 4d is non-trivial in the continuum?

Paul Romatschke, CU Boulder

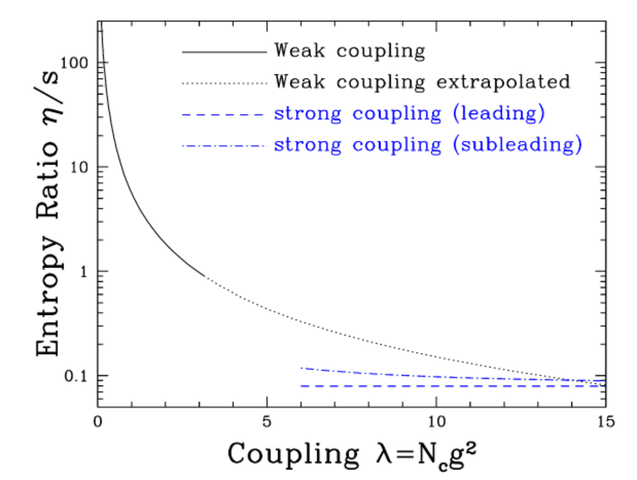


[Blaizot, Iancu, Kraemmer, Rebhan, 2006]

$\lim_{\lambda \to \infty} \frac{s}{s_0} = \frac{3}{4}$



[Huot, Jeon, Moore, 2006]



[Huot, Jeon, Moore, 2006]

[Policastro, Son, Starinets, 2001] $\lim_{\lambda \to \infty} \frac{\eta}{s} = \frac{1}{4\pi}$

Can't do QCD

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Can't really do large N $\mathcal{N} = 4$ SYM

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Can't really do large N $\mathcal{N} = 4$ SYM

Maybe we can do some other large N theory?

O(N) Model

Defined as

$$Z = \int \mathcal{D}\vec{\phi} e^{-S_E}\,, \quad S_E = \int_x \left[\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi} + \frac{\lambda}{N}\left(\vec{\phi}^2\right)^2\right]\,,$$

with $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)$.
Examples:

O(N) Model

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Examples:

- 0+1d is quantum mechanics in N dimensions (any N)
- 2+1d: conjectured AdS_4 gravity dual for $N \to \infty$ [hep-th/0210114]
- 3+1d: N=4 is Higgs case

Large N Limit

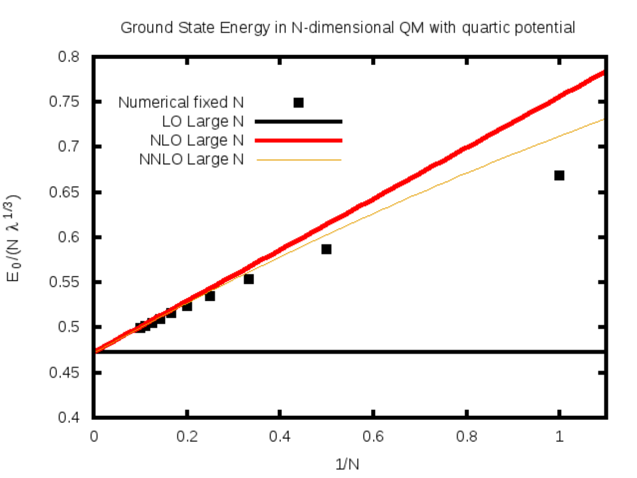
- O(N) model provides solvable interacting field theory for $N \gg 1$

  See review By Moshe and Zinn-Justin [hep-th/0306133]

- Fully non-perturbative, not a weak-coupling expansion!

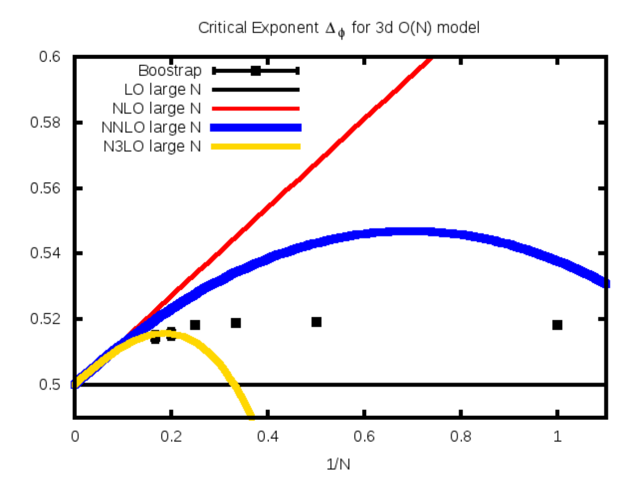
Quantitative tests of large N method in less than 4d

0+1d: Quantum Mechanics



Large N passes QM test even down to N=1!

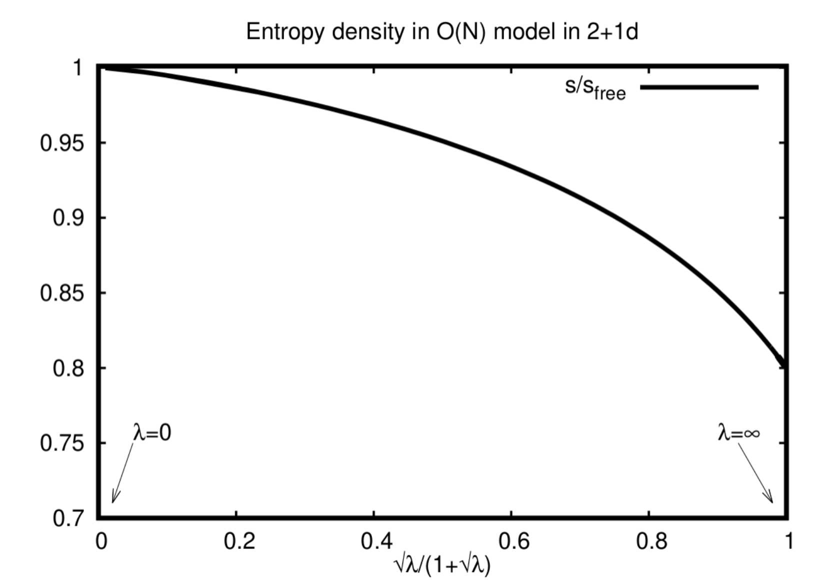
2+1d: superrenormalizable QFT



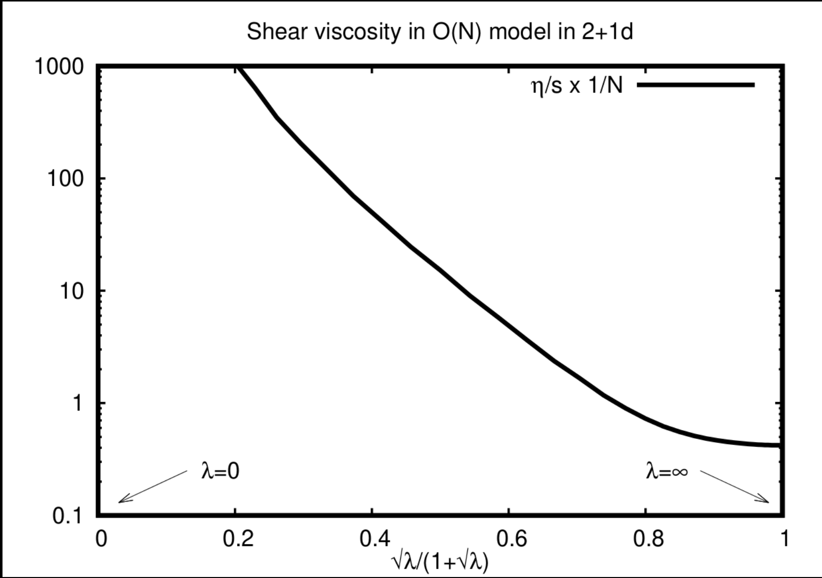
adapted from Kos et al., [1307.6856]

Large N passes test down to $N \sim 4$!

2+1d finite temperature: thermodynamics and transport


Entropy density in O(N) model in 2+1d
s/s_free
λ=0
λ=∞
1
0.95
0.9
0.85
0.8
0.75
0.7
0
0.2
0.4
0.6
0.8
1
√λ/(1+√λ)

[PR, 4/2019]


Shear viscosity in O(N) model in 2+1d
η/s x 1/N
1000
100
10
1
0.1
0
0.2
0.4
0.6
0.8
1
√λ/(1+√λ)
λ=0
λ=∞

[PR, 5/2021]

scalars in less than 4d:

- Large N math is solid
- Large N is quantitatively reliable down to $N \sim \mathcal{O}(1)$
- Large N results for $c_s^2, \frac{\eta}{s}$ for ALL $\lambda \in [0, \infty]$

scalars in 3+1d: the mental roadblock

Scalar ϕ^4 theory is trivial in four dimensions:

Scalar ϕ^4 theory is trivial in four dimensions:

$$\lim_{\Lambda_{\mathrm{UV}} \to 0} \lambda_{IR} = 0$$

physics arguments/proofs for $d = 4 + \varepsilon$: [Wilson; Fröhlich; Lüscher; Weisz;...]
math proof for $d = 4$: [Aizenman, Duminil-Copin 2019]

“Loopholes” in proofs:

- assume one or two component scalar fields
- assume $\lambda_{\mathrm{UV}} > 0$

Here's what actually happens for 4d $O(N \gg 1)$:

O($N \gg 1$) model in 4d in continuum

- O(N) model renormalization is non-perturbative
- In the **continuum limit** $\Lambda_{\rm UV} \to \infty$, running coupling is

$$\lambda_R(\mu) = \frac{(2\pi)^2}{\ln \frac{\Lambda^2_{\overline{\rm MS}}}{\mu^2}}$$

- Non-vanishing coupling in the continuum. Theory is non-trivial!
- Trick to avoiding triviality:

$$\lim_{\mu\to\infty} \lambda_R(\mu) = 0^- \, ,$$

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- Trick to avoiding triviality:

$$\lim_{\mu\to\infty} \lambda_R(\mu) = 0^- \,,$$

- Bare coupling is **negative**; situation explicitly excluded in all previous math proofs and physics assumptions!

O($N \gg 1$) model in 4d in continuum

Raises new questions:

- How does one make sense of QFT with unbounded potential (e.g. $V(x) = -gx^4$?)

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  [A: non-Hermitian or PT-symmetric construction; see extensive work on this by C. Bender!]

O($N \gg 1$) model in 4d in continuum

Raises new questions:

- How does one make sense of QFT with unbounded potential (e.g. $V(x) = -gx^4$?)

  [A: non-Hermitian or PT-symmetric construction; see extensive work on this by C. Bender!]
- If we set aside problems with 'intuitive/traditional' QFT interpretation, are there actual problems with this theory for any observable?

O(N$\gg$ 1) model in 4d – Results

- Symmetric Mass Generation in O(N) model: radiative corrections generate a VEV for $\vec{\phi}^2$ (Coleman-Weinberg)
- Acts like a mass term for vector field, but NO O(N) symmetry breaking
- At large N, vector (e.g. Higgs for N=4) mass

$$m = \sqrt{e}\Lambda_{\overline{\mathrm{MS}}} ,$$

is prediction of theory (not a parameter)

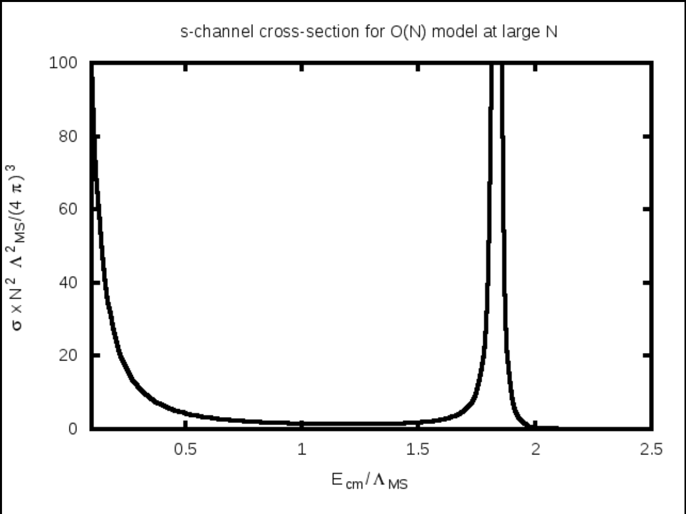
O(N$\gg$ 1) model in 4d – Results

- Finite vacuum free energy density

$$\mathcal{F} = -\frac{N\Lambda^4_{\overline{\mathrm{MS}}}}{64\pi^2}$$

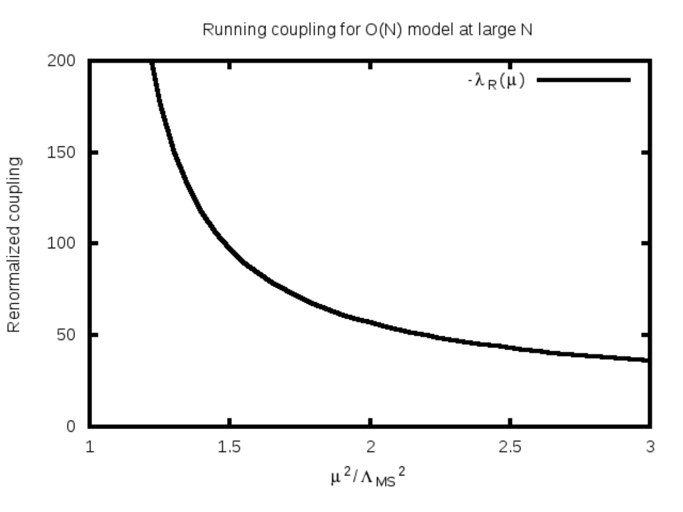
- dominates over perturbative vacuum ($\mathcal{F} = 0$)

O(N$\gg$ 1) model in 4d – Results



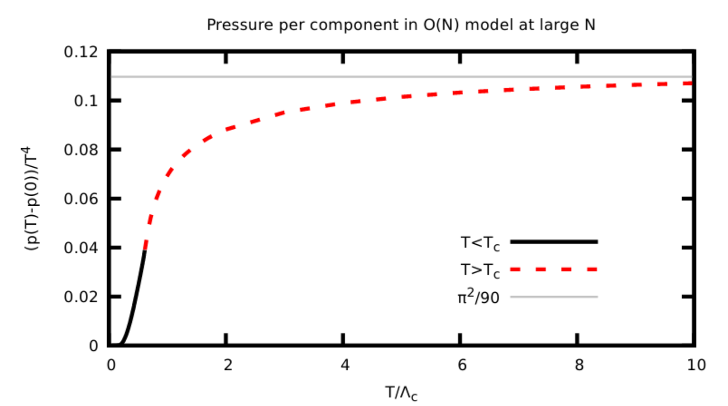
Well behaved scattering cross-section for any CM energy; prediction for scalar bound state at $m \simeq 3\Lambda_{\overline{\mathrm{MS}}}$

O(N$\gg$ 1) model in 4d – Results



Asymptotic freedom in terms of $-\lambda_R(\bar{\mu})$

O(N) model at finite temperature



low T and high T phase separated by 2^{nd} order phase transition

Talk Summary

ϕ^4 theory in 4 dimensions IS non-trivial in the continuum if

- we allow for negative bare coupling $\lambda_0 < 0$
- we stop repeating (false) beliefs about triviality proofs
- we stop equating perturbative QFT with QFT

Stop using QFT 'intuition'!
Calculate observables and check!

Thank You

Bonus Material

References & Hyperlinks

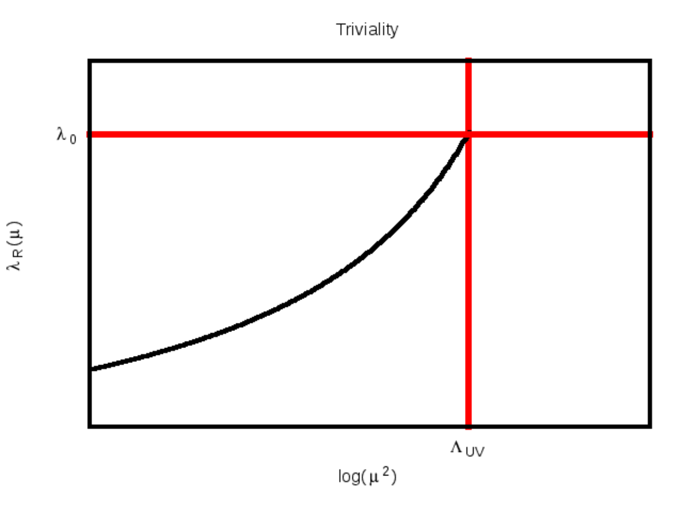
Continuum large N scalar field theory

- PR, "A solvable quantum field theory with asymptotic freedom in 3+1 dimensions", hyperlink: [2211.15683]
- PR, "Life at the Landau pole", [2212.03254]
- Grable and Weiner, "A Fully Solvable Model of Fermionic Interaction in 3+1d3+1d", [2302.08603]
- PR, "What if ϕ^4 theory in 4 dimensions is non-trivial in the continuum?", [2305.05678]

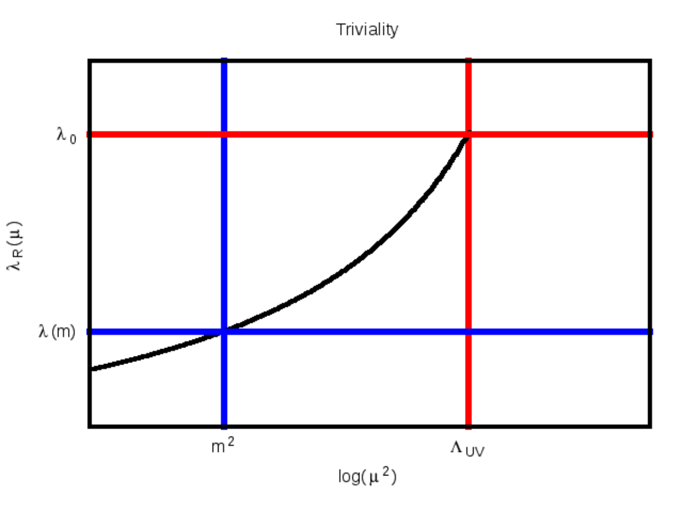
$\mathcal{PT}$-symmetric Quantum Mechanics and QFT relations

- Bender and Böttcher, "Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry", [physics/9712001]
- Ai, Bender and Sarkar, "PT-symmetric -g ϕ^4 theory", [2209.07897]
- Lawrence, Peterson, PR and Weller, "Instantons, analytic continuation, and PT-symmetric field theory", [2303.01470]

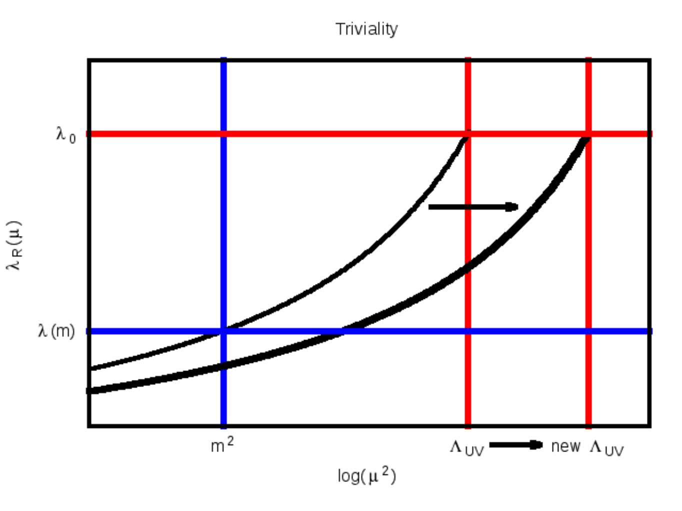
Quantum Triviality



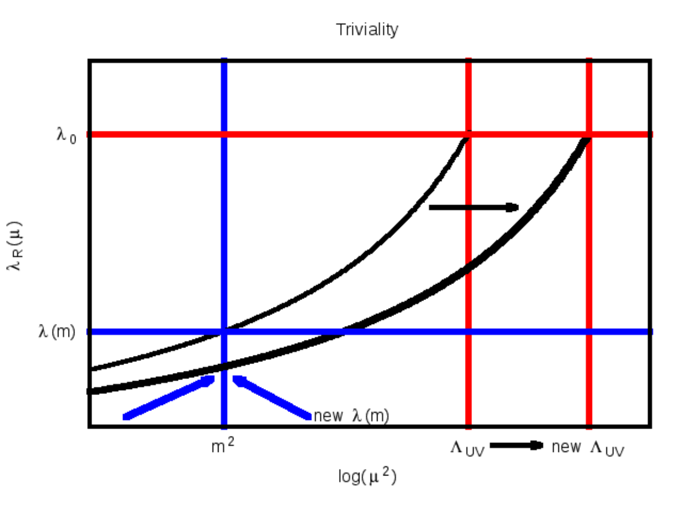
Quantum Triviality



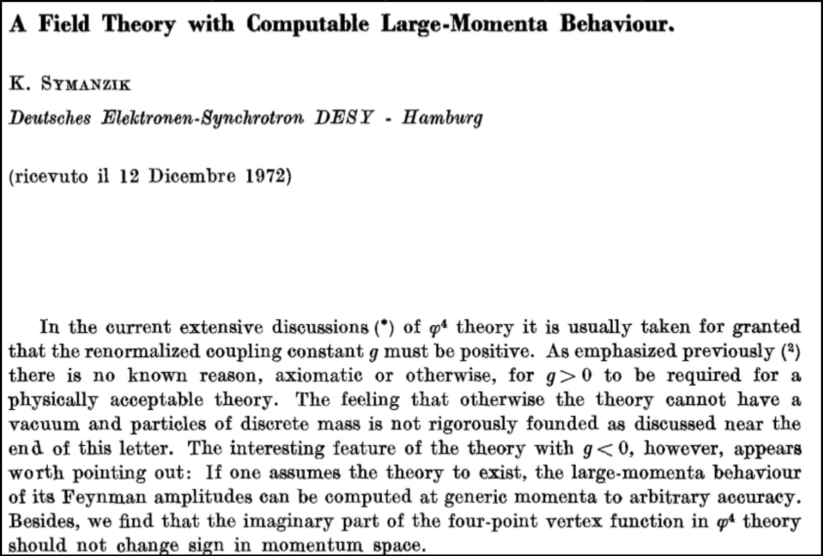
Quantum Triviality



Quantum Triviality



Negative Coupling Field Theory History



A Field Theory with Computable Large-Momenta Behaviour.

K. SYMANZIK

Deutsches Elektronen-Synchrotron DESY - Hamburg

(ricevuto il 12 Dicembre 1972)

In the current extensive discussions (*) of φ^4 theory it is usually taken for granted that the renormalized coupling constant g must be positive. As emphasized previously (2) there is no known reason, axiomatic or otherwise, for $g > 0$ to be required for a physically acceptable theory. The feeling that otherwise the theory cannot have a vacuum and particles of discrete mass is not rigorously founded as discussed near the end of this letter. The interesting feature of the theory with $g < 0$, however, appears worth pointing out: If one assumes the theory to exist, the large-momenta behaviour of its Feynman amplitudes can be computed at generic momenta to arbitrary accuracy. Besides, we find that the imaginary part of the four-point vertex function in φ^4 theory should not change sign in momentum space.

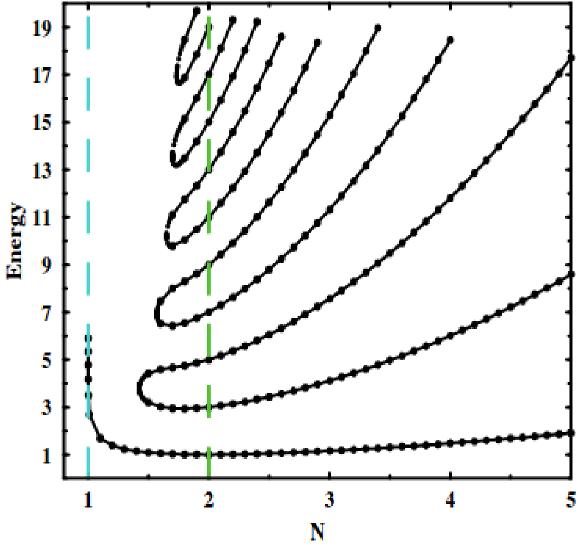
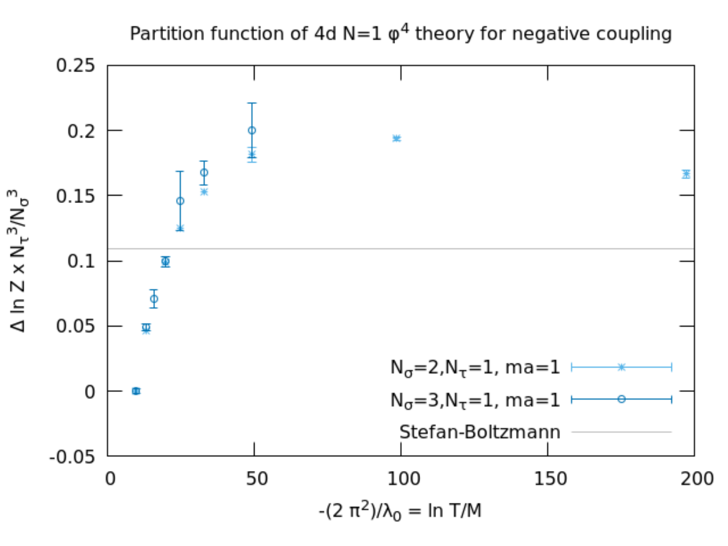


FIG. 1. Energy levels of the Hamiltonian $H = p^2 - (ix)^N$ as a function of the parameter N. There are three regions:

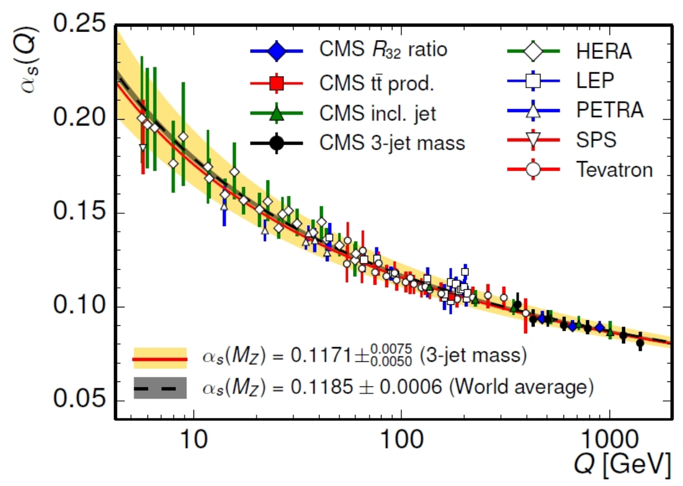
[Bender & Böttcher, 1997]

Negative coupling ϕ^4 in 4d on the lattice



adapted from [2305.05678]

QCD running coupling



Somewhat misleading: really a fit of perturbation theory to experimental measurements

QCD at infinite coupling

- In pQCD, $\alpha_s(\bar{\mu})$ does diverge at $\bar{\mu} = \Lambda_{\overline{\rm MS}} \sim 0.3$ GeV
- Usually dismissed as an artifact of perturbation theory
- Non-perturbative extractions (lattice+NRQCD) exist down to $\bar{\mu} = 1.5$ GeV where

$$\alpha_s(1.5\mathrm{GeV}) \simeq 0.336$$

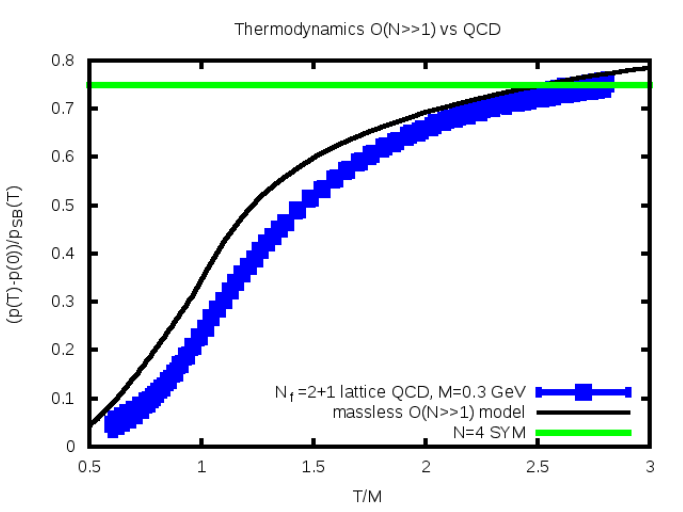
[Bazavov et al, 1407.8437]

- QCD could have a Landau pole at $\Lambda_{\overline{\rm MS}} \sim 0.3$ GeV
- No issues in QCD

The O(N$\gg$ 1) Model as a Model for QCD

- Only one scale M
- Is M the same as $\Lambda_{\overline{\rm MS}}$ in QCD?
- Let's compare!

Parameter-free comparison to QCD



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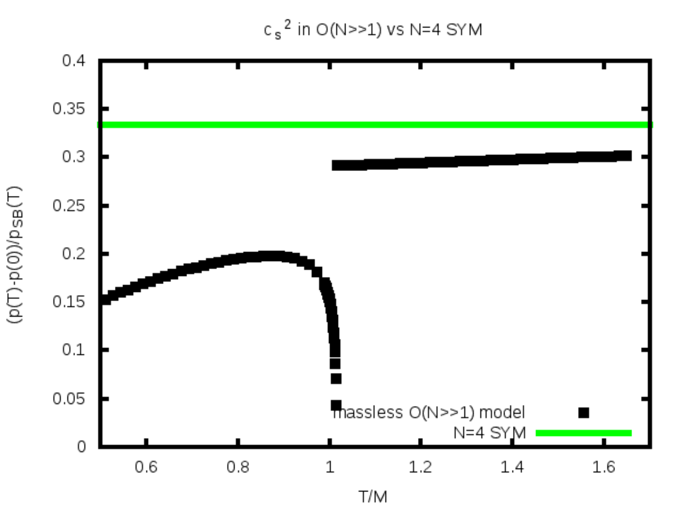
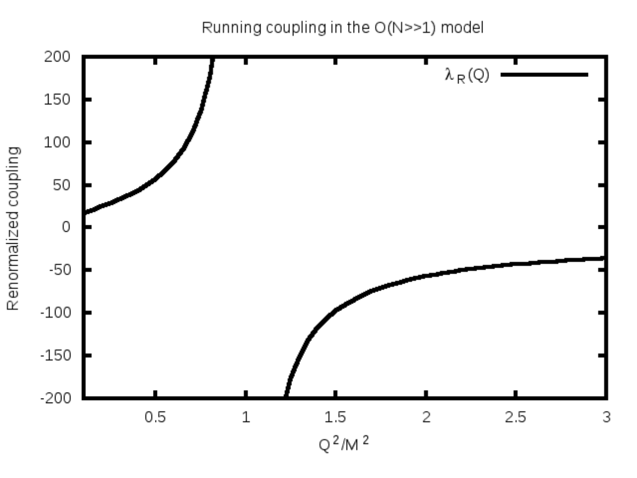


Exact Running coupling in O(N) Model



[2305.05678]

Scattering for NR fermions

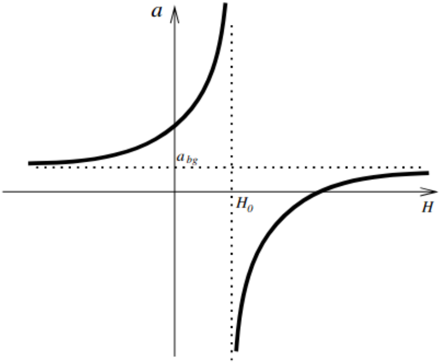


Fig. 2. A schematic of a typical, experimentally observed behavior of an s-wave scattering length $a(H)$ as a function of magnetic field H in a vicinity of a Feshbach resonance.

[Gurarie, Radzihovsky, 2007]

Analytic Continuation for Path Integral

Let's consider 0d field theory defined by

$$Z(\lambda) = \int dx e^{-\lambda x^4}$$

- Well defined for $\mathrm{Re}(\lambda) > 0$,

$$Z(\lambda) = 2\lambda^{\frac{1}{4}}\Gamma\left(\frac{5}{4}\right)$$

- $Z(\lambda)$ is formally divergent for $\mathrm{Re}(\lambda) < 0$
- But it is an analytic function with a well-defined analytic continuation

Analytic Continuation for Path Integral

Let's consider 0d field theory with negative coupling

$$Z_{\mathcal{P}T}(g) = \int dx e^{+gx^4}$$

- Particular analytic continuation: "the cone"
- Instead of integrating along real x axis, deform the integration contour into the complex plane so that integral converges

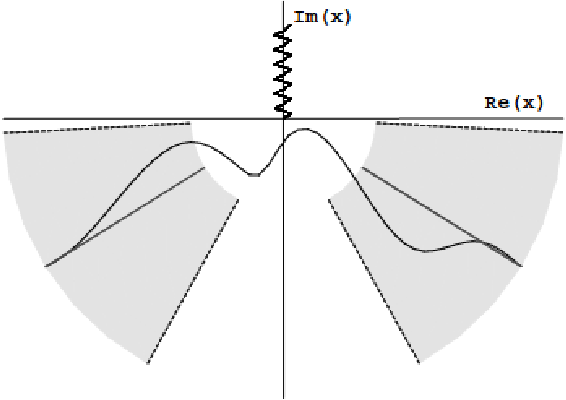


FIG. 2. Wedges in the complex-x plane containing the contour on which the eigenvalue problem for the differential

[Bender & Böttcher, 1997]

Analytic Continuation for Path Integral

$$Z_{\mathcal{P}T}(g) = \int_{\mathcal{C}} dx e^{+gx^4}$$

- Result for $Z_{\mathcal{P}T}(g)$ does not depend on details of $\mathcal{C}$ as long as is inside Bender's wedges
- Result is

$$Z_{\mathcal{P}T}(g) = \sqrt{2} g^{\frac{1}{4}} \Gamma\left(\frac{5}{4}\right) = \mathrm{Re} Z(\lambda \to -g + i0^+)$$

- Finite and well defined!