Non-Gaussian Fluctuations in Relativistic Fluids

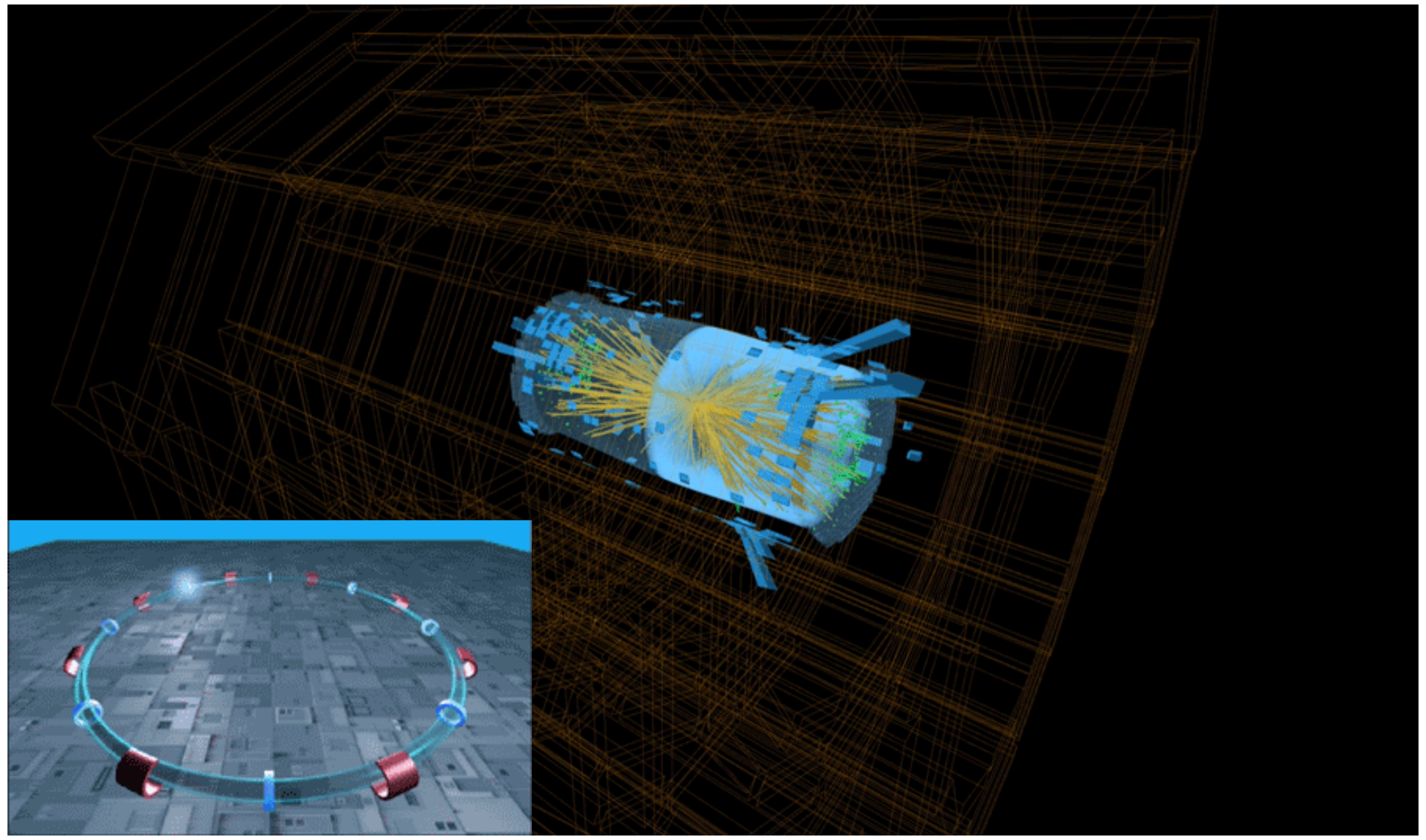
Xin An

Based on work with Basar, Stephanov and Yee

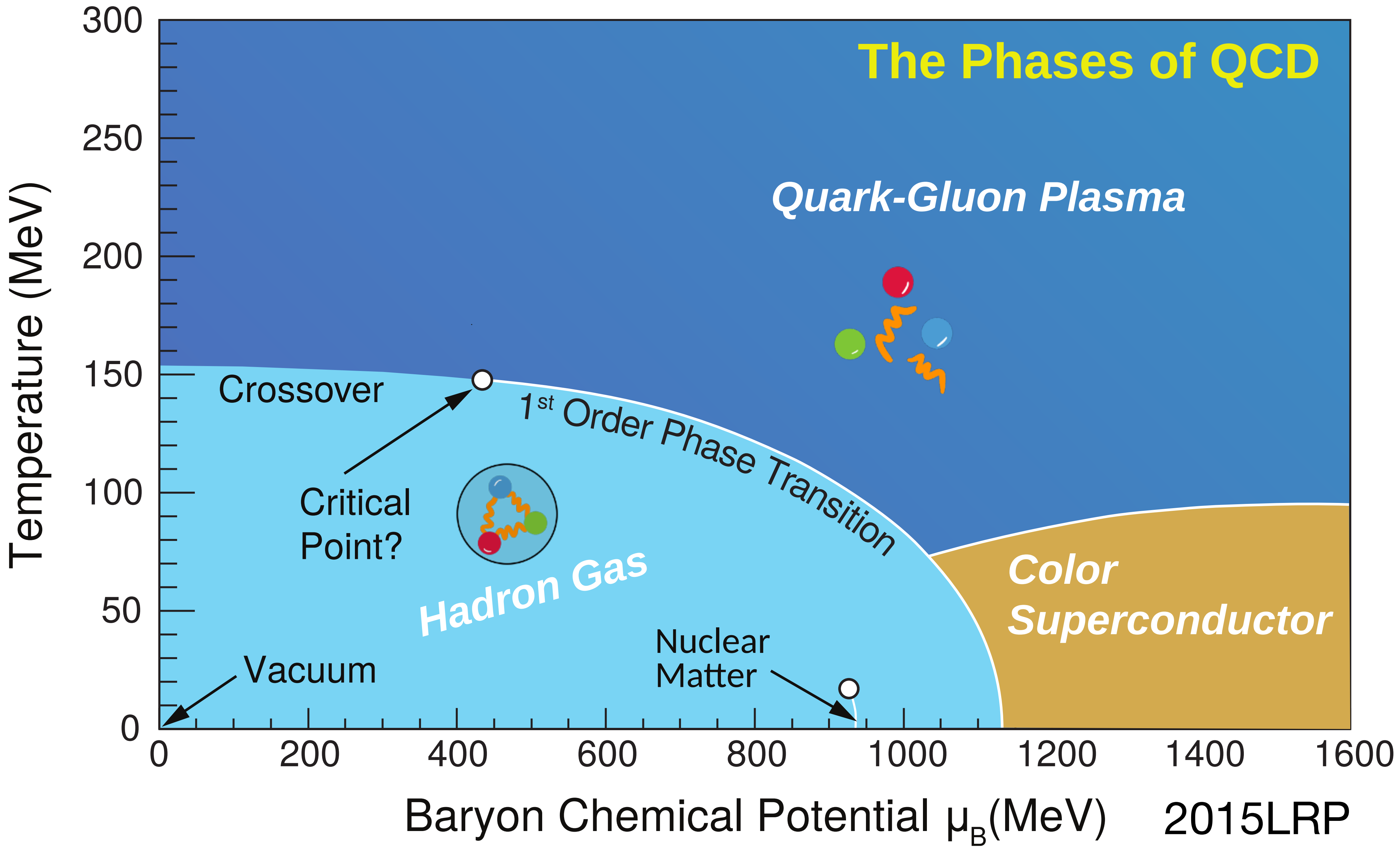
Sep 5 2023, Houston

Motivation

Filling the gap



Collision event simulation at LHC (CERN)



We are using ***non-equilibrium*** **techniques**

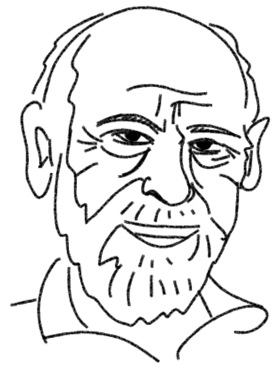
to explore the ***equilibrium*** **QCD phase structure**

via **fluctuations!**

Fluctuations in equilibrium

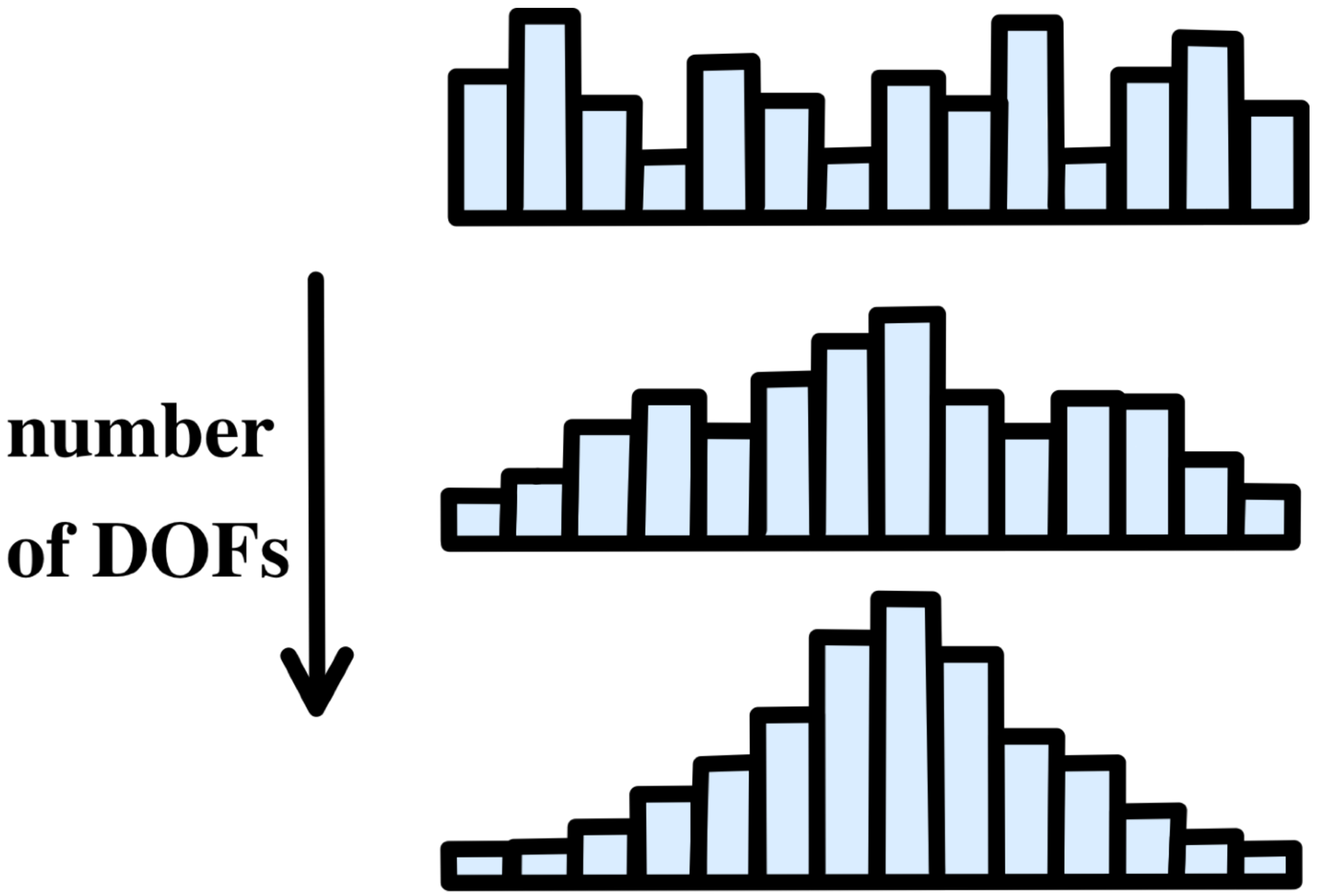
- Thermal fluctuations: systems possess *large* number of DOFs; *small* deviation from Gaussian due to CLT.

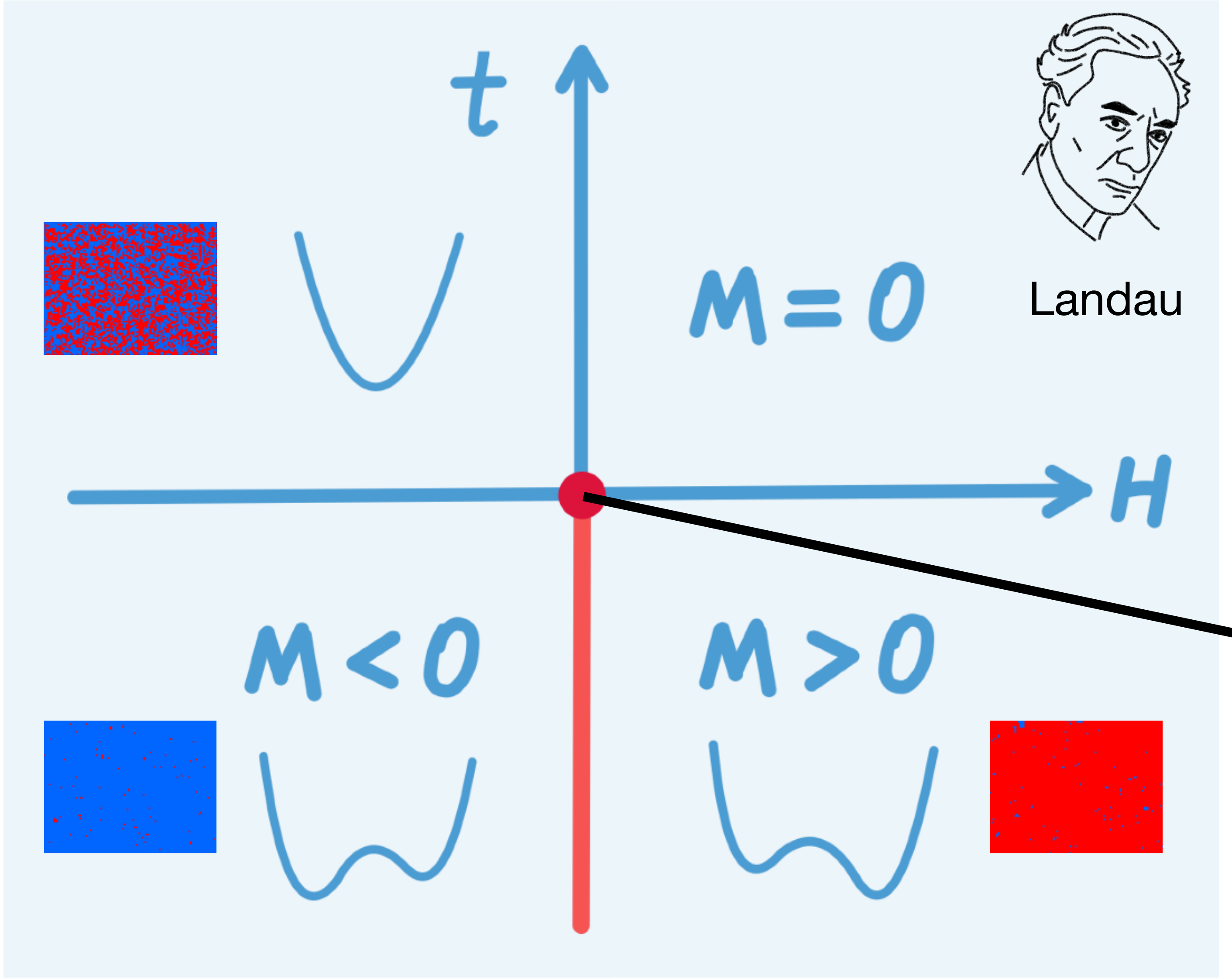
Thermal equilibrium is extremely boring.



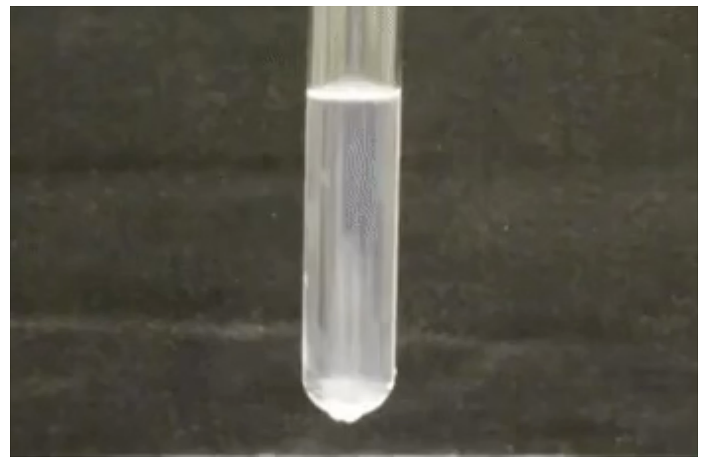
Susskind

- Non-Gaussian fluctuations become more important when systems possess *smaller* number of DOFs (e.g., *closer* to the critical point).

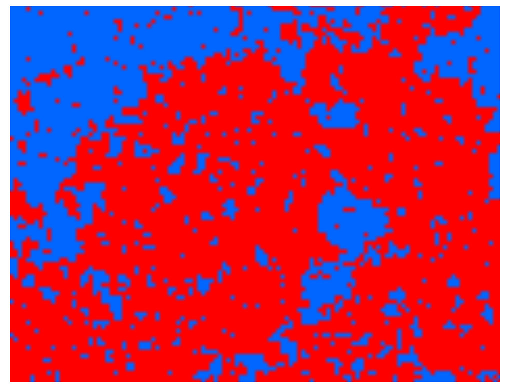




Ising phase diagram



$\xi \leftrightarrow \lambda_{\text{light}}$

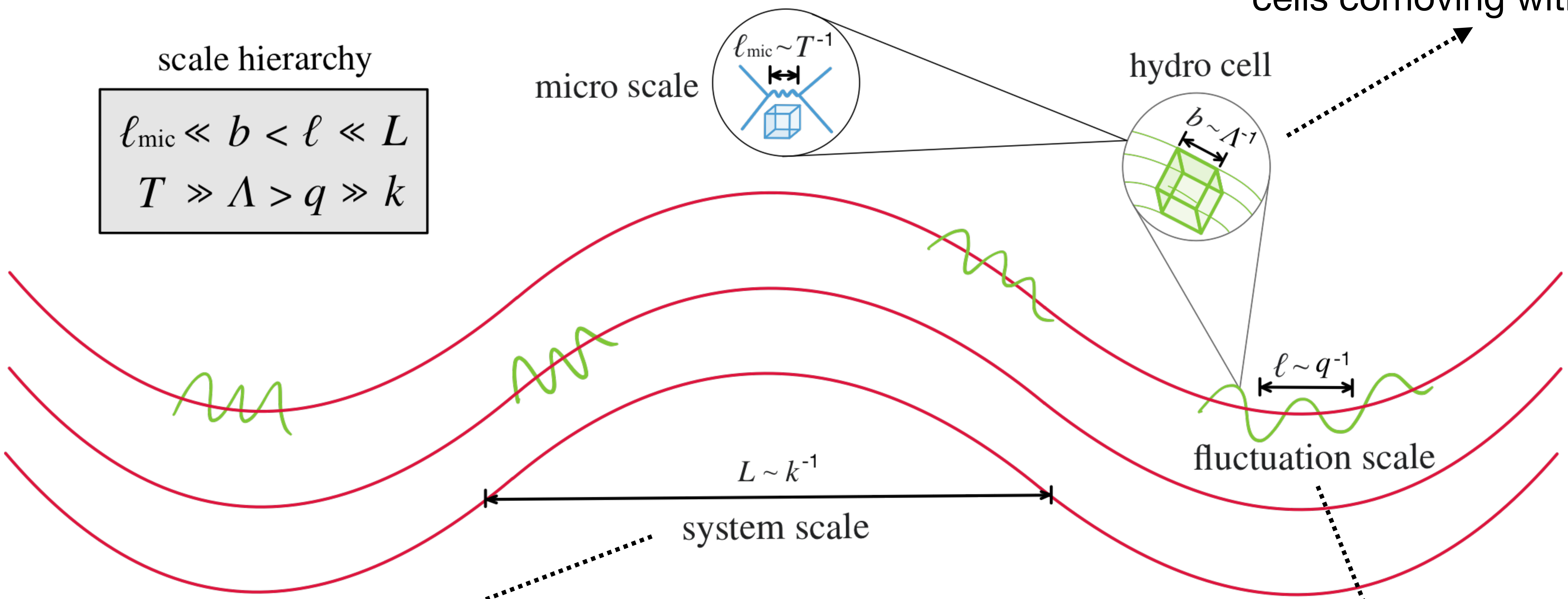


$\xi \to \infty$

Fluctuations out of equilibrium

- Hydrodynamic fluctuations:

large number of *locally* thermalized cells comoving with fluid



fluctuations *not* equilibrated at *large* scales

evolution described by a set of conservation equations

$$\partial_t \psi = \nabla \cdot (\text{flux}\,[\psi]) \quad \text{where} \quad \psi = (\,\text{n}, \epsilon, \pi_{\text{i}}\,)$$

Fluctuating hydro description of QGP

- QGP in heavy-ion collisions:

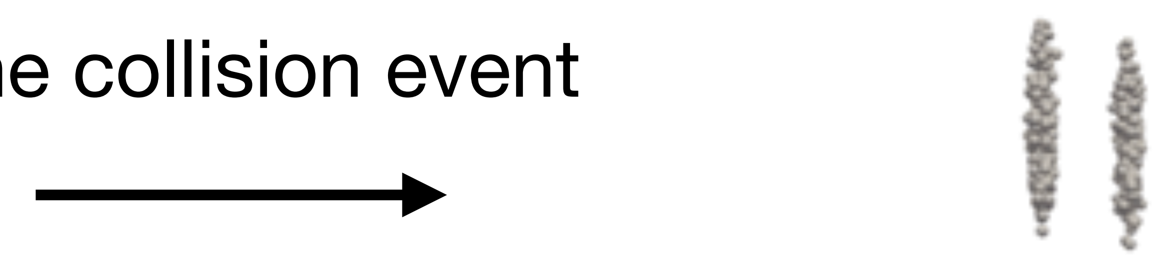
Size of the fire balls ~ 10 fm

small enough for fluctuations to be important

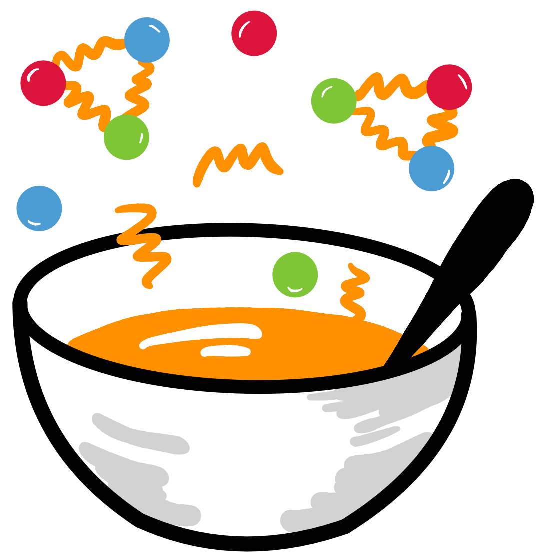
Number of particles $\sim 10^2 - 10^4$

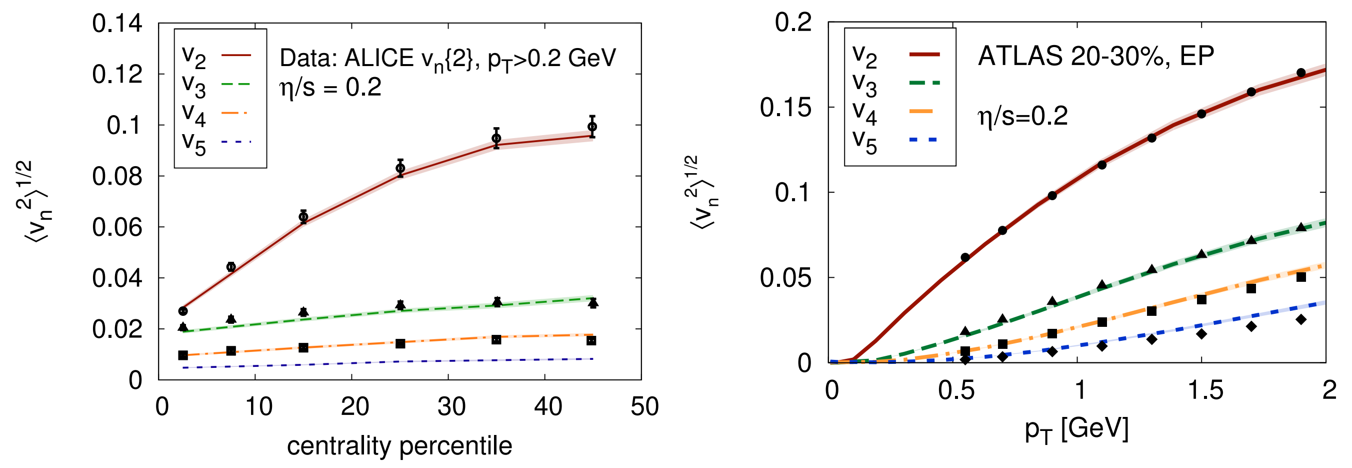
large enough for hydro to be applicable

One collision event



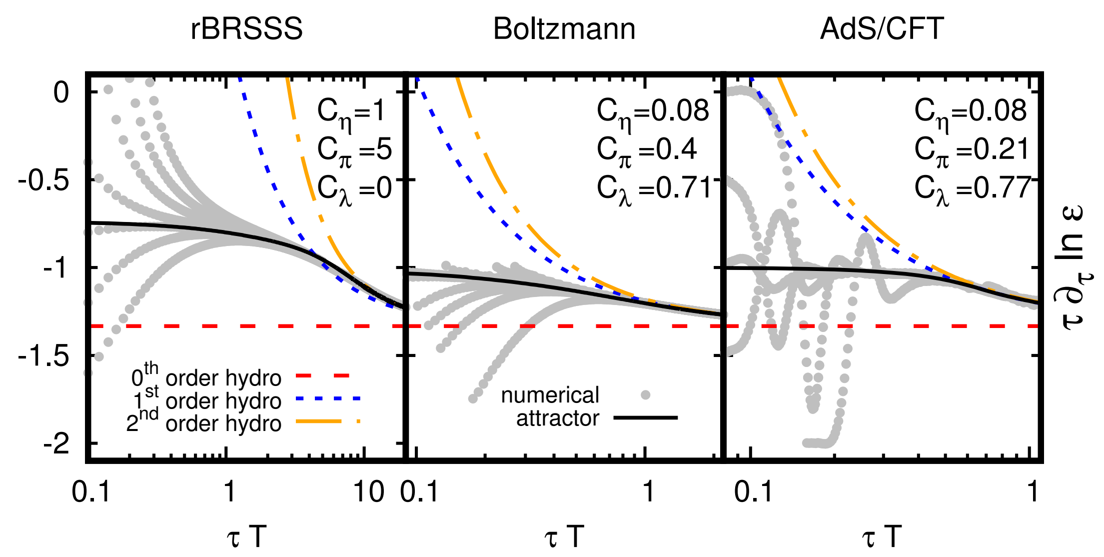
Observables obtained from samples fluctuate event-by-event





Flow collectivity manifests QGP as a *perfect fluid*

Gale et al, 1301.5893



Hydrodynamic *attractor* even far from equilibrium

Florkowski et al, 1707.02282, Romatschke et al, 1712.05815

General theory of fluctuation dynamics

Theories

Top-down like (EFTs)

Starting from effective action with first principles

e.g., Martin-Siggia-Rose (MSR), Schwinger-Keldysh (SK), Hohenberg-Halperin (HH), n-particle irreducible (nPI), etc.

Glorioso et al, 1805.09331

Jain et al, 2009.01356

Sogabe et al, 2111.14667

Chao et al, 2302.00720

…

bottom-up like (PDEs)

Starting from phenomenological equations with required properties

e.g., Langevin equations in *stochastic* description, Fokker-Planck (FP) equations in *deterministic* description.

Akamatsu et al, 1606.07742

Nahrgang et al, 1804.05728

Singh et al, 1807.05451

Chattopadhyay et al, 2304.07279

…

Two approaches in PDEs

Stochastic

Langevin equation

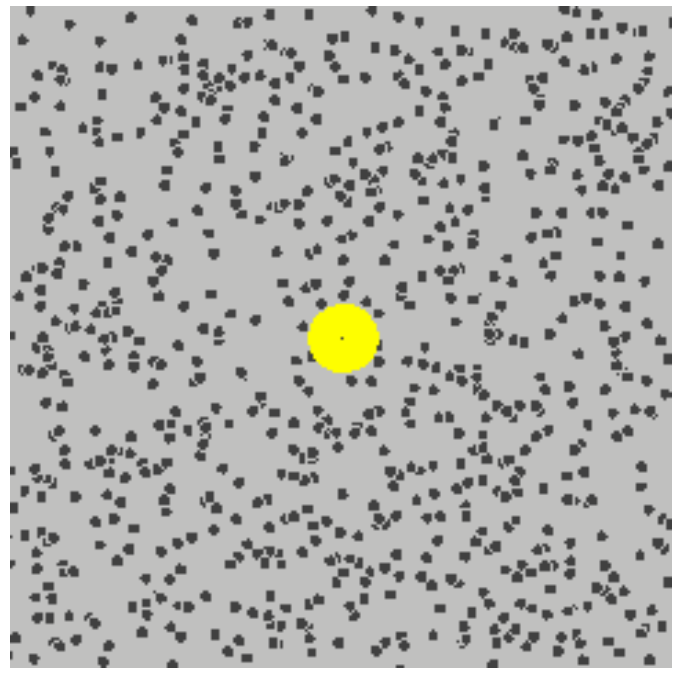
Newton's equation + noise

$$\partial_t \psi_i = F_i[\psi] + \eta_i$$

$$\langle \eta_i(x_1)\, \eta_j(x_2) \rangle = 2Q_{ij}\, \delta^{(3)}(x_1 - x_2)$$

Newton

Langevin



Brownian motion

One equation

Millions of samples

Deterministic

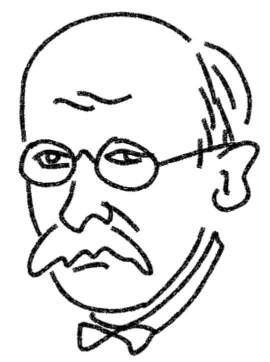
Fokker-Planck equation

probability evolution equation

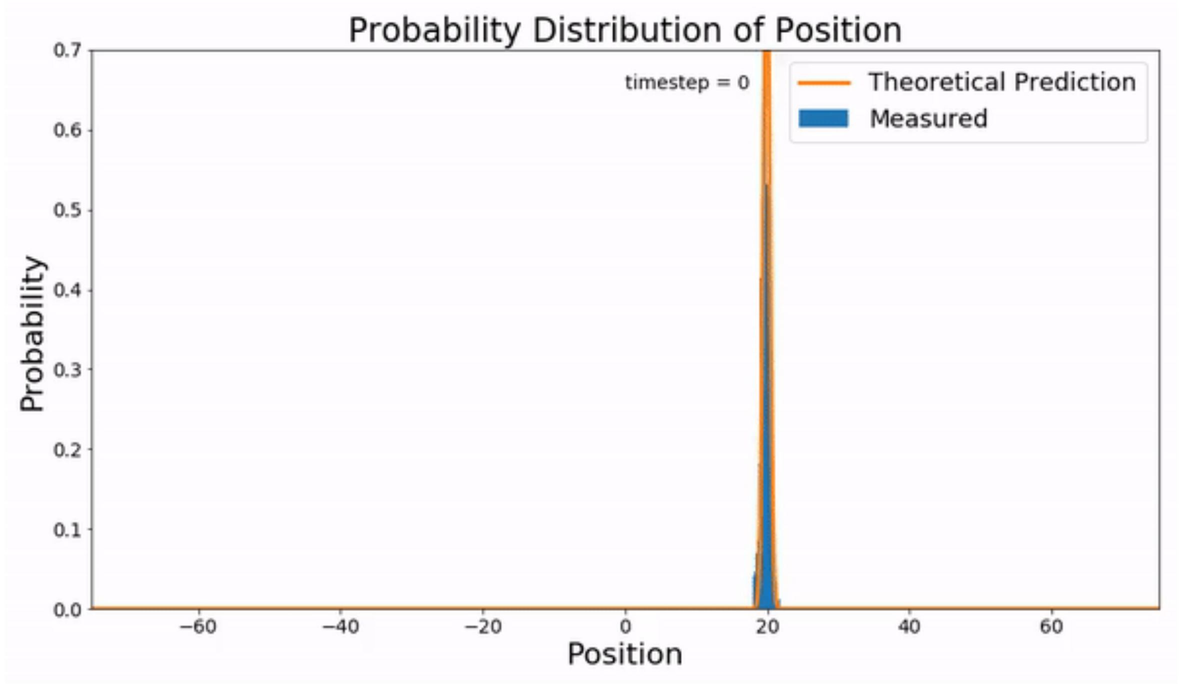
$$\partial_t P[\psi] = \partial_\psi \left(\mathrm{flux}[\psi] \right)$$

$$\mathrm{flux}[\psi] = -\,\mathrm{FP} + \partial_\psi(\mathrm{QP})$$

Fokker



Planck



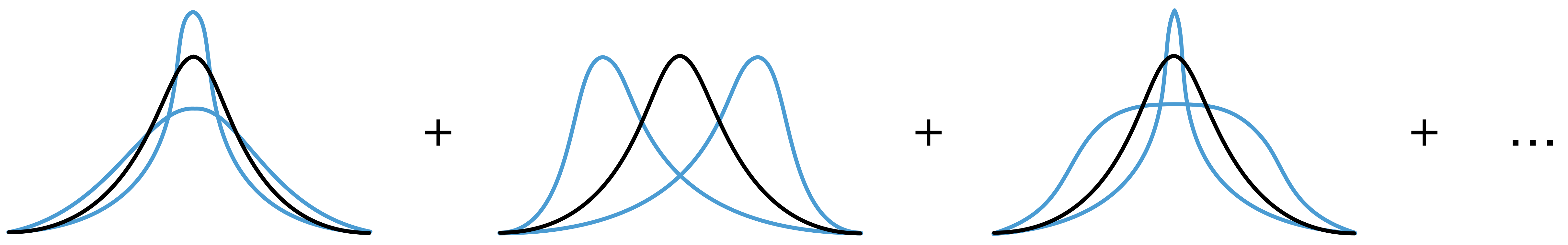
(Wikipedia)

One sample

Millions of equations

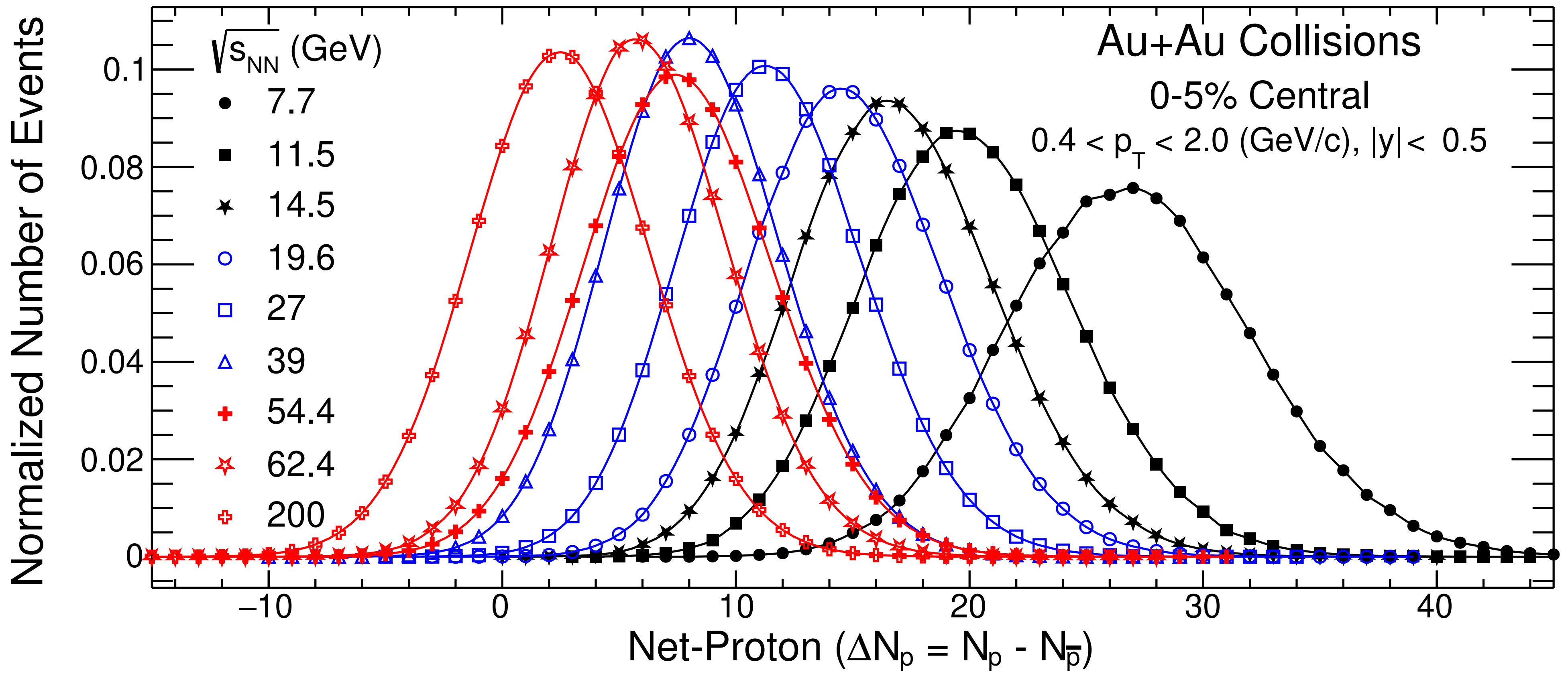
Correlators

- Both approaches consider n-pt correlators $G_n \equiv \langle \underbrace{\phi \ldots \phi}_{n} \rangle \equiv \int d\psi \, P[\psi] \, \underbrace{\phi \ldots \phi}_{n}$ where $\phi \equiv \psi - \langle \psi \rangle$.

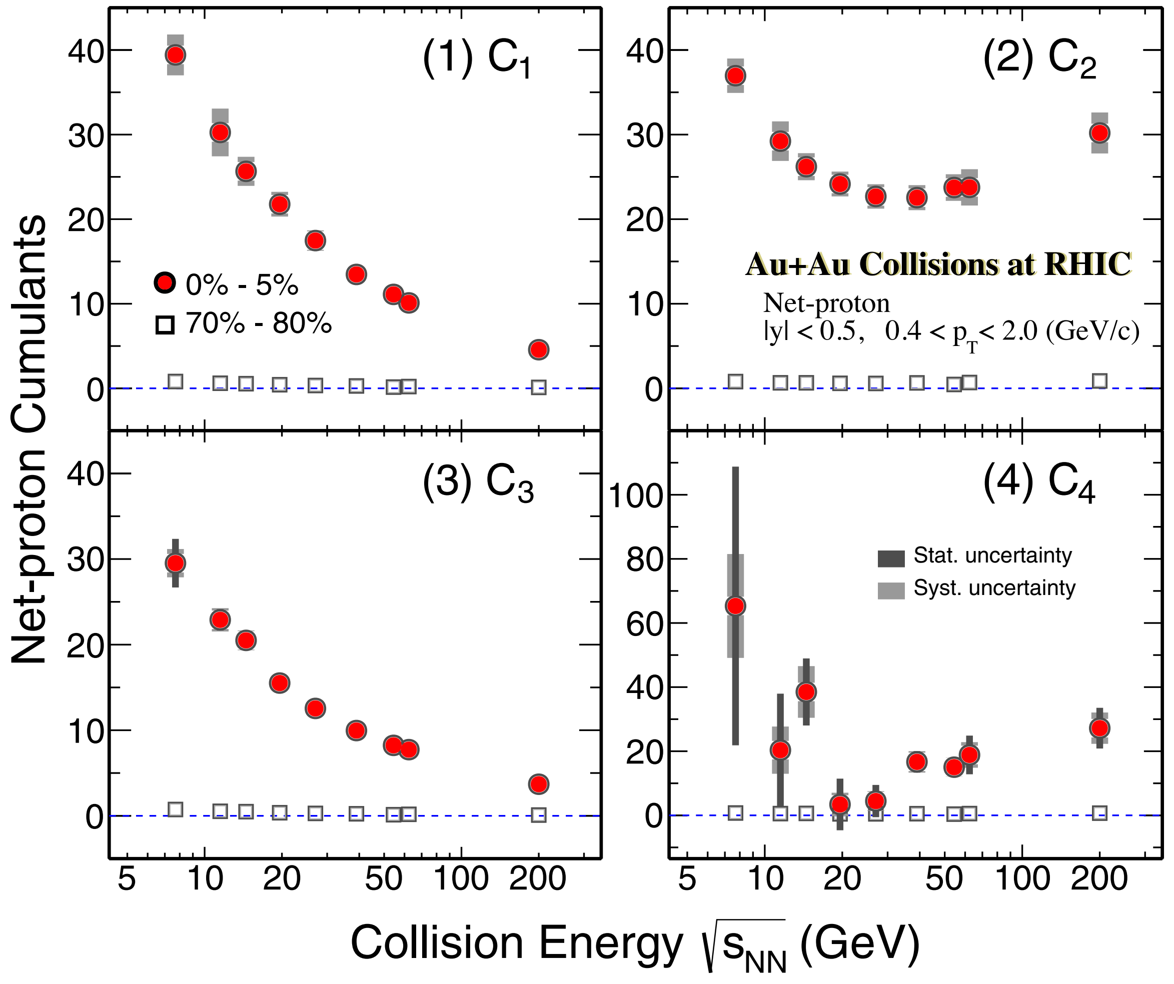


variance → width | skewness → lopsidedness | kurtosis → sharpness

n-pt correlators are related to *cumulants* measured in HIC



Events number vs net-proton yields (STAR)



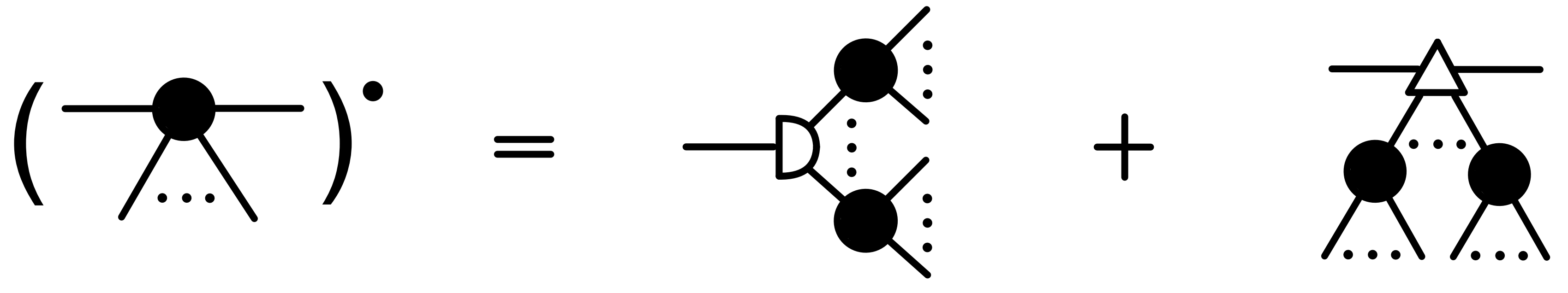
Net-proton cumulants vs energy

Dynamics of correlators

- Evolution equations for n-pt correlators: XA et al, 2009.10742, 2212.14029

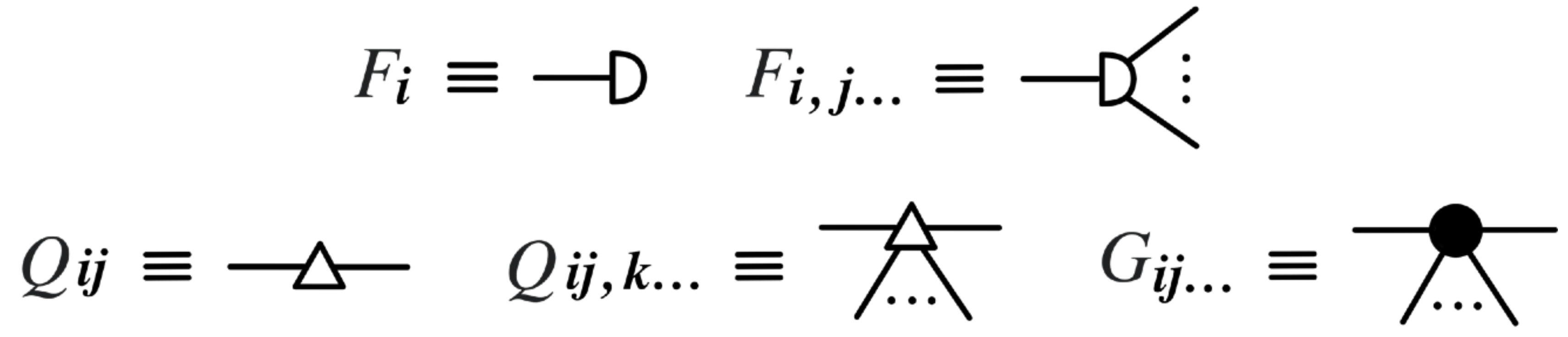
$$\partial_t G_n = \mathscr{F}\,[\psi, G_2, G_3, \ldots, G_n, G_{n+1}, \ldots, G_\infty]$$

need ∞ equations to close the system!



drift noise

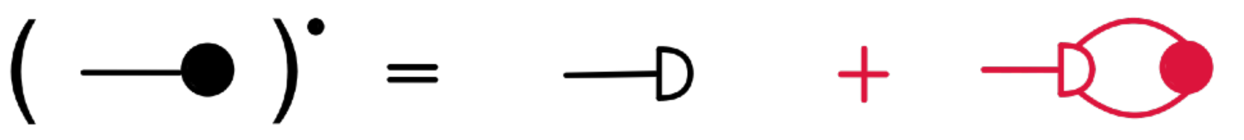
all *combinatorial* configurations of trees



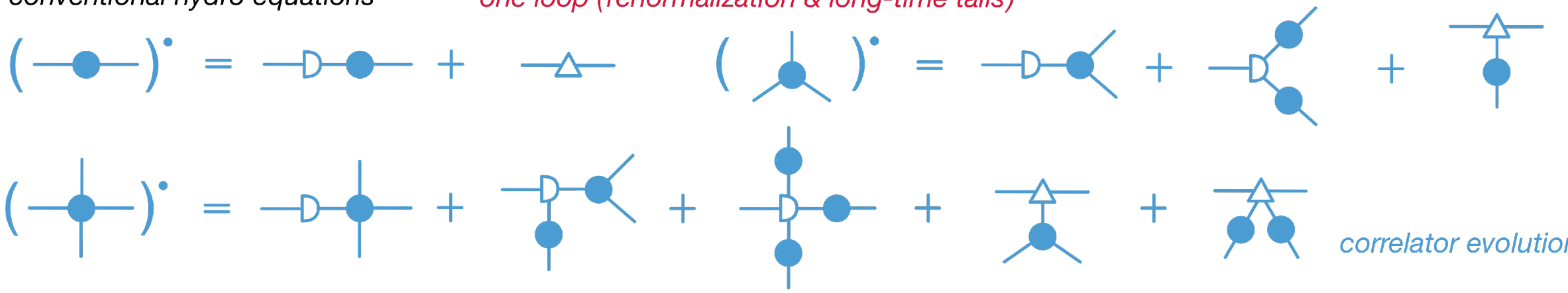
- Introducing the loop expansion parameters $\varepsilon \sim 1/\mathrm{DOFs}$, Correlator evolution equations can be truncated and iteratively solved: XA et al, 2009.10742

$$\partial_t G_n = \mathscr{F}\,[\psi, G_2, G_3, \ldots, G_n] + \mathcal{O}(\varepsilon^n)$$

where $G_n \sim \varepsilon^{n-1}$, $F \sim 1$, $Q \sim \varepsilon$.



conventional hydro equations *one loop (renormalization & long-time tails)*

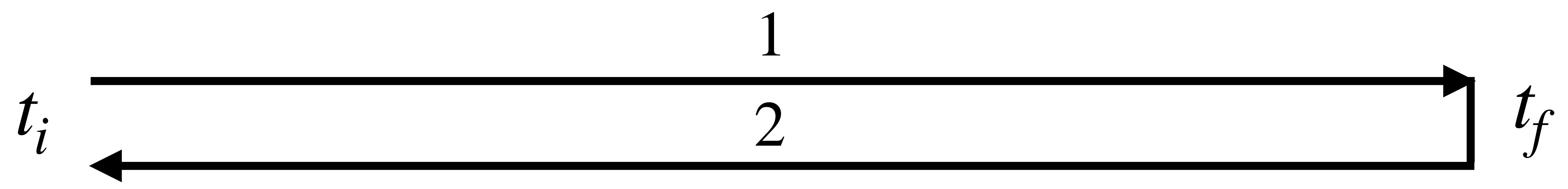


correlator evolution equations

extendable straightforwardly to *higher-pt* correlators (related to C5, C6, …)

Connection to EFTs

- Schwinger-Keldysh formalism



Schwinger Keldysh

$$Z = \int \mathscr{D}\psi_1 \mathscr{D}\psi_2 \mathscr{D}\chi_1 \mathscr{D}\chi_2 \, e^{\, i I_0(\psi_1, \chi_1) - i I_0(\psi_2, \chi_2)} = \int \mathscr{D}\psi_1 \mathscr{D}\psi_2 \, e^{\, i \int_\tau \mathscr{L}_{\text{EFT}}}$$

- The effective Lagrangian is constructed following *fundamental symmetries*:

Glorioso et al, 1805.09331; Jain et al, 2009.01356

$$\mathscr{L}_{\text{EFT}}(\psi_r, \psi_a) = \psi_a Q^{-1}(F - \dot{\psi}_r) + i\psi_a Q^{-1}\psi_a \quad \text{where} \quad \psi_r = \frac{1}{2}(\psi_1 + \psi_2), \quad \psi_a = \psi_1 - \psi_2$$

$$P[\psi] = \int_{\psi_r = \psi(t)} \mathscr{D}\psi_r \mathscr{D}\psi_a \, J(\psi_r) \, e^{\, i \int_{-\infty}^{t} d\tau \mathscr{L}_{\text{EFT}}} \longrightarrow \partial_t P[\psi] = \frac{\partial}{\partial \psi}\left(\text{flux}[\psi]\right)$$

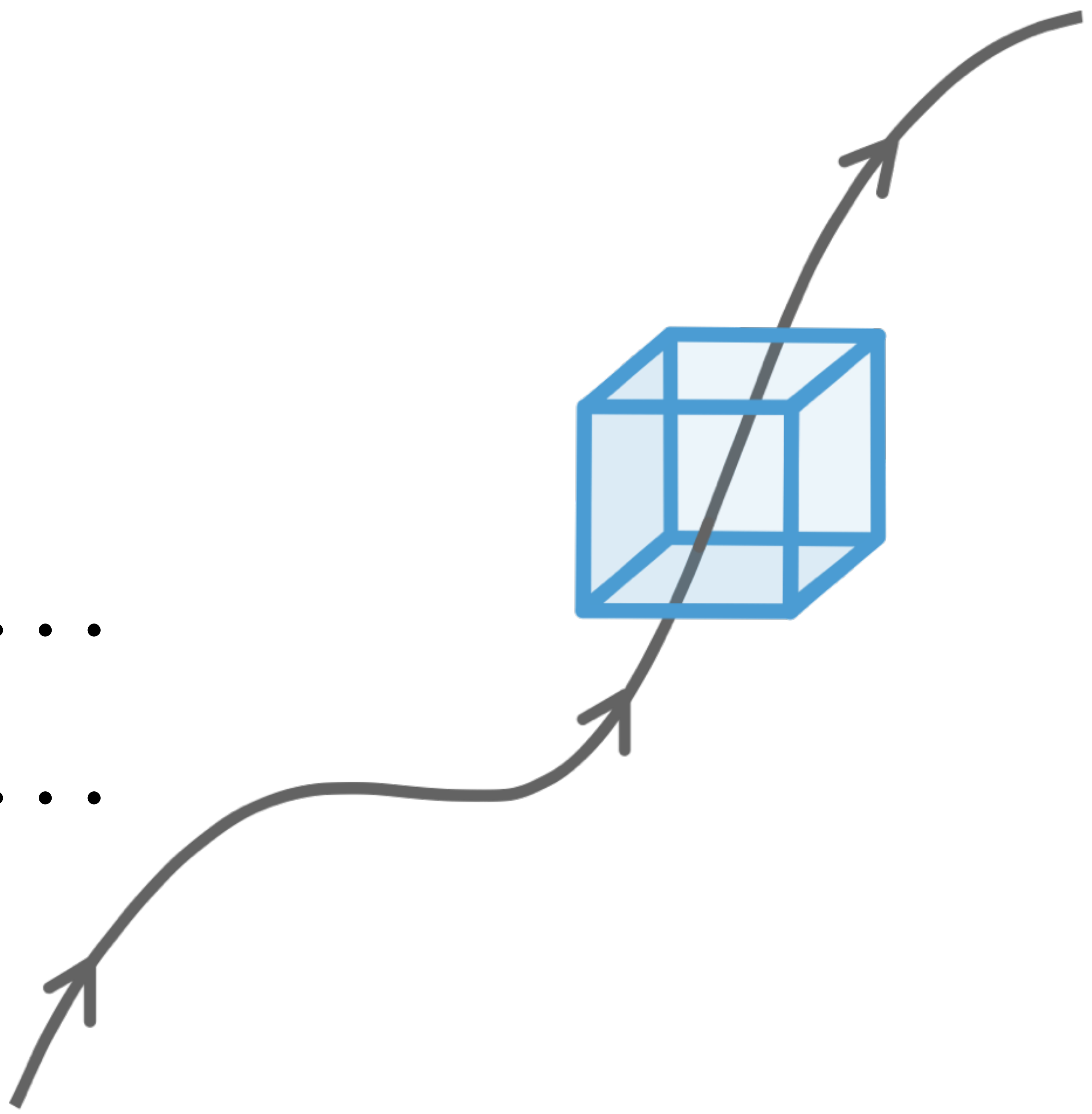
XA et al, in progress

Fluctuation dynamics in relativistic fluids

Relativistic dynamics

Eulerian specification

more often used in non-relativistic theory

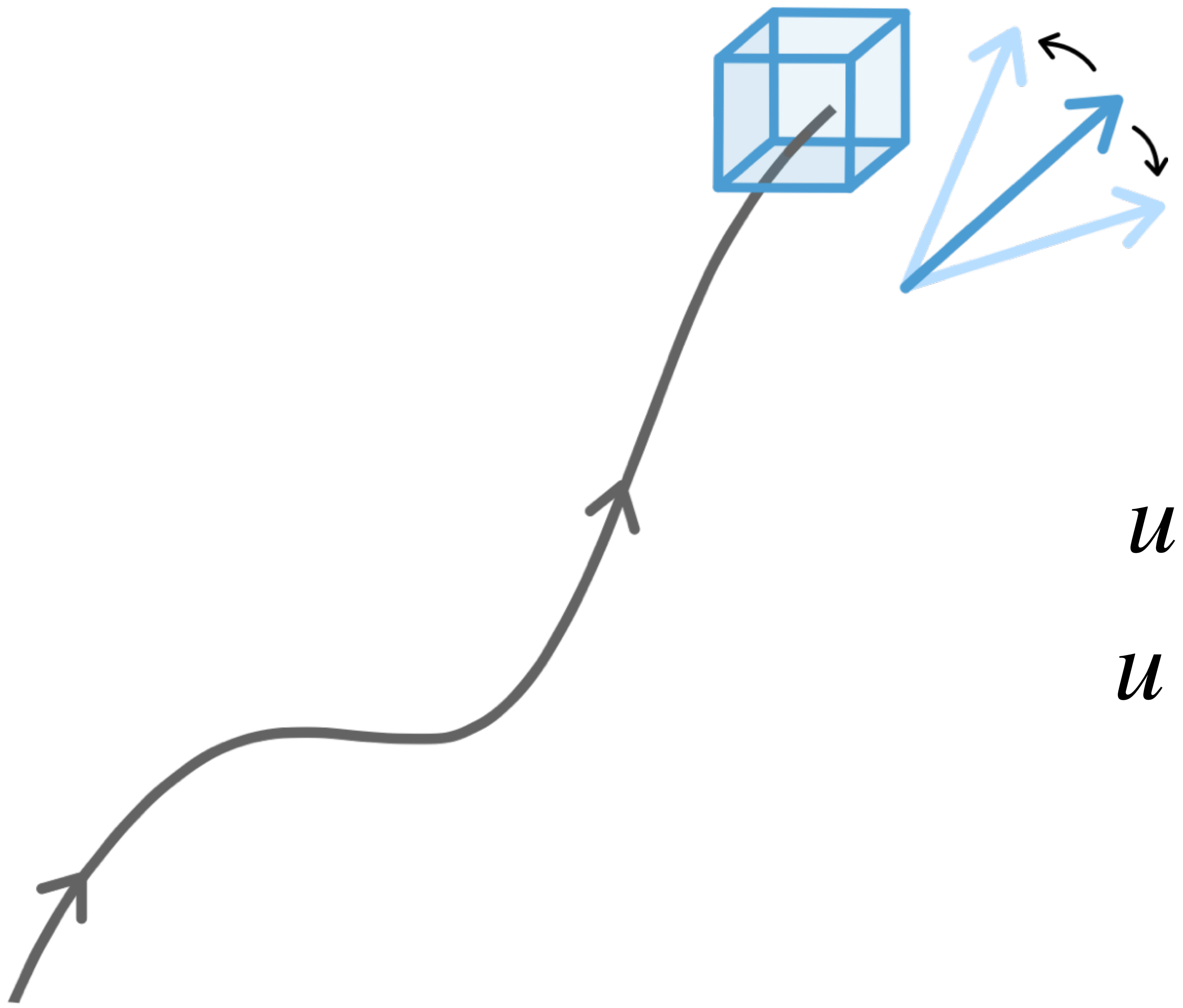


$$(\partial_t + v \cdot \nabla)\,\psi_i = \ldots$$
$$(\partial_t + v \cdot \nabla)\,G_n = \ldots$$

There is a global time for every observer.
All correlators G_n can be measured at the same time in the same frame (lab).

Lagrangian specification

more convenient for relativistic theory



$$u = u(\psi)$$

$$u \cdot \partial\,\psi_i = \ldots$$
$$u \cdot \partial G_n = \ldots$$

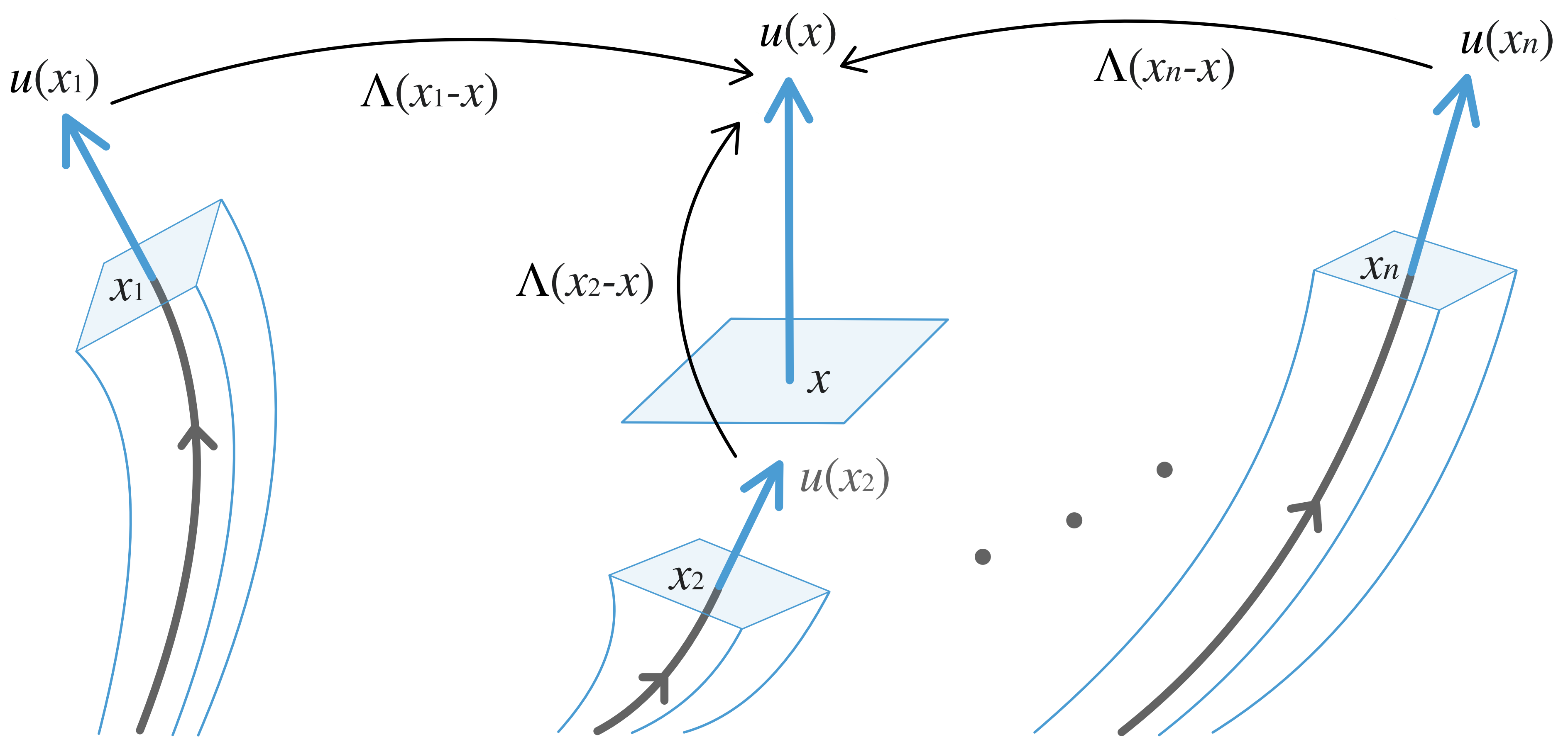
Each fluid cell has its own clock (proper time).
How to define the analogous *equal-time* correlator G_n in relativistic theory?

Confluent formulation: correlator and derivative

- Confluent formulation: covariant description for the comoving fluctuations.

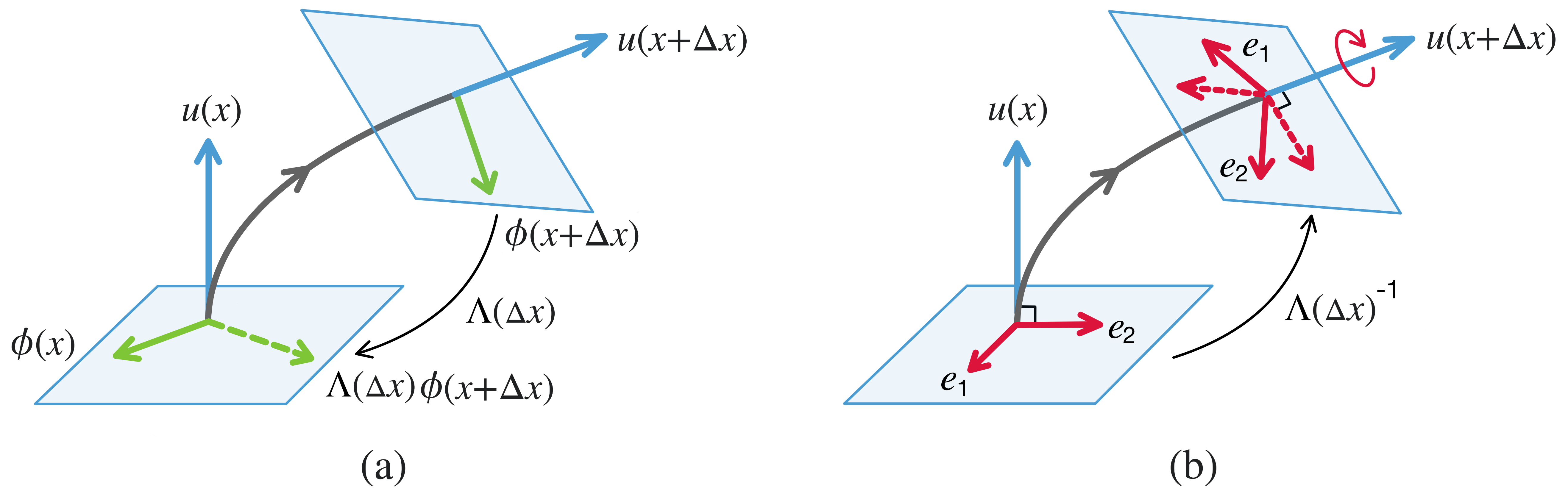
  See XA et al, 2212.14029 for more details

Confluent correlator $\bar{G}$



boost all fields (measured at their own local rest frame) to one common frame (chosen at their midpoint)

Confluent derivative $\bar{\nabla}$

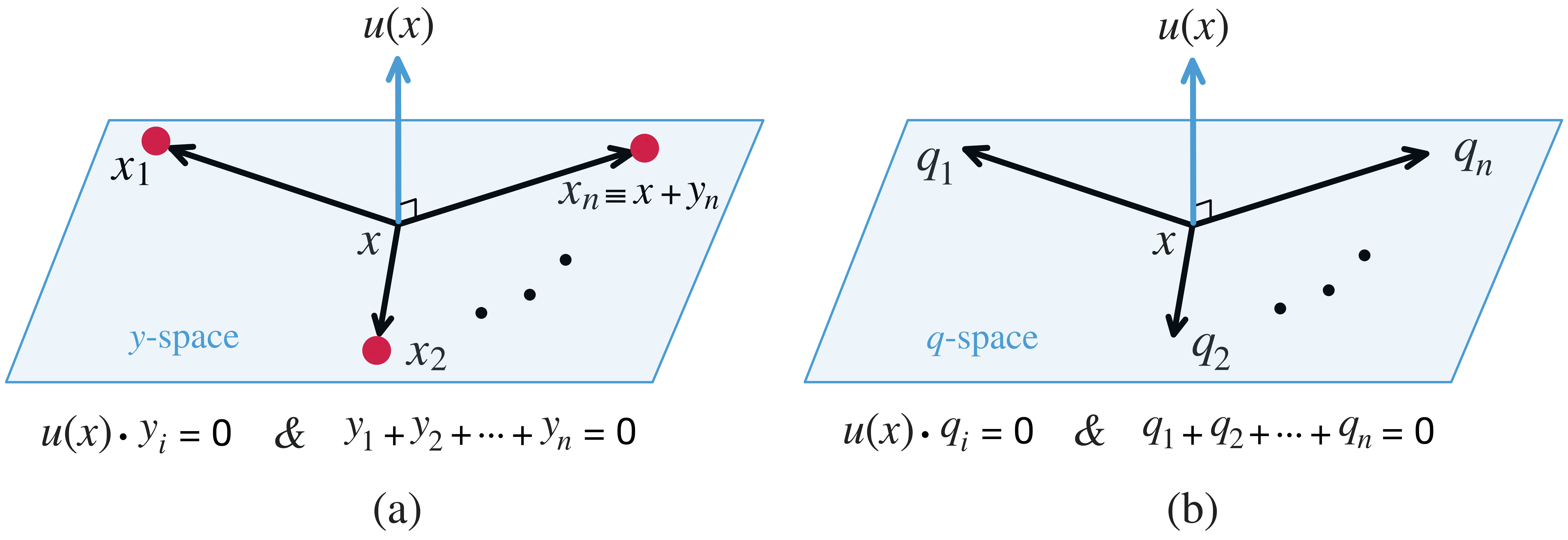


the frame at midpoint moves accordingly as the n points move, the difference of a given field before and after the movement is calculated in one same frame, with the equal-time constraint preserved by introducing the local triad e_a^μ with $a = 1,2,3$

Confluent formulation: Wigner function

- The confluent n-pt Wigner transform from x-independent variable $y^a = e^a_\mu(x)\, y^\mu$ to q^a with $a = 1,2,3$. XA et al, 2212.14029

$$W_n(x; q_1^a, \ldots, q_n^a) = \int \prod_{i=1}^{n} \left(d^3 y_i^a\, e^{-iq_{ia}y_i^a}\right) \delta^{(3)}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^a\right) \bar{G}_n(x + e_a y_1^a, \ldots, x + e_a y_n^a)$$



y-space q-space

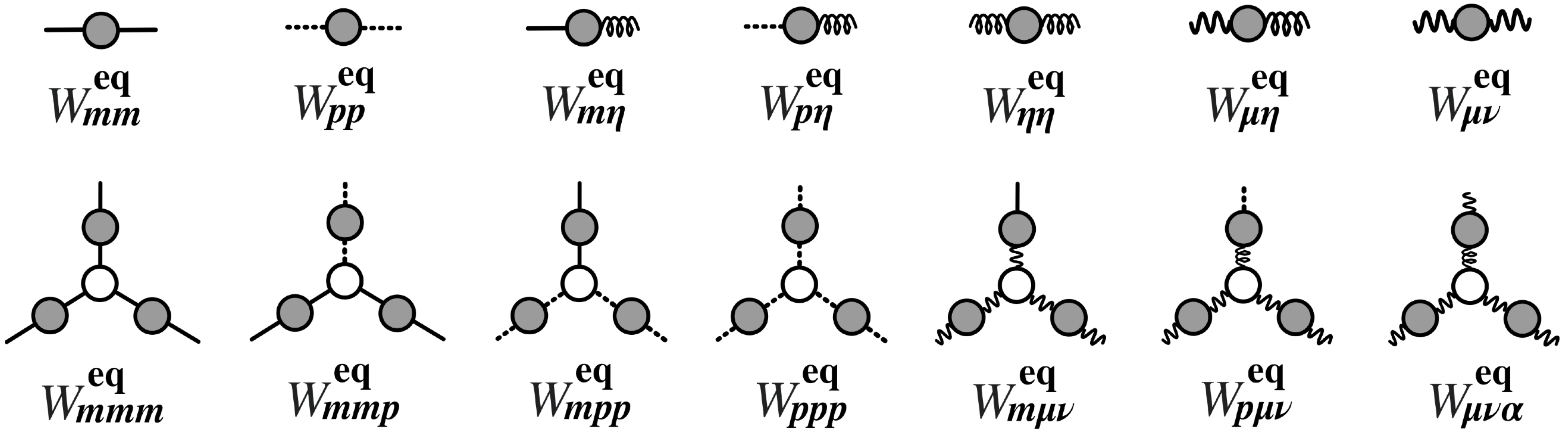
Confluent fluctuation evolution equations

- Fluctuation evolution equations in the *impressionistic* form: XA et al, in progress

$$\mathscr{L}W_n = ic_s q(W_n - \dots) + \gamma q^2(W_n - \dots) + kW_n + \dots \quad \text{where} \quad \mathscr{L} = u\cdot\bar{\nabla}_x + f\cdot\nabla_q$$

of which the solutions match thermodynamics with entropy $S(m, p, u_\mu, \eta)$.

m: entropy per baryon; p: pressure; η: Lagrange multiplier for $u^2 = -1$.



Equilibrium solutions in diagrammatic representation

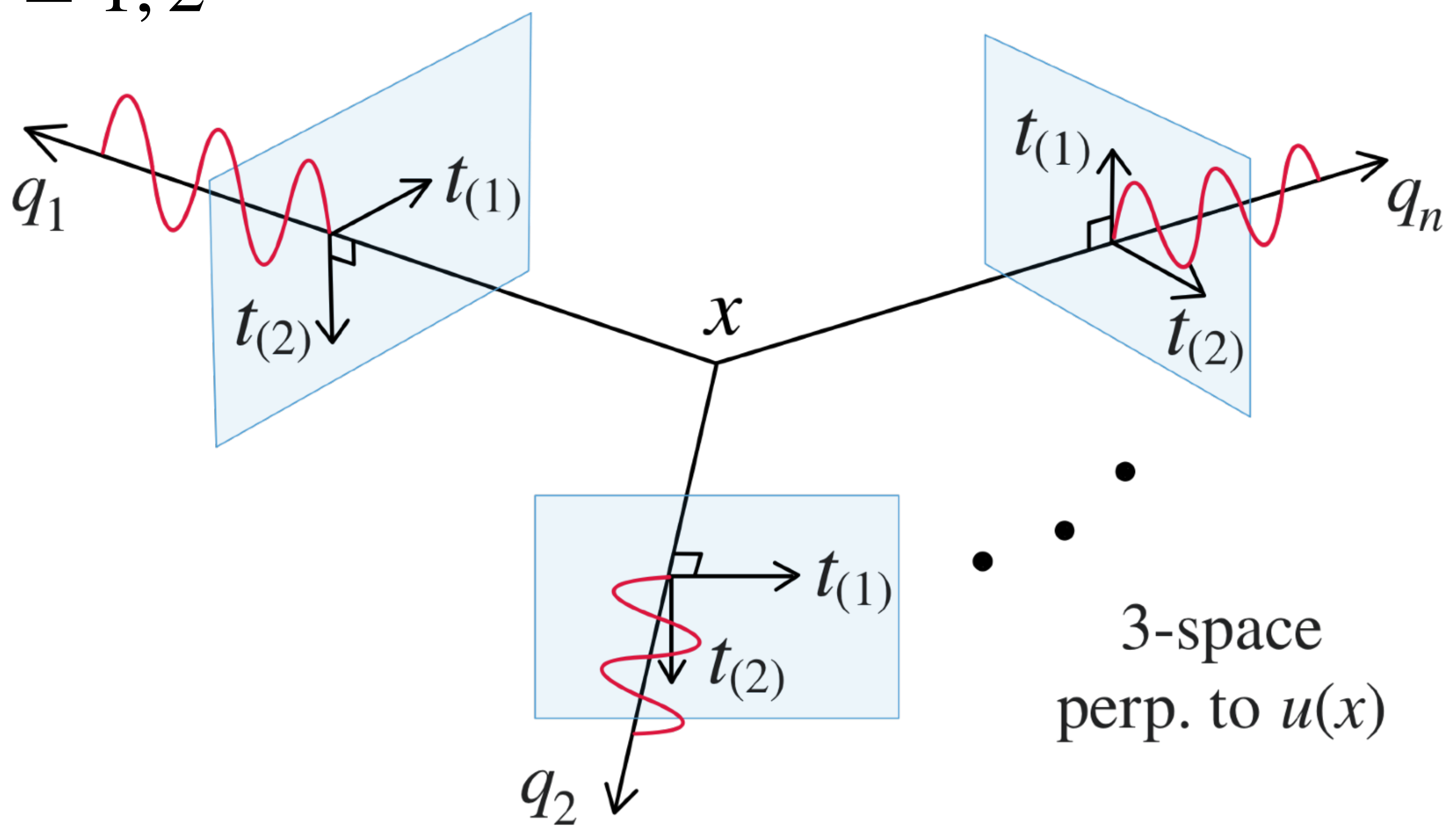
For $\phi = (\delta m, \delta p, \delta u_\mu)$, there are 21+56+126=**203** equations (for the 2-pt, 3-pt and 4-pt correlators) to solve——bite off more than one can chew!

Rotating phase approximation

- Step 1: choose a set of new bases in Fock space s.t. the ideal hydrodynamic equations are diagonalized with eigenvalues $\lambda_\pm(q) = \pm c_s |q|$, $\lambda_m(q) = \lambda_{(i)}(q) = 0$.

$$\phi = \begin{pmatrix} \phi_m \\ \phi_p \\ \phi_\mu \end{pmatrix} = \begin{pmatrix} \delta m \\ \delta p \\ \delta u_\mu \end{pmatrix} \longrightarrow \quad \Phi = \begin{pmatrix} \Phi_m \\ \Phi_\pm \\ \Phi_{(i)} \end{pmatrix} \sim \begin{pmatrix} \delta m \\ \delta p \pm c_s w \hat{q} \cdot \delta u \\ t_{(i)} \cdot \delta u \end{pmatrix} \quad i = 1, 2$$

NB: n-pt correlators are analogous to n-particle quantum states lying in the Fock space.



- Step 2: for n-pt correlators $W_{\Phi_1 \ldots \Phi_n}(q_1, \ldots, q_n)$,

$$\text{if } \sum_{i=1}^{n} \lambda_{\Phi_i}(q_i) \begin{cases} = 0 & \longrightarrow \quad \text{slow mode (kept)} \\ \neq 0 & \longrightarrow \quad \text{fast mode (averaged out)} \end{cases}$$

E.g., $W_{+-}(q_1, q_2)$ is a slow mode since $\lambda_+(q_1) + \lambda_-(q_2) = c_s(|q_1| - |q_2|) = 0$;

$W_{+++}(q_1, q_2, q_3)$ is *not* a slow mode since $\lambda_+(q_1) + \lambda_+(q_2) + \lambda_+(q_3) = c_s(|q_1| + |q_2| + |q_3|) \neq 0$.

As a result, we end up with 7+10+15=**32** equations to solve.

E.g., the 7 independent 2-pt slow modes are W_{mm}, $W_{m(i)}$, $W_{(i)(j)}$, W_{+-}.

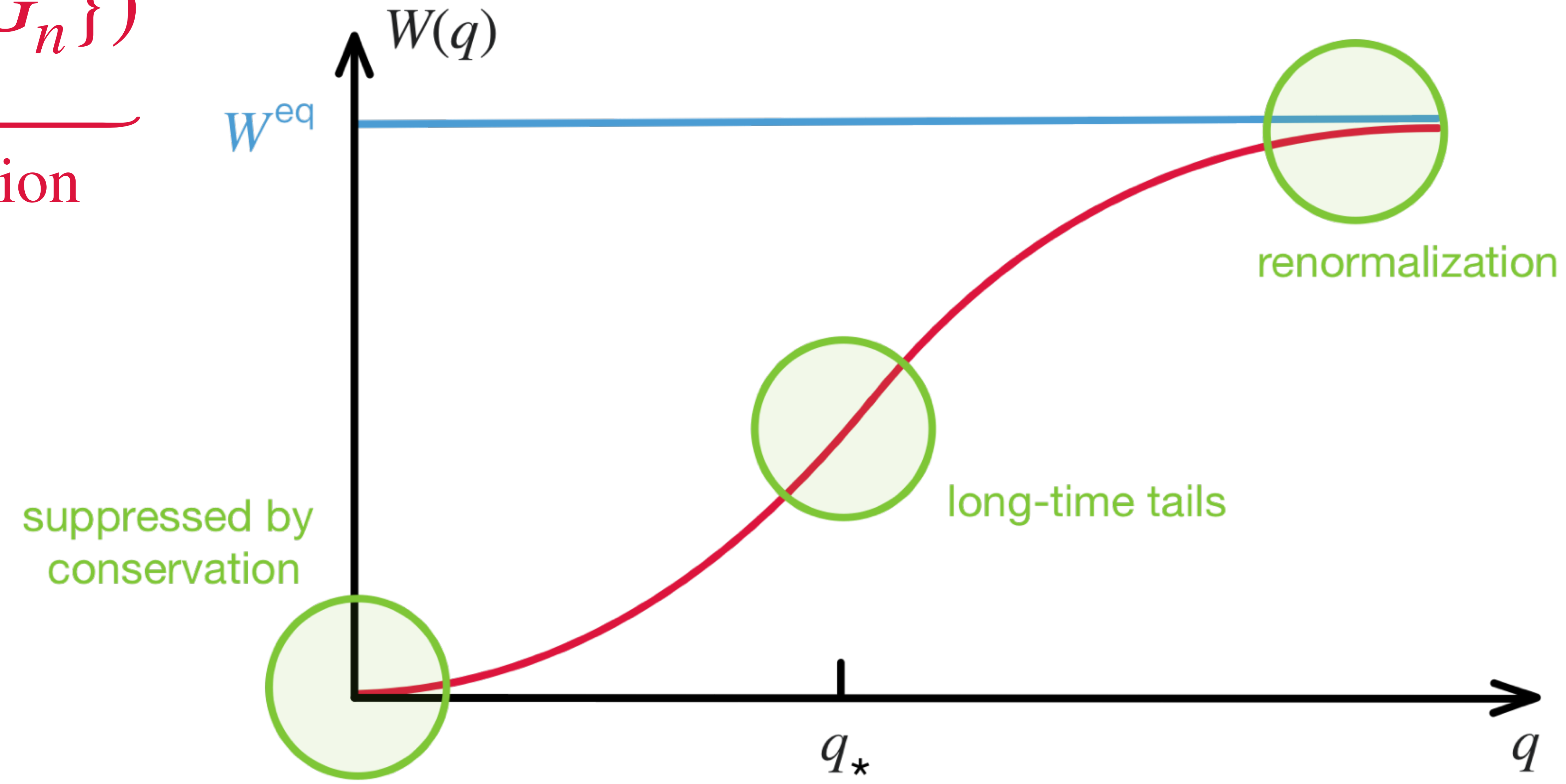
Fluctuation feedback

- Hydrodynamic fluctuations renormalize bare quantities order by order in gradient expansion.

$$T^{\text{physical}}_{\mu\nu} = \underbrace{T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + \ldots}_{\text{bare}} + \underbrace{\delta T_{\mu\nu}(\{G_n\})}_{\text{fluctuation}}$$

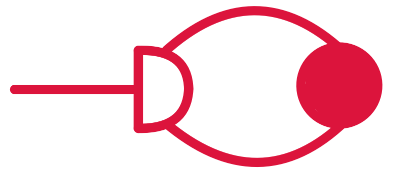
$$= \underbrace{T^{R(0)}_{\mu\nu} + T^{R(1)}_{\mu\nu} + T^{R(2)}_{\mu\nu}}_{\text{renormalized}}$$

$$\underbrace{+\tilde{T}^{(3/2)}_{\mu\nu} + \tilde{T}^{(3)}_{\mu\nu} + \tilde{T}^{(9/2)}_{\mu\nu} + \ldots}_{\text{long-time tails}}$$



where $G_n(x) \sim \int d^3q_1 \ldots d^3q_n \delta^{(3)}(q_1 + \ldots + q_n) W_n(x, q_1, \ldots, q_n)$

Long-time tail due to n-pt correlators is of order $\varepsilon^{n-1} \sim q^{3(n-1)} \sim k^{3(n-1)/2}$,

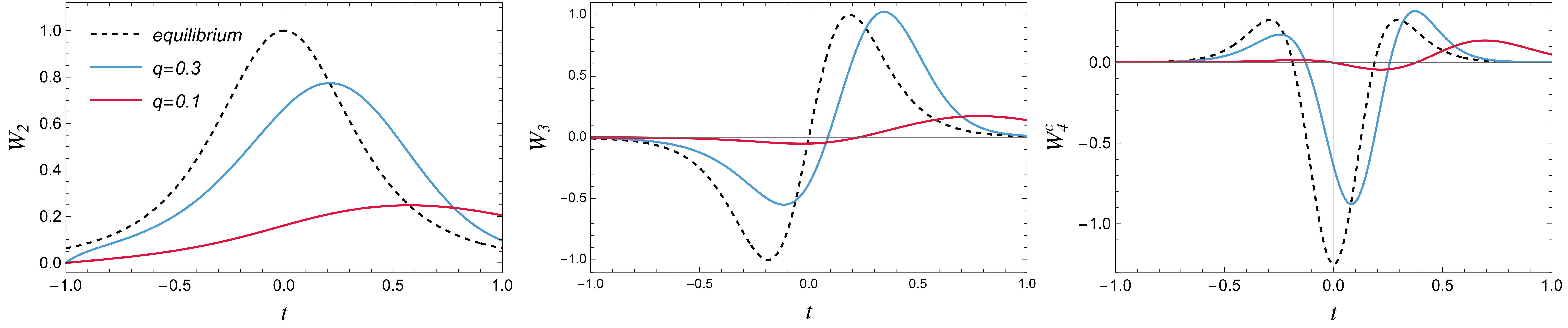
the leading $k^{3/2}$ behavior results from 2-pt correlators via  .

Fluctuation dynamics near critical point

Critical dynamics

- Different slow modes may relax with different time scales near critical point due to critical slowing down. Stephanov, 1104.1627; Berdnikov et al, 9912274; XA, 2003.02828

- The slowest modes are fluctuations of entropy (W_{mm}, W_{mmm}, W_{mmmm}, ...) with $\tau_{\rm rel} \sim \xi^3$.



Evolution of the slowest modes (i.e., W_{mm}, W_{mmm}, W_{mmmm}) in a toy model XA et al, 2009.10742, 2209.15005

for recent numerical implementation and freeze-out procedure, see talk by M. Pradeep (CP1, Tue)
Pradeep et al, 2204.00639, 2211.09142

Conclusion

Recap

- Various approaches for fluctuating hydro have been developed, each with its own pros and cons.

- Our framework for fluctuation dynamics now incorporates non-Gaussian fluctuations of fluid velocity covariantly.

Outlook

- Need efforts to simulate the fluctuation equations with background.

- Need freeze-out prescription for the connection to observables.

- More…