THE FREEZEOUT PROCEDURE WITH THE METHOD OF MOMENTS

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Abstract

We demonstrate how the relativistic Boltzmann equation can be solved using a generalization of the method of moments. First, we show how to obtain a general equation of motion for the irreducible moments of a generic distribution function for arbitrary flow configurations. Then, we analyze a system of classical massless particles in Bjorken flow, a regime in which these equations assume a simple form, and show the consistency of this approach with numerical solutions of the Boltzmann equation.

I. Boltzmann equation

The relativistic Boltzmann equation is given by [1]

$$k^\mu \partial_\mu \underbrace{f_{\mathbf{k}}}_{\text{single-particle distribution function}} = C[f]$$

where C[f] is the **collision term**, defined as

$$C[f] = \frac{1}{2} \int dK' dP dP' \mathcal{W}_{\mathbf{k}\mathbf{k}' \leftrightarrow \mathbf{p}\mathbf{p}'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \right)$$

It describes a dilute gas taking into account only binary collisions.
How to solve this equation?

II. Generalized method of moments

Replace the integro-differential equation by a set of coupled equations of motion for the irreducible moments of the distribution function.

First, the single-particle distribution function is factorized as

$$f_{\mathbf{k}} = \underbrace{f_{0\mathbf{k}}}_{\text{equilibrium}} + \underbrace{\delta f_{\mathbf{k}}}_{\text{non-equilibrium}} = f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

$\phi_{\mathbf{k}}$ is expanded in terms of a *complete* basis of **irreducible** momenta [2]

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{\langle \mu_1 \cdots \mu_\ell \rangle} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}$$

- brackets denote irreducible projection: $A^{\langle \mu_1 \cdots \mu_\ell \rangle} \equiv \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} A^{\nu_1 \cdots \nu_\ell}$
- in practice, this expansion must be truncated at a given rank ℓ
- second-order [2] → $\ell = 2$; third-order [3] → $\ell = 4$

The **symmetric**, **traceless** and **orthogonal** projection operator is [4]

$$\Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} = \sum_{q=0}^{[\ell/2]} \frac{C(\ell, q)}{\mathcal{N}_{\ell,q}} \sum_{\mathcal{P}^\ell_\mu \mathcal{P}^\ell_\nu} \Delta^{\mu_1 \mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} \Delta_{\nu_1 \nu_2} \cdots \Delta_{\nu_{2q-1}\nu_{2q}} \Delta^{\mu_{2q+1}}_{\nu_{2q+1}} \cdots \Delta^{\mu_\ell}_{\nu_\ell}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$

with the coefficients being

$$C(\ell, q) = (-1)^q \frac{(\ell!)^2}{(2\ell)!} \frac{(2\ell - 2q)!}{q!(\ell - q)!(\ell - 2q)!}$$

(ensures the traceless property)

$$\mathcal{N}_{\ell,q} = \frac{1}{(\ell - 2q)!} \left(\frac{\ell!}{2^q q!} \right)^2$$

(inverse number of permutations)

Furthermore,

$$\lambda_{\mathbf{k}}^{\langle \mu_1 \cdots \mu_\ell \rangle} = \sum_{n=0}^{\infty} \Phi_n^{\langle \mu_1 \cdots \mu_\ell \rangle} P^{(\ell)}_{\mathbf{k}n}$$

orthogonal functions of $E_{\mathbf{k}}$

$$P^{(\ell)}_{\mathbf{k}n} = \sum_{r=0}^{n} a^{(\ell)}_{nr} E^r_{\mathbf{k}}$$

The single-particle distribution function then reads

$$f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \sum_{r=0}^{n} \frac{\mathcal{N}^{(\ell)}}{\ell!} a^{(\ell)}_{nr} P^{(\ell)}_{\mathbf{k}n} \rho_r^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \right)$$

where the irreducible moments of the **non-equilibrium** distribution function are

$$\rho_r^{\mu_1 \cdots \mu_\ell} = \int dK E^r_{\mathbf{k}} k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f_{\mathbf{k}}$$

How to obtain an expression for these moments?

From the Boltzmann equation, we can derive the equations of motion for the irreducible moments of **arbitrary rank ℓ**

$$\begin{aligned}
\dot{\varrho}_r^{\langle \mu_1 \cdots \mu_\ell \rangle} &= \mathcal{C}^{\mu_1 \cdots \mu_\ell}_{r-1} + r \varrho_{r-1}^{\mu_1 \cdots \mu_{\ell+1}} \dot{u}_{\mu_{\ell+1}} - \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} \nabla_{\nu_{\ell+1}} \varrho_{r-1}^{\nu_1 \cdots \nu_{\ell+1}} + (r-1) \varrho_{r-2}^{\mu_1 \cdots \mu_{\ell+2}} \sigma_{\mu_{\ell+1}\mu_{\ell+2}} + \ell \varrho_r^{\alpha \langle \mu_1 \cdots \mu_{\ell-1}} \omega^{\mu_\ell \rangle}{}_\alpha + \\
&+ \frac{\ell}{2\ell+1} \left[r m^2 \varrho_{r-1}^{\langle \mu_1 \cdots \mu_{\ell-1}} - (r + 2\ell + 1) \varrho_{r+1}^{\langle \mu_1 \cdots \mu_{\ell-1}} \right] \dot{u}^{\mu_\ell \rangle} + \frac{1}{3} \left[(r-1) m^2 \varrho_{r-2}^{\mu_1 \cdots \mu_\ell} - (r + \ell + 2) \varrho_r^{\mu_1 \cdots \mu_\ell} \right] \theta + \\
&+ \frac{\ell}{2\ell+3} \left[(2r-2) m^2 \varrho_{r-2}^{\alpha \langle \mu_1 \cdots \mu_{\ell-1}} - (2r + 2\ell + 1) \varrho_r^{\alpha \langle \mu_1 \cdots \mu_{\ell-1}} \right] \sigma^{\mu_\ell \rangle}_\alpha - \frac{\ell}{2\ell+1} \nabla^{\langle \mu_1} \left(m^2 \varrho_{r-1}^{\mu_2 \cdots \mu_\ell \rangle} - \varrho_{r+1}^{\mu_2 \cdots \mu_\ell \rangle} \right) + \\
&+ \frac{\ell(\ell-1)}{4\ell^2 - 1} \left[(r-1) m^4 \varrho_{r-2}^{\langle \mu_1 \cdots \mu_{\ell-2}} - (2r + 2\ell - 1) m^2 \varrho_r^{\langle \mu_1 \cdots \mu_{\ell-2}} + (r + 2\ell) \varrho_{r+2}^{\langle \mu_1 \cdots \mu_{\ell-2}} \right] \sigma^{\mu_{\ell-1}\mu_\ell \rangle}
\end{aligned}$$

- $\varrho_r^{\mu_1 \ldots \mu_\ell}$ are moments of a generic distribution function
- Navier-Stokes terms must be calculated separately for $\ell = 0, 1, 2$ [2].
- We recover the results obtained for second- [2] and third-order [3] theories, for $\ell = 0, 1, 2$ and $\ell = 3, 4$, respectively.

III. Bjorken flow

A highly symmetric, simplified framework to study heavy-ion collisions.

Milne coordinates: $\underbrace{\tau = \sqrt{t^2 - z^2}}_{\text{proper time}}$, $\underbrace{\eta_s = \frac{1}{2} \ln \left(\frac{t+z}{t-z} \right)}_{\text{spacetime rapidity}}$ $\underbrace{g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -\tau^2)}_{\text{metric tensor}}$

Assumptions:
- invariance under reflections with respect to η_s-axis
- invariance under translations along η_s-axis
- homogeneity and isotropy in the transverse plane

The distribution function for massless particles is $f_{\mathbf{k}} = f_{\mathbf{k}}(\tau, u^\mu k_\mu, z^\mu k_\mu)$

Then, the irreducible moments can be expressed as

$$\varrho_n^{\mu_1 \cdots \mu_{2\ell}} = \varrho_{n+2\ell,\ell} z^{\langle \mu_1} \cdots z^{\mu_{2\ell} \rangle} \xrightarrow[z_\mu = (0,0,0,-\tau)]{} \varrho_{n+2\ell,\ell} = \int dK k_0^{n+2\ell} P_{2\ell}(\cos\Theta) f_{\mathbf{k}}$$

In Bjorken flow, the equations of motion reduce simply to [4]

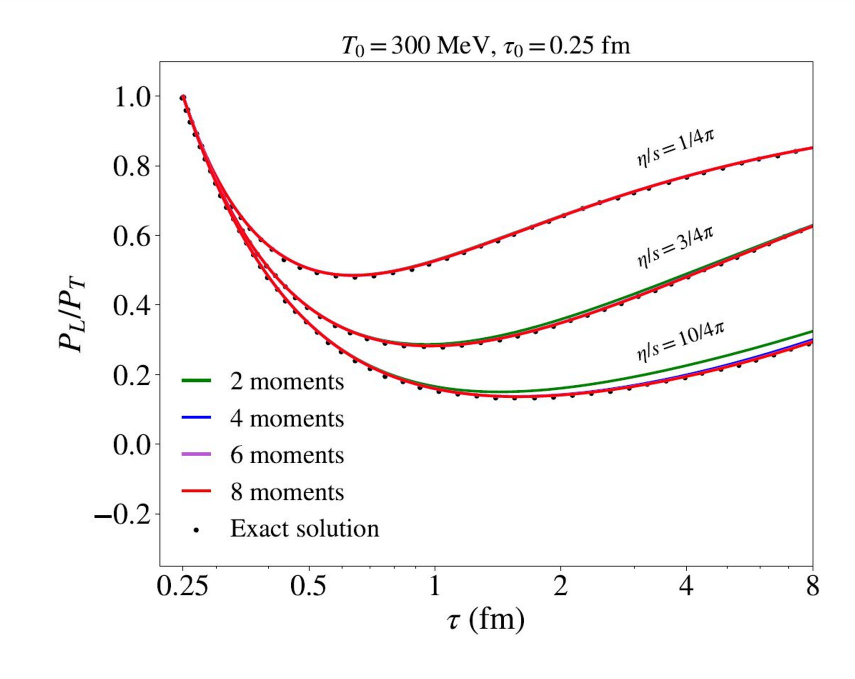
$$\begin{aligned}
D_\tau \varrho_{m,\ell} &= -2\ell \frac{(m + 2\ell)(2\ell - 1)}{(4\ell+1)(4\ell-1)} \frac{\varrho_{n,\ell-1}}{\tau} - \left[\frac{2\ell(2\ell+1) + m(24\ell^2 + 12\ell - 3)}{3(4\ell-1)(4\ell+3)} + \frac{2}{3} \right] \frac{\varrho_{m,\ell}}{\tau} \\
&- \frac{1}{\tau_R} \left(\varrho_{m,\ell} - \varrho^{\mathrm{eq}}_{m,\ell} \right) - (m - 2\ell - 1) \frac{(2\ell+1)(2\ell+2)}{(4\ell+1)(4\ell+3)} \frac{\varrho_{m,\ell+1}}{\tau}
\end{aligned}$$

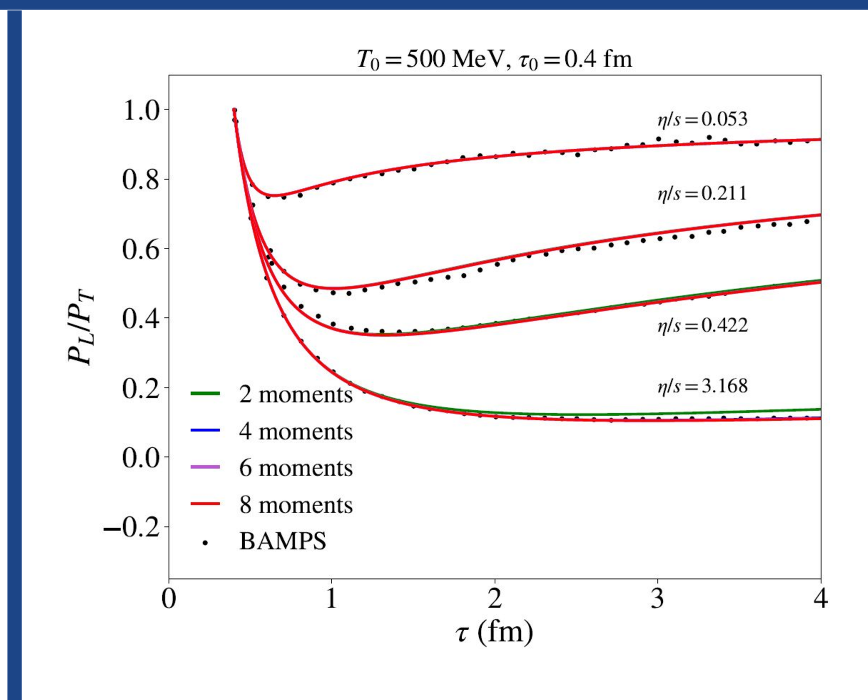
using the **relaxation time approximation** [5]: $C[f] = -\frac{E_{\mathbf{k}}}{t_R} \delta f_{\mathbf{k}}$

Finally, we are able to compute the dynamics of *all* irreducible moments and thus obtain a consistent expression for the single-particle distribution function.

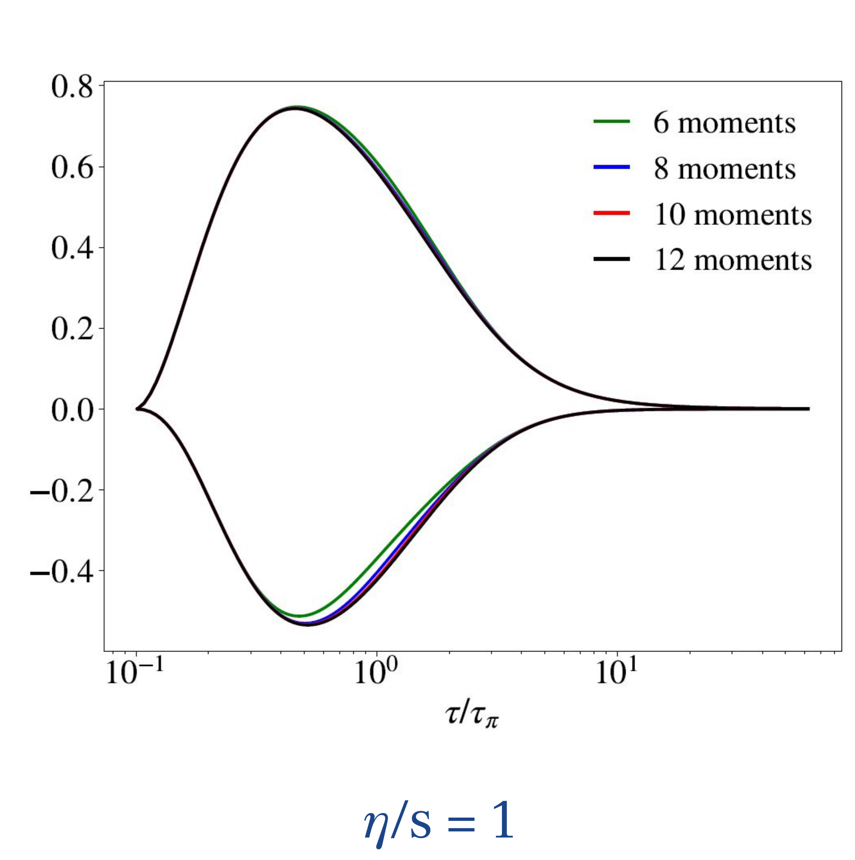
IV. Results

Pressure anisotropy

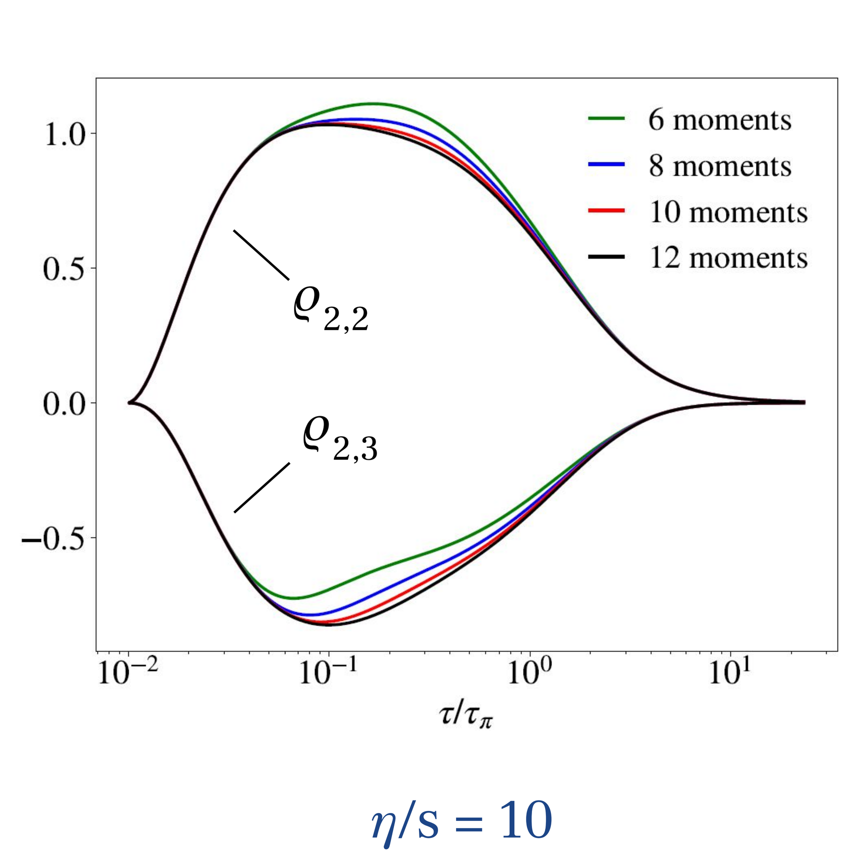




First two non-hydrodynamic moments



$\eta/s = 1$



$\eta/s = 10$

A system of classical massless particles in Bjorken flow, assuming $\tau_R = \tau_\pi = 5\eta/(Ts)$ [2].

References

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