Exact Polarization of Particles of Any Spin at Global Equilibrium

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Abstract. The polarization of the Λ particle offers the unique opportunity to study the hydrodynamic gradients in the Quark-Gluon Plasma formed in heavy-ion collisions. However, the theoretical formula commonly used to calculate polarization is only a linear order expansion in thermal vorticity and neglects higher-order corrections. Here, I present an exact calculation to all orders in (constant) thermal vorticity at global equilibrium, obtaining the analytic form of the spin density matrix and the polarization vector for massive particles of any spin. Finally, I extend these results to local equilibrium and assess their phenomenological impact by numerically calculating the polarization vector in a 3+1 hydrodynamic simulation.

1 Introduction

The experimental measurement of the polarization vector of Λ particles in heavy ion collisions paved a new way in the hydrodynamic studies of the QGP [1–14]. In fact, with spin polarization one can access the thermodynamic gradients, such as the thermal vorticity (to be defined soon), in the plasma at freeze-out. According to the hydrodynamic model, it was found early on that the polarization vector of massive spin-S fields is [15–17]:

$$S^\mu(p) = -\frac{S(S+1)}{3}\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}p_\sigma\frac{\int \mathrm{d}\Sigma\cdot p\,\varpi_{\nu\rho}n(p)[1+(-1)^{2S}n(p)]}{\int \mathrm{d}\Sigma\cdot p\,n(p)}, \tag{1}$$

where p is the four-momentum of the particle at hand, the integral is calculated on the decoupling hypersurface, $n(p) = [\mathrm{e}^{\beta\cdot p} + (-1)^{2S+1}]^{-1}$ is the Fermi-Dirac or the Bose-Einstein distribution function and ϖ is the thermal vorticity:

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu), \tag{2}$$

β^μ being the four-temperature vector. Additional thermodynamic gradients contribute to spin polarization, such as the thermal shear [18–21] and the gradient of the chemical potential (spin hall effect) [22–24], which however I will not discuss further. It is important to stress that *all* the terms mentioned so far are the leading order contributions to the spin vector in linear response theory. Even in the case of global equilibrium, where the only contribution

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to the spin vector comes from the constant thermal vorticity, there are no results beyond the first order of approximation in ϖ. Here, I show how to compute the spin vector to all orders in ϖ for fields of any spin at global equilibrium [25]. I will also provide a finite and analytic expression of the spin density matrix, which can be used to compute any spin observable, including the vector meson spin alignment.

2 Spin density matrix and polarization vector

To compute the polarization vector or any other spin-related quantity, one uses the spin density matrix Θ. For massive fields of spin S, the spin density matrix is a $(2S+1)\times(2S+1)$ hermitian matrix, characterized by S^2-1 real numbers. The definition of the spin density matrix adopted in the context of relativistic heavy-ion collisions is [26]:

$$\Theta_{sr}(p) = \frac{\mathrm{Tr}\left(\widehat{\rho}\,\widehat{a}_r^\dagger(p)\widehat{a}_s(p)\right)}{\sum_l \mathrm{Tr}\left(\widehat{\rho}\,\widehat{a}_l^\dagger(p)\widehat{a}_l(p)\right)} = \frac{\langle\widehat{a}_r^\dagger(p)\widehat{a}_s(p)\rangle}{\sum_l\langle\widehat{a}_l^\dagger(p)\widehat{a}_l(p)\rangle}. \tag{3}$$

In the above equation, $\widehat{\rho}$ is the statistical operator, and $\widehat{a}$ and $\widehat{a}^\dagger$ are the annihilation and creation operators of states with momentum p and spin r, s.

The spin density matrix is related to the spin vector S^μ through [26]:

$$S^\mu(p) = \sum_{i=1}^{3}[p]^\mu_{\ i}\mathrm{tr}\left(\Theta(p)D^S(\mathrm{J}^i)\right), \tag{4}$$

where $D^S(\mathrm{J}^i)$ is the i-th generator of SO(3) in the S-representation. The transformation denoted as $[p]$ is the so-called *standard boost*: a Lorentz transformation that maps the rest frame momentum $\mathfrak{p} = (m, \mathbf{0})$ to the momentum $p = (\varepsilon, \boldsymbol{p})$, i.e. $p^\mu = [p]^\mu_{\ \nu}\mathfrak{p}^\nu$. Other formulae for the polarization vector, such as those involving the Wigner function, are always derived from eqs. (3) and (4).

3 Exact spin physics at global equilibrium

From equation (4), the polarization vector can be computed once the spin density matrix is known. Here, I will compute exactly the spin density matrix for fields of any spin at global equilibrium. This state is described by the density operator:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-b\cdot\widehat{P} + \frac{\varpi:\widehat{J}}{2} + \zeta\widehat{Q}\right], \tag{5}$$

where $\widehat{Q}$ is the charge operator, and $\widehat{P}$ and $\widehat{J}$ are the generators of the Poincaré group: the four-momentum and the angular momentum-boost operator, respectively. The vector b^μ and antisymmetric tensor $\varpi^{\mu\nu}$, that is the thermal vorticity, are constants, and they define the four-temperature vector as the Killing vector:

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu}x_\nu.$$

In addition, $\zeta = \mu/T$ is also constant, μ being the chemical potential. The proper temperature is $T = \left(\sqrt{\beta\cdot\beta}\right)^{-1}$.

In Refs. [27, 28], it was realized that factorizing the operator (5) and performing the analytic continuation $\varpi \mapsto -i\phi$ provides a powerful technique to compute exact expectation values. This method allows the calculation of $\langle \widehat{a}_r^\dagger(p)\widehat{a}_s(p)\rangle$:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\mathbf{p}-\mathbf{p}')D(W(\Lambda^n,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^n\Lambda^k p}\mathrm{e}^{n\zeta}, \tag{6}$$

where $\Lambda = \exp\left[-i\phi : J/2\right]$ is the Lorentz transformation with parameter ϕ equal to the *imaginary* vorticity and $W(\Lambda, p) = [\Lambda p]^{-1}\Lambda[p]$ is the so-called Wigner rotation. The vector $\tilde{b}(\phi)$ is:

$$\tilde{b}^\mu(\phi) = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\phi^{\mu}{}_{\nu_1}\phi^{\nu_1}{}_{\nu_2}\dots\phi^{\nu_{k-1}}{}_{\nu_k}b^{\nu_k}, \tag{7}$$

and its form is a consequence of the fact that the Poincaré group is non-Abelian.

Plugging eq. (6) into eq. (3), yields:

$$\Theta_{rs}(p) = \frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^n\Lambda^k p}\mathrm{e}^{n\zeta}D_{rs}(W(\Lambda^n,p))\delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^n\Lambda^k p}\mathrm{e}^{n\zeta}\mathrm{tr}[D(W(\Lambda^n,p))]\delta^3(\Lambda^n p - p)}. \tag{8}$$

To proceed further in the calculation, and as long as the thermal vorticity is imaginary, the constraint $\Lambda p = p$ must be enforced. Indeed, from eq. (6), one sees that if this is not the case $\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p)\rangle = 0$, and the spin density matrix would become trivial.

Imposing the constraint $\Lambda p = p$, it follows that:

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\xi_\rho\frac{p_\sigma}{m}, \qquad\qquad \xi^\rho = -\frac{1}{2m}\epsilon^{\rho\mu\nu\sigma}\phi_{\mu\nu}p_\sigma. \tag{9}$$

Notice that, since we are still using the imaginary vorticity, these quantities are purely auxiliary mathematical constructions. Thanks to the constraint $\Lambda p = p$, and after some algebra (see Ref. [25] for the full derivation), the spin density matrix becomes:

$$\Theta(p) = \frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{n\zeta}\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{n\zeta}\mathrm{tr}\left(\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}\right)},$$

where $\xi_0^\mu = [p]^{-1\mu}{}_\nu\xi^\nu$, $\boldsymbol{\xi}_0$ being its spatial part, and $D^S(\mathbf{J})$ is the three-vector of the generators of the rotation group in the S-representation. The series can be summed as a geometric series [29], and after the summation it can be analytically continued outside of the radius of convergence. By analytically continuing ϕ, back to the physical vorticity, $\phi \mapsto i\varpi$, then $-i\xi \mapsto \theta$, where I defined:

$$\theta^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}\frac{p_\sigma}{m}. \tag{10}$$

This quantity is physical, and represents the angular velocity (over temperature) of the fluid, as perceived by the particle with momentum p^μ. Finally, the exact spin density matrix for massive particles of *any* spin at global equilibrium with rotation and acceleration reads:

$$\Theta(p) = \frac{\left[(-1)^{2S}\mathbb{I} - \mathrm{e}^{-b\cdot p+\boldsymbol{\theta}_0\cdot D^S(\mathbf{J})}\right]^{-1} - (-1)^{2S}\mathbb{I}}{\sum_{k=-S}^{S}\left(\mathrm{e}^{b\cdot p - k\sqrt{-\theta^2}} - (-1)^{2S}\right)^{-1}}. \tag{11}$$

From eq. (11), one can compute all spin-related quantities, including spin polarization and spin alignment. In particular, the mean spin vector reads, using eq. (4):

$$S^\mu(p) = \frac{\theta^\mu}{\sqrt{-\theta^2}}\frac{\sum_{k=-S}^{S}k\left[\mathrm{e}^{b\cdot p-\zeta-k\sqrt{-\theta^2}} - (-1)^{2S}\right]^{-1}}{\sum_{k=-S}^{S}\left[\mathrm{e}^{b\cdot p-\zeta-k\sqrt{-\theta^2}} - (-1)^{2S}\right]^{-1}}. \tag{12}$$

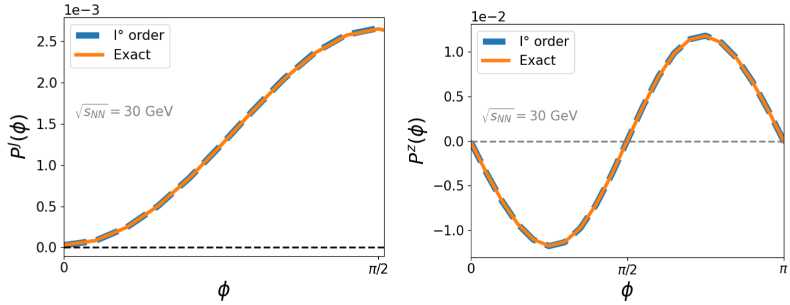

Figure 1. The components of the polarization vector along the total angular momentum (left panel) and the beam direction (right panel) as a function of the azimuthal angle. The blue dashed line is the linear approximation (1), whereas the solid orange line corresponds to eq. (13).

This formula reproduces the previous literature, including the small vorticity approximation and the case of Boltzmann statistics. In the case of very large thermal vorticity, it predicts a polarization equal to one [25].

4 Polarization in heavy-ion collisions

One can now evaluate the effect of the formula (12) in phenomenological applications. Although the derivation holds in global equilibrium, eq. (12) can be generalized to local equilibrium by promoting the thermal vorticity to be a local variable, i.e. $\varpi \mapsto \varpi(x)$, and integrating eq. (12) over the decoupling hypersurface.

$$S^\mu_{LE}(p) = \frac{\int \mathrm{d}\Sigma \cdot p\, n(p)\, S^\mu(p)}{\int \mathrm{d}\Sigma \cdot p\, n(p)}. \tag{13}$$

In Fig. 1, I report the result of eq. (13) compared to that of eq. (1) in Au-Au collisions at $\sqrt{s_{NN}} = 30$ GeV, where the vorticity should be larger compared to $\sqrt{s_{NN}} = 200$ GeV and hence the impact of higher order corrections may be more significant. I simulated the centrality class 10-60% using the hydrodynamic code vHLLE [30]. The initial state is an averaged entropy density profile generated by GLISSANDO v.2.702 [31]. For the figure, I consider the case of Dirac fermions at midrapidity. I remind the reader that polarization is defined as $P^\mu = S^\mu/S$, where S is the spin. One can see that the two formulae give the same result, and it follows that the linear approximation is a good one, at least for $\sqrt{s_{NN}} > 30$ GeV.

5 Conclusions

In this work, I have shown how to calculate the spin density matrix and the polarization vector for massive spin-S fields to all orders in thermal vorticity, and I obtained an analytic finite expression for both quantities. Using the spin density matrix, all spin observables can be computed. The spin vector is a finite sum, representing the average of the spin state weighted by Bose or Fermi distribution functions.

I have shown a numerical calculation for AuAu collisions at 30 GeV, demonstrating that the linear approximation of the polarization vector is an excellent one in most practical applications to heavy-ion collisions. The exact spin vector, eq. 13, may turn out to be more important in even lower-energy collisions, where the thermal vorticity is larger.

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