Equation of state of a hot-and-dense quark gluon plasma: lattice simulations at real μ_B vs. extrapolations

[Phys. Rev. D 107, L091503]

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Quark Matter 2023

Introduction

- Equation of State of Quark-Gluon Plasma at $\mu \geq 0$ is important in heavy-ion phenomenology
- Common lore:
  - A direct lattice simulation is impossible due to sign problem
  - $\Rightarrow$ only extrapolations from imaginary or zero μ can be used and different schemes have been proposed
- In this talk, we would:
  - Demonstrate that direct simulation is actually expensive but not impossible
  - Compare the extrapolation schemes with direct results at $\mu_B > 0$
- Outline:
  - Taylor expansion and resummation
  - Reweighting methods
  - Comparison with Taylor expansion and resummation

Introduction

- Simulation: $N_f = 2+1$ QCD at $\mu_B > 0$ and $T > 0$
- Grand canonical partition function:

$$Z = \mathrm{Tr}\left[e^{-(H_{\mathrm{QCD}} - \mu_u N_u - \mu_d N_d - \mu_s N_s)/T}\right]$$

$$p = \frac{T}{V}\ \ln\ Z \equiv \hat{p}\ T^4$$

- Scenario considered in this talk:

$$\hat{\mu}_q\ T \equiv \mu_q \equiv \mu_u = \mu_d = \frac{1}{3}\mu_B,\ \mu_s = 0$$

- Observables of interest:
  - Light quark density $\hat{n}_L \equiv \frac{\partial \hat{p}}{\partial \hat{\mu}_q}$
  - Susceptibility $\chi_n^l = \frac{\partial^n \hat{p}}{\partial \hat{\mu}_q^n}$

Taylor expansion

- $\hat{p}$ and its derivatives can be expanded in Taylor series in $\hat{\mu}_q$

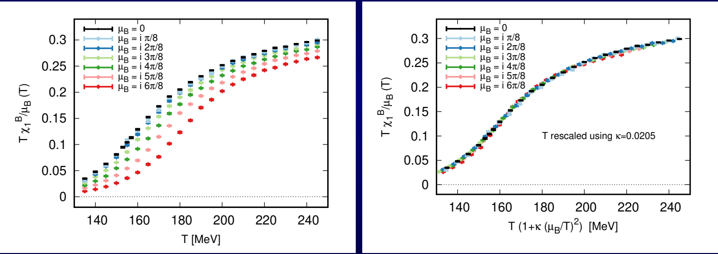
$$\hat{p}(T,\hat{\mu}_B) = p_0(T) + p_2(T)\hat{\mu}_B^2 + p_4(T)\hat{\mu}_B^4 + \ldots$$

- Taylor coefficients can be obtained as:
  - Moments at $\mu_B = 0$ simulation (e.g.[A. Bazavov et al, Phys. Rev. D 101, 074502(2020)])
  - Fit parameters from simulations at a series of imaginary values of μ_B (e.g. [M D'Elia et al, Phys. Rev. D 95, 094503 (2017)])

Resummations with shifting methods

- Empirical observation: There is an approximate scaling variable $T(1+\kappa\hat{\mu}_B^2)$ such that [S. Borsanyi et al, Phys. Rev. Lett. 126 232001 (2021)]

$$\frac{\chi_1^B}{\hat{\mu}_B} = \frac{\hat{n}_L(T,\hat{\mu}_B)}{\hat{\mu}_B} = \frac{d\hat{n}_L}{d\hat{\mu}_B}(T(1+\kappa_2(T)\hat{\mu}_B^2+\dots),0)$$



- It can be made more applicable to higher T by normalizing with the Stefan-Boltzmann limit [S. Borsanyi et al, Phys. Rev. D 105, 114504 (2022)]

$$\frac{\hat{n}_L(T,\hat{\mu}_B)}{\hat{\mu}_B} \to \frac{\hat{n}_L(T,\hat{\mu}_B)}{\hat{n}_L^{\mathrm{SBL}}(\hat{\mu}_B)}$$

Reweighting

- Simulation is impossible/hard in target ("t") action
  $\Rightarrow$ reweight from simulated ("s") action:

$$\langle O \rangle_t = \frac{\int D\phi w_t(\phi) O(\phi)}{\int D\phi w_t(\phi)} = \frac{\int D\phi w_s(\phi) \frac{w_t(\phi)}{w_s(\phi)} O(\phi)}{\int D\phi w_s(\phi) \frac{w_t(\phi)}{w_s(\phi)}} = \frac{\left\langle \frac{w_t}{w_s} O \right\rangle_s}{\left\langle \frac{w_t}{w_s} \right\rangle_s},$$

$$\frac{Z_t}{Z_s} = \left\langle \frac{w_t}{w_s} \right\rangle_s,$$

$$Z_t = \int D\phi w_t(\phi),\ w_t(\phi) \in \mathbb{C},\ \ Z_s = \int D\phi w_s(\phi),\ w_s(\phi) > 0$$

- Problems getting exponentially hard as V increases:
  - Sign problem: $\frac{w_t}{w_s} \in \mathbb{C}$
  - Overlap problem: $\rho\left(\frac{w_t}{w_s}\right)$ has long tail

Simulation Details

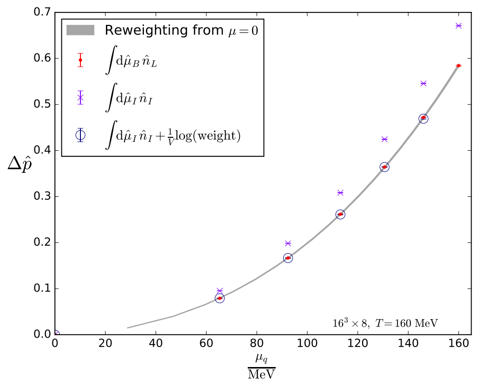
- Tree-level Symanzik improved gauge action and 2-step stout smearing of $\rho = 0.15$.
- $N_s^3 \times N_\tau = 16^3 \times 8$ at temperatures $145 \leq T \leq 240$ MeV and $\hat{\mu}_B^2 = 1.5, 3, 4.5, 6, 7.5, 9$
- $\mu_u = \mu_d = \mu_q = \mu = \mu_B/3, \mu_s = 0$
- Two reweighting schemes are considered:
  - Reweighting from $\mu_B = 0$: $\frac{w_t}{w_s} = \frac{\det M(\mu_B)}{\det M(0)}$

$$w_t = e^{-S_g} \det M(\mu_B),\ w_s = e^{-S_g} \det M(0)$$

  - Reweighting from Phase Quenched(PQ): $\frac{w_t}{w_s} = e^{i\theta(\mu_B)}$

$$w_t = e^{-S_g} \mid \det\ M(\mu_B) \mid e^{i\theta(\mu_B)},\ w_s = e^{-S_g} \mid \det\ M(\mu_B) \mid$$

Reweighting schemes



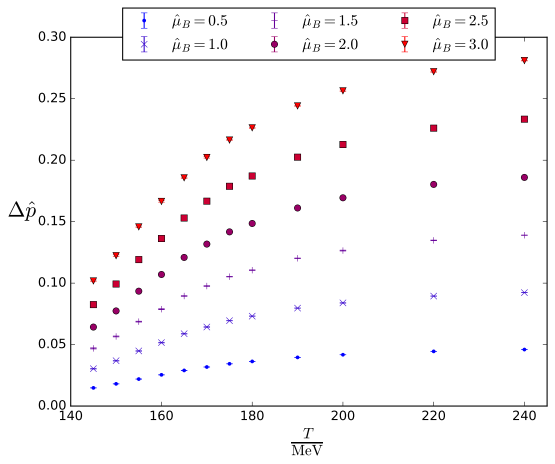
- $\Delta\hat{p}$ can be computed in two ways:

$$\Delta\hat{p} = \int_0^{\hat{\mu}_B/3} \hat{n}_I(\hat{\mu}_q, T) d\hat{\mu}_q + \frac{1}{(LT)^3} \ln\langle e^{i\theta}\rangle_{\mathrm{PQ}},\ \hat{n}_I \equiv \left(\frac{\partial \hat{p}}{\partial \hat{\mu}_q}\right)_{\mu_I=0}$$

$$\Delta\hat{p} = \int_0^{\hat{\mu}_B/3} \hat{n}_L d\hat{\mu}_q = \int_0^{\hat{\mu}_B/3} \frac{1}{(LT)^3 \langle e^{i\theta}\rangle_{\mathrm{PQ}}} \left\langle e^{i\theta} \frac{\partial}{\partial \hat{\mu}_B} \ln\det M \right\rangle_{\mathrm{PQ}} d\hat{\mu}_q$$

Reweighting from $\mu_B = 0$ agrees with both estimates

Reweighting schemes

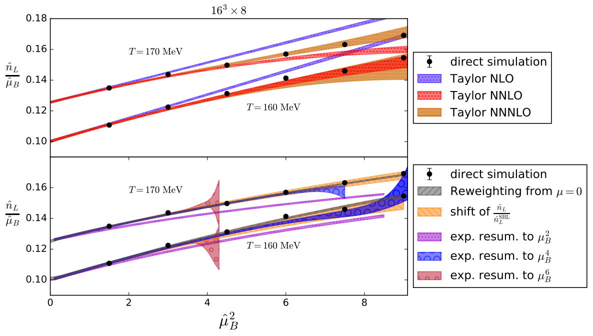


- Reweighting from Phase Quenched

Reweighting Vs Taylor expansions and Resummation

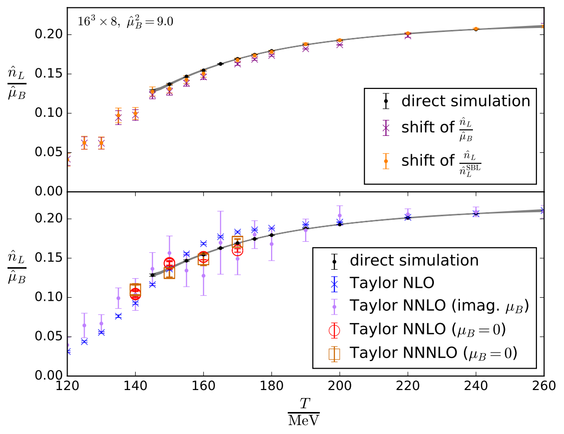
- We use the same simulation setup as reweighting simulations, but here we simulate at $\hat{\mu}_B = \{0, 4, 6, 7, 8, 9, 10, 12\}\frac{i\pi}{16}$
- Taylor coefficients: obtained via $\frac{d^2\hat{p}}{d\hat{\mu}_B^2}$ and $\frac{d^4\hat{p}}{d\hat{\mu}_B^4}$ at $\mu_B = 0$ and fitting $\frac{\hat{n}_L}{\hat{\mu}_B}$ with polynomial of order $\hat{\mu}_B^6$
- Shift method: obtain κ_2 and κ_4 (λ_2 and λ_4) from fitting $\frac{\hat{n}_L(T,\hat{\mu}_B)}{\hat{\mu}_B}$ $\left(\frac{\hat{n}_L(T,\hat{\mu}_B)}{\hat{n}_L^{\mathrm{SBL}}(\hat{\mu}_B)}\right)$

Reweighting Vs Taylor expansions and Resummation



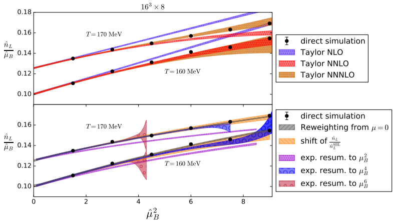
- Taylor expansion: agrees with phase-reweighting at NNLO. It is true in the whole range of simulated T up to $\hat{\mu}_B \approx 3 (\approx 500\text{MeV})$
- Shift method: agrees with phase-reweighting in the whole range
- Shift method requires coefficients up to $\hat{\mu}_B^6$, lower than NNNLO Taylor which needs $\hat{\mu}_B^8$
- Exponential Resummation [S. Mondal et al, Phys. Rev. Lett. 128 022001 (2022)] : Alternative summation method not discussed here

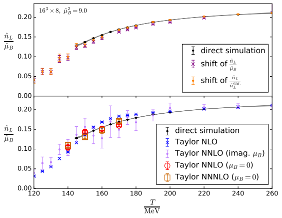
Reweighting Vs Taylor expansions and Resummation



- The shift method with SB correction agrees better with the one without at higher T, as designed

Summary





- It is possible to simulations at real values of μ_B with reweighting methods.
- By comparison with such simulations in the range of simulated T values, we have studied the validity of Taylor expansion and Shift methods.
- Taylor expansion NNNLO and Shift methods agree with the reweighted results, and they work beyond the entire RHIC Beam Energy Scan $0 \leq \hat{\mu}_B \leq 3$
- In order to facillitate the studies of the transition line and location of critical end point, it is essential to extend the validity of the methods to lower T and higher $\hat{\mu}_B$

Staggered Rooting at $\mu_B > 0$

- For $N_f = 4$, $\det M(\hat{\mu})$ can be expressed as follows in the temporal gauge: [Hasenfratz, Toussaint 1991]

$$\det M(\hat{\mu}) = e^{-3V\hat{\mu}} \prod_{i=1}^{6V} \left(\xi_i - e^{\hat{\mu}}\right)$$

where ξ_i are $6V = 6N_xN_yN_z$ eigenvalues of reduced matrix P that depend only on U and not μ

$$P = -\left(\prod_{i=0}^{N_t-1} P_i\right) L,\ P_i = \begin{pmatrix} B_i & 1 \\ 1 & 0 \end{pmatrix}\ B_i = \eta_4 (D^{(3)} + am)|_{t=i},\ L = \begin{pmatrix} U_4 & 0 \\ 0 & U_4 \end{pmatrix}|_{t=N_t-1}$$

- Rooting becomes:

$$[\det M_l(\hat{\mu})]^{1/2} = (\det M_l(0))^{1/2} \prod_{i=1}^{6V} \sqrt[\mathbb{C}]{\frac{\xi_i e^{\frac{\hat{\mu}}{2}} - e^{-\frac{\hat{\mu}}{2}}}{\xi_i - 1}}$$

- $[\det M(\hat{\mu})]^{1/2}$ therefore has a branchcut that creates ambiguity of which root is to be taken
- Such ambiguity leads to observable numerical consequences [S. Borsanyi et al, arXiv: 2308.06105] that will not be discussed in this talk. We stay at values of T and μ_B where this issue is absent.

Exponential Resummations [S. Mondal et al, Phys. Rev. Lett. 128 022001 (2022)]

- The difference between pressures at $\mu_B = 0$ and finite μ_B can be expressed as:

$$\Delta\hat{p}(T,\hat{\mu}_B) \equiv \hat{p}(T,\hat{\mu}_B) - \hat{p}(T,0) = \frac{1}{(LT)^3}\ln\left\langle \frac{\det M(\hat{\mu}_B)}{\det M(0)} \right\rangle_{\hat{\mu}_B=0}$$

- The approximation $\frac{\det M(\hat{\mu}_B)}{\det M(0)} \approx \exp(\sum_{n=1}^{N}\frac{1}{n!}D_n\hat{\mu}_B^n)$ can be done at some truncation order N:

$$\Delta\hat{p}(T,\hat{\mu}_B) \approx \frac{1}{(LT)^3}\ln\left\langle \exp\left(\sum_{n=1}^{N}\frac{1}{n!}D_n\hat{\mu}_B^n\right)\right\rangle_{\hat{\mu}_B=0}$$

- $D_n \equiv \frac{\partial^n}{\partial\hat{\mu}_B^n}\ln\det M(\hat{\mu}_B)$ is also needed in Taylor coefficients p_{2n}, thus is obtained for free if the latter is already computed.
- D_n is computed exactly for each configuration using the reduced matrix formalism [Hasenfratz, Toussaint 1991]