# Hydrodynamic theories for a system of weakly-interacting ultra-relativistic scalar particles

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## Abstract

In the latter decades, the study of heavy ion collisions has stimulated research on the emergence of relativistic fluid dynamics from a microscopic theory and on its domain of applicability. For dissipative fluids, the fundamental hydrodynamic equations of motion, $\partial_\mu N^\mu = 0, \partial_\mu T^{\mu\nu} = 0$ must be complemented by constitutive relations or further equations of motion for the dissipative currents. After the recognition that relativistic Navier-Stokes (NS) theory is acausal and unstable [1], second order transient hydrodynamic [2] (TT) and, more recently, Bemfica-Disconzi-Noronha-Kovtun (BDNK) [3] theories were proposed to circumvent this problem. In a Kinetic theory framework, these equations of motion emerge from the relativistic Boltzmann equation

$$p^\mu \partial_\mu f_{\mathbf{p}} = \int dQ\, dQ'\, dP'\, W_{pp' \leftrightarrow qq'} (f_q f_{q'} - f_p f_{p'}), \quad W_{\mathbf{pp'} \leftrightarrow \mathbf{qq'}} = (2\pi)^6 s' \sigma(s', \Theta) \delta^{(4)}(p + p' - q - q'), \tag{1}$$

where $\sigma(s',\Theta)$ is cross-section and $s' = (p+p')^2$, through power-counting procedures implementing the so-called gradient expansion. The corresponding transport coefficients usually require the inversion of the linearized collision term, $f_{0\mathbf{p}} \widehat{L} \phi_{\mathbf{p}} \equiv \int dQ\, dQ'\, dP'\, W_{pp' \leftrightarrow qq'} f_{0\mathbf{p}} f_{0\mathbf{p}'} (\phi_{\mathbf{q}} + \phi_{\mathbf{q}'} - \phi_{\mathbf{p}} - \phi_{\mathbf{p}'})$, where $\phi_{\mathbf{p}} \equiv \delta f_{\mathbf{p}}/f_{0\mathbf{p}} \equiv (f_{\mathbf{p}} - f_{0\mathbf{p}})/f_{0\mathbf{p}}$ and $f_{0\mathbf{p}} = \exp(\alpha - \beta u_\mu p^\mu)$ is the local equilibrium distribution. This is usually a highly non-trivial task. However, in Ref. [4] it was shown that for scalar particles whose Lagrangian density is $\mathcal{L} = (1/2)\partial_\mu \varphi \partial^\mu \varphi - (\lambda/4!)\varphi^4$, the eigensystem of $\widehat{L}$ can be computed in exact form,

$$\widehat{L} \,|\chi_{n\ell}\rangle = \chi_{n\ell} \,|\chi_{n\ell}\rangle \quad |\chi_{n\ell}\rangle = L^{(2\ell+1)}_{n,\mathbf{p}} p^{\langle \mu_1} \cdots p^{\mu_\ell \rangle}, \quad \chi_{n\ell} = -\frac{g e^\alpha}{4\pi^2 \beta^2} \left( \frac{n+\ell-1}{n+\ell+1} + \delta_{n0}\delta_{\ell 0} \right), \quad g = \frac{\lambda^2}{32\pi}, \tag{2}$$

where $L^{(\alpha)}_{n\mathbf{p}} \equiv L^{(\alpha)}_n(\beta E_{\mathbf{p}})$ denotes associated Laguerre polynomials and $p^{\langle \mu_1} \cdots p^{\mu_\ell \rangle}$ denotes irreducible tensors constructed from 4-momentum [5]. This result emerges as a consequence of the form of the cross-section $\sigma(s',\Theta) = g/(2\pi s')$, which possesses no angular dependence. In the present work [6], we use this extremely rare result to compute the transport coefficients of NS, BDNK and TT *analytically*.

## Equations of motion and transport coefficients

In kinetic theory, the particle 4-current and the energy-momentum tensor are identified, respectively, as $N^\mu = \int dP\, p^\mu f_{\mathbf{p}}$, $T^{\mu\nu} = \int dP\, p^\mu p^\nu f_{\mathbf{p}}$, which can be decomposed in equilibrium and non-equilibrium parts as $N^\mu = N^\mu_{\text{eq}} + \delta N^\mu$, $T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \delta T^{\mu\nu}$, where $N^\mu_{\text{eq}} = n_0 u^\mu$, $\delta N^\mu = \delta n u^\mu + \nu^\mu$, $T^{\mu\nu}_{\text{eq}} = \varepsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}$, and $\delta T^{\mu\nu} = \delta\varepsilon u^\mu u^\nu - \Pi \Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu}$. However, the separation $f_{\mathbf{p}} = f_{0\mathbf{p}} + \delta f_{\mathbf{p}}$, is not unique and the thermodynamic variables $\beta$, $\alpha$, and $u^\mu$ must be defined using matching conditions. In a kinetic theory framework, an ensemble of matching conditions can be formulated in the following way

$$\int dP\, E^q_{\mathbf{p}} \delta f_{\mathbf{p}} \equiv 0, \quad \int dP\, E^s_{\mathbf{p}} \delta f_{\mathbf{p}} \equiv 0, \quad \int dP\, E^z_{\mathbf{p}} p^{\langle \mu \rangle} \delta f_{\mathbf{p}} \equiv 0. \tag{3}$$

The above conditions reduce to the Landau matching conditions when $q = 1$, $s = 2$, and $z = 1$ and to the Eckart conditions when $q = 1$, $s = 2$, and $z = 0$. Other values of $q$, $s$, and $z$ lead to novel matching conditions that often do not have any intuitive physical interpretation. The inspiration for employing such alternative matching conditions stems from BDNK theory. In this case, causality and stability imply that $\delta\varepsilon, h^\mu \neq 0$ [3].

Different hydrodynamic equations of motion emerge from the Boltzmann equation by different power-counting procedures. Employing the Chapman-Enskog procedure [5] and conditions (3), we have the NS relations

$$\delta n, \delta\varepsilon, \Pi = 0, \quad \nu^\mu = z \frac{3}{g\beta^2} \nabla^\mu \alpha, \quad h^\mu = (z-1) \frac{12}{g\beta^3} \nabla^\mu \alpha, \quad \pi^{\mu\nu} = \frac{96}{g\beta^3} \sigma^{\mu\nu}, \tag{4}$$

Alternatively, using the order of magnitude procedure [7] in Landau matching conditions, we obtain the transient hydrodynamic equations of motion

$$\begin{aligned} &\tau_\nu D\nu^{\langle\lambda\rangle} + \nu^\lambda = \kappa_n \nabla^\lambda \alpha - \delta_{\nu\nu} \nu^\lambda \theta - (\lambda_{\nu\pi} \nabla_\mu \alpha + \tau_{\nu\pi} \nabla_\mu P_0) \pi^{\lambda\mu} + \ell_{\nu\pi} \Delta^\lambda_{\ \alpha} \nabla_\mu \pi^{\alpha\mu} \\ &- \frac{7}{5} \tau_\nu \sigma_\mu^{\ \lambda} \nu^\mu - \tau_\nu \omega_\mu^{\ \lambda} \nu^\mu, \\ &\tau_\pi D\pi^{\langle\lambda\mu\rangle} + \pi^{\lambda\mu} = 2\eta\sigma^{\lambda\mu} + \varphi_8 \nu^{\langle\lambda} \nu^{\mu\rangle} - \delta_{\pi\pi} \pi^{\lambda\mu} \theta - \tau_{\pi\nu} \nabla^{\langle\lambda} P_0\, \nu^{\mu\rangle} + \ell_{\pi\nu} \nabla^{\langle\lambda} \nu^{\mu\rangle} \\ &+ \lambda_{\pi\nu} \nabla^{\langle\lambda} \alpha\, \nu^{\mu\rangle} - 2\tau_\pi \omega_\nu^{\ \langle\lambda} \pi^{\mu\rangle\nu} - \tau_{\pi\pi} \sigma_\nu^{\ \langle\lambda} \pi^{\mu\rangle\nu}, \end{aligned} \tag{5}$$

$$\tau_\nu = \frac{60}{g n_0 \beta^2}, \quad \kappa_n = \frac{3}{g\beta^2}, \quad \tau_\pi = \frac{72}{g n_0 \beta^2}, \quad \eta = \frac{48}{g\beta^3}, \tag{6}$$

$$\delta_{\nu\nu} = \tau_\nu, \quad \lambda_{\nu\pi} = \frac{3\tau_\nu \beta}{40}, \quad \tau_{\nu\pi} = \frac{\tau_\nu \beta}{80 P_0}, \quad \ell_{\nu\pi} = \frac{\tau_\nu \beta}{40},$$

$$\varphi_8 = \frac{4}{n_0 \beta}, \quad \delta_{\pi\pi} = \frac{4}{3}\tau_\pi, \tau_{\pi\pi} = 2\tau_\pi, \quad \ell_{\pi\nu} = -\frac{4}{3}\frac{\tau_\pi}{\beta}, \quad \lambda_{\pi\nu} = \frac{2}{3}\frac{\tau_\pi}{\beta}, \quad \tau_{\pi\nu} = -\frac{4}{3}\frac{\tau_\pi}{n_0}.$$

And, finally employing the modified Chapman-Enskog procedure [8] and conditions (3), we obtain the BDNK relations

$$\begin{aligned} &\Pi = \frac{\chi}{3}\left(\frac{D\beta}{\beta} - \frac{\theta}{3}\right), \quad \delta n = \xi \left(\frac{D\beta}{\beta} - \frac{\theta}{3}\right), \quad \delta\varepsilon = \chi \left(\frac{D\beta}{\beta} - \frac{\theta}{3}\right), \\ &\nu^\mu = \varkappa \left(\frac{\nabla^\mu \beta}{\beta} + Du^\mu\right), \quad h^\mu = \lambda \left(\frac{\nabla^\mu \beta}{\beta} + Du^\mu\right), \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \end{aligned} \tag{7}$$

$$\begin{aligned} &\xi = \frac{12}{g\beta^2}(q-1)(s-1), \quad \chi = \frac{36}{g\beta^3}(q-2)(s-2), \\ &\varkappa = \frac{12}{g\beta^2} z, \quad \lambda = \frac{48}{g\beta^3}(z-1), \quad \eta = \frac{48}{g\beta^3}. \end{aligned} \tag{8}$$

Next, we analyze solutions for the equations of motion (4), (5), and (7) in the transversely-homogeneous, longitudinally-boost invariant Bjorken flow.

## Bjorken flow solutions

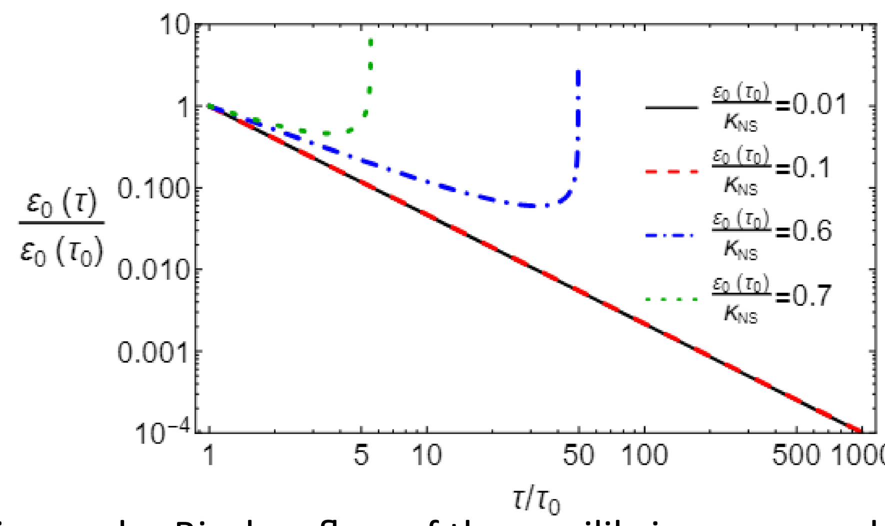


Fig. 1 – Evolution under Bjorken flow of the equilibrium energy density according to Navier-Stokes equations of motion for various values of $\varepsilon_0(\tau_0)/K_{\text{NS}} \propto$ initial Knudsen number.

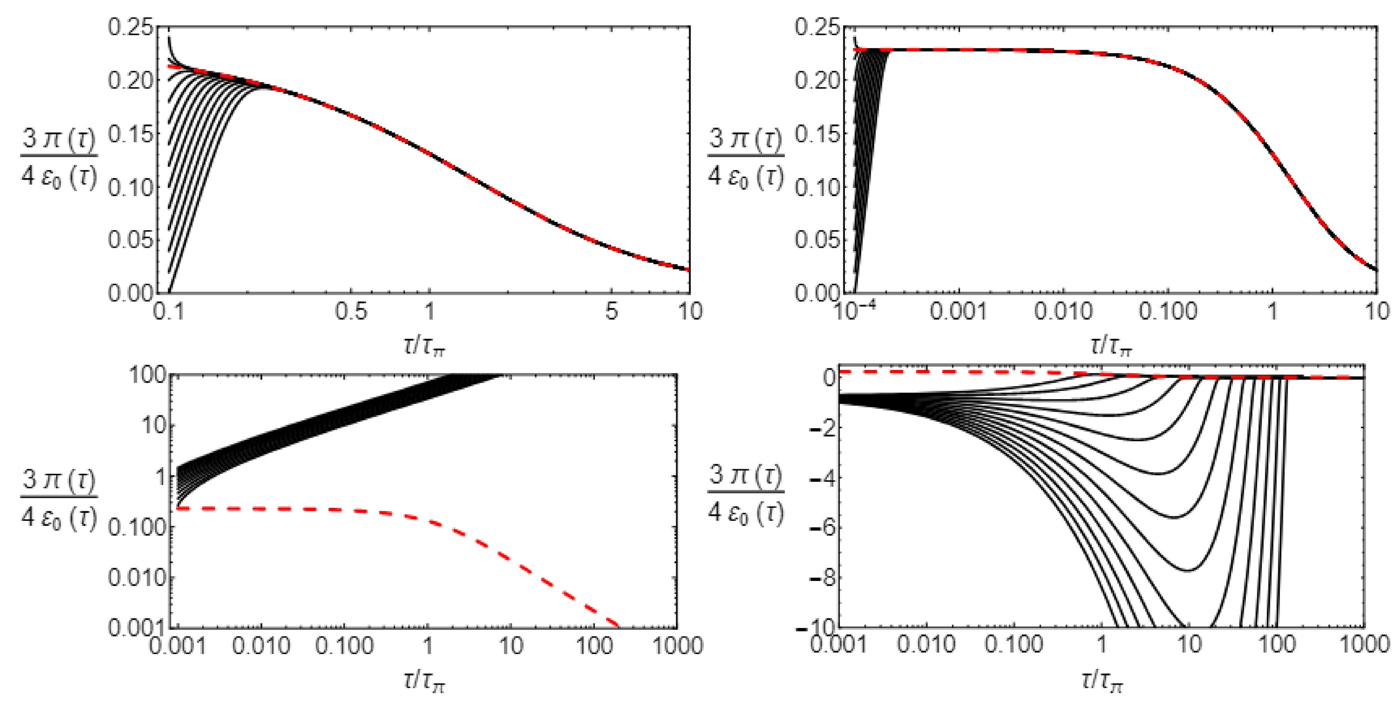


Fig. 2 – Evolution under Bjorken flow of the independent component of the shear-stress tensor according to the transient hydrodynamic theory for several initial conditions (continuous lines) in comparison with the attractor (dashed lines).

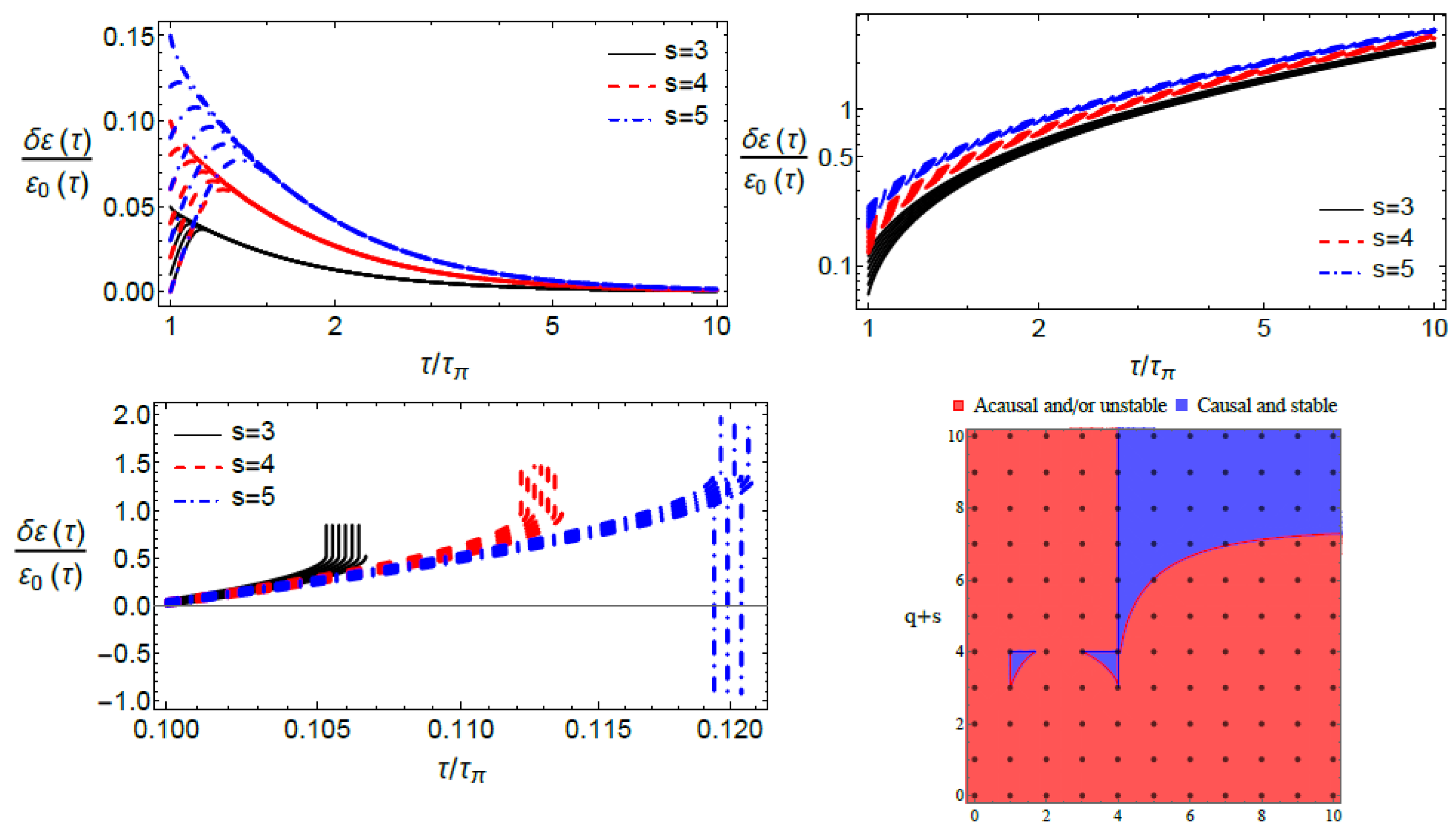


Fig. 3 – Evolution under Bjorken flow of the normalized dissipative component of the energy density according to BDNK theory for several initial conditions. (Lower right panel) Causal and stable regions in matching parameter space for BDNK theory.

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