

# Microscopic Origin of Criticality at Macroscale in QCD Chiral Phase Transition

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**Abstract.** We reveal that the criticality of the chiral phase transition in QCD at the macroscale arises from the microscopic energy levels of its fundamental constituents, the quarks. We establish a novel relation between cumulants of the chiral order parameter (i.e., chiral condensate) and correlations among the energy levels of quarks (i.e., eigenspectra of the massless Dirac operator), which naturally leads to a generalization of the Banks-Casher relation. Based on this novel relation and through (2+1)-flavor lattice QCD calculations using the HISQ action with varying light quark masses in the vicinity of the chiral phase transition, we demonstrate that the correlations among the infrared part of the Dirac eigenspectra exhibit same universal scaling behaviors as expected of the cumulants of the chiral condensate. We find that these universal scaling behaviors extend up to the physical values of the up and down quark masses.

## 1 Introduction

One of the central goals in heavy-ion collision experiments is to search for universal signatures of criticality in the phase diagram of strong-interaction matter governed by QCD. Various macroscopic observables for this purpose have been measured [1]. Even though these macroscopic critical behaviors are finally established, how such criticality at the macroscale arises from the fundamental constituents and interactions of QCD remains unanswered.

Moreover, theoretical pursuit of the microscopic origin of the criticality in QCD chiral phase transition is difficult to proceed. The universal feature of the second order chiral phase transition in QCD with massless up and down quarks and a strange quark with physical mass [2–4] is that macroscopic quantities related to the order parameter follow certain power law scaling behaviors that are uniquely characterized by the dimensionality and global symmetries of the system, irrespective of the details of its microscopic degrees of freedom and interactions. Consequently, although a simplified effective theory possessing the same dimensionality and global symmetries of QCD can be used to understand macroscopic properties of a system near the transition region, it cannot unveil properties at the microscale due to ignorance of microscopic complexities of QCD in such effective theories. In this work,

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we will present a first lattice QCD based understanding of connections between the universal features at macroscopic and microscopic scales of QCD. Detailed information about this work can be found in Ref. [5].

## 2 Theoretical developments

For (2+1)-flavor QCD with degenerate light up ( $u$ ) and down ( $d$ ) quarks having masses  $m_l = m_u = m_d$  and a heavier strange quark with physical mass  $m_s$ , for simplicity we develop the main theoretical idea by just considering QCD with degenerate light quarks in this section.

We start with a generating functional defined as

$$\mathbb{G}(m_l; \epsilon) = \ln \langle \exp \{-m_l \bar{\psi} \psi(\epsilon)\} \rangle_0, \quad (1)$$

where a probe operator  $\bar{\psi} \psi(\epsilon) \equiv 2\text{Tr}[\mathcal{D}[\mathcal{U}] + \epsilon]^{-1}$  is introduced to probe the system in the chiral limit with a valence quark mass  $\epsilon$  used to facilitate the evaluation of it. Here  $\mathcal{D}[\mathcal{U}]$  is the massless Dirac operator for a given background  $SU(3)$  gauge field  $\mathcal{U}$ ; trace is applied over the color, spin and space-time indices;  $\langle \cdot \rangle_0$  denotes an expectation value with respect to the QCD partition function in the chiral limit  $Z(0) = \int \exp\{-S[\mathcal{U}, 0]\} \mathcal{D}[\mathcal{U}]$ , where  $S[\mathcal{U}, m_l] = S_g[\mathcal{U}] + \bar{\psi} \mathcal{D}[\mathcal{U}] \psi + m_l \bar{\psi} \psi$  is the Euclidean-time QCD action,  $S_g[\mathcal{U}]$  is the pure gauge action and  $\bar{\psi} \psi = \bar{\psi}_u \psi_u + \bar{\psi}_d \psi_d$ . Then the  $n$ th order cumulants,  $\mathbb{K}_n$ , of the chiral order parameter  $\bar{\psi} \psi(m_l)$  can be obtained from derivatives of above generating functional

$$\mathbb{K}_n[\bar{\psi} \psi] = \frac{T}{V} (-1)^n \left. \frac{\partial^n \mathbb{G}(m_l; \epsilon)}{\partial m_l^n} \right|_{\epsilon=m_l}, \quad (2)$$

where  $T$  is the temperature and  $V$  is the spatial volume of the system. Recognizing  $\langle \mathcal{O} \rangle = \langle \mathcal{O} \exp\{-m_l \bar{\psi} \psi(m_l)\} \rangle_0 / \langle \exp\{-m_l \bar{\psi} \psi(m_l)\} \rangle_0$  and  $Z(m_l)/Z(0) = \langle \exp\{-m_l \bar{\psi} \psi(m_l)\} \rangle_0$ , it is easy to see that  $\mathbb{K}_n$  are the standard cumulants of  $\bar{\psi} \psi(m_l)$ ; e.g.,  $\mathbb{K}_1[\bar{\psi} \psi] = T \langle \bar{\psi} \psi(m_l) \rangle / V$ ,  $\mathbb{K}_2[\bar{\psi} \psi] = T \langle [\bar{\psi} \psi(m_l) - \langle \bar{\psi} \psi(m_l) \rangle]^2 \rangle / V$ ,  $\mathbb{K}_3[\bar{\psi} \psi] = T \langle [\bar{\psi} \psi(m_l) - \langle \bar{\psi} \psi(m_l) \rangle]^3 \rangle / V$ , etc.

The eigenvalues  $\lambda_j[\mathcal{U}]$  of  $\mathcal{D}[\mathcal{U}]$  give the energy levels of a massless quark in given background  $\mathcal{U}$ , and the probe operator is further related to eigenspectra  $\rho_U(\lambda) = \sum_j \delta(\lambda - \lambda_j)$  as  $\bar{\psi} \psi(\epsilon) \equiv 2\text{Tr}[\mathcal{D}[\mathcal{U}] + \epsilon]^{-1} = 2 \sum_j (i\lambda_j + \epsilon)^{-1} = \int_0^\infty d\lambda 4\epsilon \rho_U(\lambda) / (\lambda^2 + \epsilon^2)$ . Thus, Eq. 1 becomes

$$\mathbb{G}(m_l; \epsilon) = \ln \left\langle \exp \left\{ -m_l \int_0^\infty P_U(\lambda; \epsilon) d\lambda \right\} \right\rangle_0, \quad \text{where} \quad P_U(\lambda; \epsilon) \equiv \frac{4\epsilon \rho_U(\lambda)}{\lambda^2 + \epsilon^2}. \quad (3)$$

Hence through derivatives shown as Eq. 2 it is straightforward to obtain our main theoretical result connecting the cumulants of the order parameter to the  $n$ -point correlations of the quark energy levels  $\rho_U(\lambda)$ :

$$\mathbb{K}_n[\bar{\psi} \psi] = \int_0^\infty P_n(\lambda) d\lambda, \quad (4)$$

where  $P_1(\lambda) = K_1[P_U(\lambda; m_l)]$  for  $n = 1$ , and for  $n \geq 2$

$$P_n(\lambda) = \int_0^\infty K_1[P_U(\lambda; m_l), P_U(\lambda_2; m_l), \dots, P_U(\lambda_n; m_l)] \prod_{i=2}^n d\lambda_i. \quad (5)$$

Here  $K_1$  is the first order joint cumulant of  $n$  variables.

In the vicinity of the chiral transition of the staggered lattice QCD at nonvanishing lattice spacings [6–8] the following relation is expected [5]

$$\mathbb{K}_n[\bar{\psi} \psi] = \int_0^\infty P_n(\lambda) d\lambda \sim m_l^{1/\delta - n + 1} f_n(z). \quad (6)$$

Here the scaling variable  $z \propto z_0 m_l^{-1/\beta\delta} (T - T_c)/T_c$ , where  $T_c$  is the chiral phase transition temperature and  $z_0$  is a scale parameter; both are system specific.  $\beta$  and  $\delta$  are the universal critical exponents, and  $f_{n+1}(z) = (1/\delta - n + 1)f_n(z) - z f_n'(z)/\beta\delta$  are universal scaling functions belonging to a 3-d O(2) universality class. Here,  $n \geq 1$  and the superscript prime denotes derivative with respect to  $z$ . In our work we adopted  $\beta = 0.349$ ,  $\delta = 4.78$  and for consistency the scaling functions  $f_n(z)$  of the O(2) universality class determined in Refs. [7, 9].

Eq. 6 indicates that universal scaling properties of the macroscopic observables  $\mathbb{K}_n[\bar{\psi}\psi]$  arise from the correlations among the microscopic energy levels  $P_n(\lambda)$ . To elucidate this point, consider  $m_l \rightarrow 0$ . Recognizing  $P_U(\lambda; \epsilon \rightarrow 0) = 2\pi\rho_U(\lambda)\delta(\lambda)$  from Eq. 3, one finds  $\lim_{m_l \rightarrow 0} P_1(\lambda) = 2\pi K_1[\rho_U(\lambda)]\delta(\lambda)$  and  $\lim_{m_l \rightarrow 0} P_n(\lambda) = (2\pi)^n K_1[\rho_U(\lambda), [\rho_U(0)]^{n-1}]\delta(\lambda)$  for  $n \geq 2$  (from Eq. 5). Noting that  $K_1$  of  $n$  identical variables is equivalent to  $\mathbb{K}_n$ , Eq. 4 in the chiral limit thus becomes a novel generalization of Banks-Casher relation [10] expressed as follows:

$$\lim_{m_l \rightarrow 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]. \quad (7)$$

In the chiral limit and around  $T_c$ , according to Eq. 7 the universal scaling behavior manifested in  $\mathbb{K}_n[\bar{\psi}\psi]$  must arise from the universal behaviors of the  $\lambda$ -independent  $\mathbb{K}_n[\rho_U(0)]$ . Thus, it is natural to expect for small  $m_l$  within the scaling window the critical scaling of  $\mathbb{K}_n[\bar{\psi}\psi]$  in Eq. 6 arises from the universal behaviors of the amplitudes of  $P_n(\lambda)$  at the infrared but not from its system-specific  $\lambda$  dependence; i.e.,  $P_n(\lambda) = m_l^{1/\delta-n+1} f_n(z) g_n(\lambda)$ , where  $g_n(\lambda)$  are non-universal functions encoding the properties of specific system under consideration.

### 3 Results

To check our conjecture for  $P_n(\lambda)$  numerically, (2+1)-flavor lattice QCD calculations with the HISQ action were performed for  $T \in [135, 176]$  MeV, where  $m_s$  was fixed to its physical value with a varying  $m_l = m_s/27, m_s/40, m_s/80, m_s/160$  that correspond to Goldstone pion masses  $m_\pi \approx 140, 110, 80, 55$  MeV. Details about the lattice setup can be found in Ref. [5].

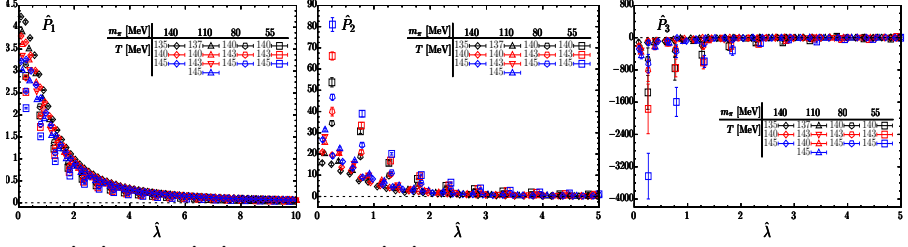
Owing to Eq. 7 we expect the relevant infrared energy scale is  $\lambda \sim m_l$  for small values of  $m_l$ . It is natural to express all quantities as functions of the dimensionless and renormalization group invariant  $\lambda/m_l$  and following notations are used hereafter:

$$\hat{\lambda} = \lambda/m_l, \quad \hat{m}_l = m_l/m_s, \quad z = z_0 \hat{m}_l^{-1/\beta\delta} (T - T_c)/T_c, \\ \hat{P}_n(\hat{\lambda}) = m_s^{n+1} \hat{m}_l P_n(\lambda)/T_c^4, \quad \text{and} \quad \hat{\mathbb{K}}_n[\bar{\psi}\psi] = m_s^n \mathbb{K}_n[\bar{\psi}\psi]/T_c^4 = \int_0^\infty \hat{P}_n(\hat{\lambda}) d\hat{\lambda}. \quad (8)$$

The system-specific parameters  $T_c = 144.2(6)$  MeV and  $z_0 = 1.83(9)$  needed below to obtain  $f_n(z)$  were taken from Ref. [8], where 3-dimensional O(2) scaling fits were carried out for the same lattice ensembles but using an entirely different macroscopic observable, namely the  $m_l$  dependence of the static quark free energy.

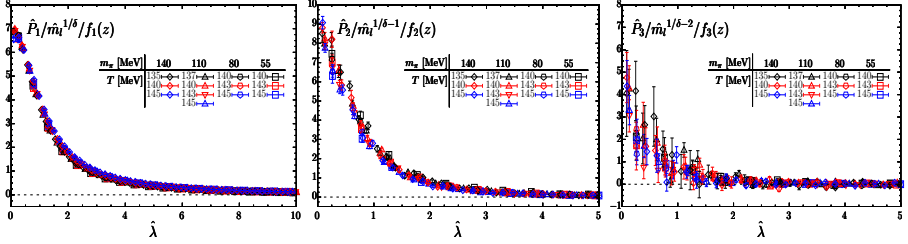
Fig. 1 shows  $\hat{P}_n(\hat{\lambda})$  for  $n \leq 3$  as a function of  $\hat{\lambda}$  in the proximity of  $T_c$ .  $\hat{P}_n(\hat{\lambda})$  rapidly vanishes for  $\hat{\lambda} \gtrsim 1$ , and the regions where  $\hat{P}_n(\hat{\lambda}) \neq 0$  get smaller with increasing  $n$ . This reinforces that the relevant infrared energy scale turns out to be  $\hat{\lambda} \sim 1$ . In this infrared region  $\hat{P}_n(\hat{\lambda})$  at a fixed  $T$  shows clear dependences on  $m_l$ , which becomes stronger for increasing  $n$ . The form of  $m_l$  dependence of  $\hat{P}_n(\hat{\lambda})$  also changes with varying  $T$ . Expectedly, our results become increasingly noisy with increasing  $n$  and decreasing  $m_l$ . With our present statistics we cannot access correlation functions with  $n > 3$ , particularly for smaller  $m_l$ .

The  $m_l$  and  $T$  dependence of  $\hat{P}_n(\hat{\lambda})$  shown in Fig. 1 can be understood in terms of the 3-d O(2) scaling properties. Once the  $\hat{P}_n(\hat{\lambda})$  are rescaled with respective  $\hat{m}_l^{1/\delta+1-n} f_n(z)$  the data in Fig. 1 magically collapse onto each other (see Fig. 2). Thus, our expectations for  $P_n(\lambda)$  are



**Figure 1.**  $\hat{P}_1(\hat{\lambda})$  (left),  $\hat{P}_2(\hat{\lambda})$  (middle) and  $\hat{P}_3(\hat{\lambda})$  (right) for  $135 \text{ MeV} \leq T \leq 145 \text{ MeV}$  and  $55 \text{ MeV} \leq m_\pi \leq 140 \text{ MeV}$ .

clearly borne out in Fig. 2, namely  $\hat{P}_n(\hat{\lambda}) = \hat{m}_l^{1/\delta-n+1} f_n(z) \hat{g}_n(\hat{\lambda})$ . Here  $\hat{g}_n(\hat{\lambda})$  characterize the system specific of the  $n$ th order energy-level correlations. To satisfy our generalized Banks-Casher relations of Eq. 7 the  $\hat{g}_n(\hat{\lambda})$  must also satisfy  $\lim_{V \rightarrow \infty} \lim_{a \rightarrow 0} \lim_{m_l \rightarrow 0} \hat{g}_n(\hat{\lambda}) \rightarrow \delta(\hat{\lambda})$ , such that  $\mathbb{K}_n[\bar{\psi}\psi]$  has the correct scaling behavior in  $(T - T_c)/T_c$ . It is noteworthy that the physical QCD with  $m_\pi \approx 140 \text{ MeV}$  also shows same scaling for  $T \in [135, 145] \text{ MeV}$ . Outside this temperature window we do not observe the scaling.



**Figure 2.**  $\hat{P}_n(\hat{\lambda})$  in Fig. 1 rescaled by  $\hat{m}_l^{1/\delta+1-n} f_n(z)$  for  $n = 1$  (left),  $n = 2$  (middle) and  $n = 3$  (right).

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