Equation of state of a hot-and-dense quark gluon plasma: lattice simulations at real μ_B vs. extrapolations

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Abstract. The equation of state of the quark gluon plasma is a key ingredient of heavy ion phenomenology. In addition to the traditional Taylor method, several novel approximation schemes have been proposed with the aim of calculating it at finite baryon density. In order to gain a pragmatic understanding of the limits of these schemes, we compare them to direct results at $\mu_B > 0$, using reweighting techniques free from an overlap problem. We use 2stout improved staggered fermions with 8 time-slices and cover the entire RHIC BES range in the baryochemical potential, up to $\mu_B/T = 3$. This work is based on Ref. [1].

1 Introduction

The equation of state of strongly interacting matter under extreme conditions such as high temperatures or baryon densities plays a key role in many physical systems, such as the early Universe, heavy ion collisions and neutron stars. The most well established first-principles method to study the strongly coupled regime is lattice QCD [2]. While many properties of strongly interacting matter at zero baryon density have been elucidated using this method, studies at finite baryon density are, however, hampered by the sign problem. Thus, most lattice results on the properties of hot-and-dense QCD matter rely on extrapolations from zero or purely imaginary chemical potential, two situations with no sign problem. Details can be found in reviewing literatures such as Ref. [3]. There has recently been considerable progress in the calculation of the equation of state of a hot-and-dense quark gluon plasma (QGP). In addition to ordinary Taylor expansions, several resummation schemes have been proposed [4–8], with a promise of better convergence properties. It is important for phenomenological applications to determine the region of validity of these techniques. This is the purpose of this work. Here we compare three such schemes: the Taylor expansions, shifting sigmoid resummation [4, 8] and exponential resummation [5] with reweighted results. In this paper we did not do sign reweighting. We did reweighting from mu=0 and phase reweighting [9, 10] using a 2 stout improved staggered lattice action with $2+1$ fermions at physical quark

masses, for which details can be found in Ref. [1]. In this work we consider the quark chemical potentials satisfying $\mu_q \equiv \mu_u = \mu_d$ and $\mu_s = 0$. The observable of interest is the light quark density $\hat{n}_L = d\hat{p}/d\hat{\mu}_B$, where $\hat{p} = p/T$ with pressure p, as a function of the temperature T and dimensionless chemical potential $\hat{\mu}_B \equiv \mu_B/T = 3\mu_q/T \equiv 3\hat{\mu}_q$. We simulate the phase quenched ensemble for $\mu_B^2 = 1.5, 3, 4.5, 6, 7.5, 9$ with a subsequent reweighting to real baryo-chemical potental, following the method of Ref. [10]. We will refer to this approach as 'direct simulation' in the following discussion. We also simulate $\mu = 0$ for $T = 140, 150, 160, 170$ MeV to perform reweighting from $\mu_B = 0$ to calculate the Taylor coefficients and exponential resummation using the reduced matrix formalism [11].

2 Comparison between reweighted data and extrapolation schemes

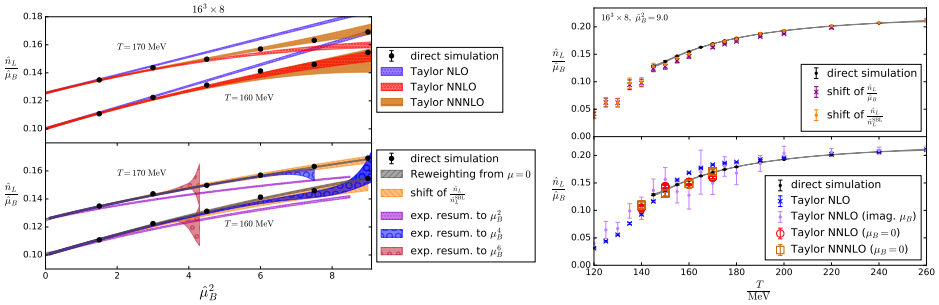

Figure 1. Left: The direct results for $\hat{n}_L$ at non-zero μ_B compared with different approximations (or extrapolations): the Taylor method to different orders up to p_8 and the exponential resummations [5] to order $N = 2, 4$ and 6, calculated from the ensemble at $\mu_B = 0$, as well as the shifting n_L/n_L^{SBL} method calculated from imaginary chemical potential simulations and reweighting from $\mu_B = 0$. Right: The density $\hat{n}_L$ as a function of the temperature T for $\hat{\mu}_B = 3$ for the ordinary Taylor expansion (bottom) and the resummation schemes based on shifting $\hat{n}_L/\hat{\mu}_B$ and $\hat{n}_L/\hat{n}_L^{\text{SBL}}$ (top). The direct Taylor data from $\mu_B = 0$ simulations has smaller errors than the fit to imaginary μ_B data. This is mainly due to the small volume in our study. For larger volumes, the signal-to-noise ratio of the direct p_6 and p_8 would be considerably larger. A spline interpolation of the direct results is included to lead the eye.

Using simulations at imaginary chemical potentials: $\text{Im}\,\hat{\mu}_B \frac{16}{\pi} = 0, 4, 6, 7, 8, 9, 10$ and 12, we computed the normalized shifting sigmoid resummation schemes of Refs. [4, 8] up to order κ_4 and λ_4 (for the Stefan-Boltzmann normalized variant [8]). The systematic error includes the fit range in imaginary μ_B, the ansatz in μ_B^2, and the interpolation of the light quark susceptibility at $\mu_B = 0$. We also demonstrate a more straightforward use of the imaginary chemical potential data by performing a second determination of the Taylor expansion coefficients, fitting $\frac{\hat{n}_L}{\hat{\mu}_B}$ with a polynomial of order $\hat{\mu}_B^6$. For the fits we also include $\frac{d^2\hat{p}}{d\hat{\mu}_B^2}$ and $\frac{d^4\hat{p}}{d\hat{\mu}_B^4}$ at $\mu_B = 0$ as further data points. We show the comparison of the different reweighting schemes with the direct data in Fig. 1, as a function of T at a fixed $\hat{\mu}_B = 3$, the largest value where we have direct data. The Taylor expansion at next-to-leading order - $O(\mu_B^4)$ in the pressure - is not consistent with the direct data, systematically underestimating n_L below 150 MeV, and systematically overestimating it above 150 MeV. This is due to a peak in $p_4(T)$ slightly above the crossover temperature. Including the next term in the expansion, with the coefficient $p_6(T)$, the Taylor method agrees with the direct data up to $\hat{\mu}_B = 3$ at $T = 160$ MeV and up to $\hat{\mu}_B \approx 1.2$ at $T = 170$ MeV. Including the next-to-next-to-next-to-leading order term $p_8(T)$,

the expansion agrees with the direct data at all studied temperatures up to $\hat{\mu}_B = 3$. In contrast, the exponential resummation scheme shows bad convergence properties from $\hat{\mu}_B^2 \approx 4$ for all temperatures. While the $N = 2$ truncation of the scheme remains close to the direct results in the entire range, the higher orders make the agreement better only below this value, but not above. At large T, the method based on shifting $\hat{n}_L/\hat{n}_L^{\mathrm{SBL}}$ outperforms the method of shifting $\hat{n}_L/\hat{\mu}_B$. This is not surprising, as the Stefan-Boltzmann correction was introduced as a way to improve the convergence properties of the scheme at high T.

In summary, we can say that both the Taylor expansion to order $\hat{\mu}_B^8$(NNNLO) and the resummation based on shifting $\hat{n}_L/\hat{n}_L^{\mathrm{SBL}}$ to order λ_4 accurately describe the direct data for the equation of state in the range $0 \leq \hat{\mu}_B \leq 3$ - which includes the entire range of the RHIC Beam Energy Scan. Note the faster convergence of the resummed expansion, as the calculation of the coefficient λ_4 only requires the determination of the Taylor coefficients up to order $\hat{\mu}_B^6$. On the other hand, the shifting $\hat{n}_L/\hat{\mu}_B$ method at order κ_4 has a slight systematic discrepancy with the direct data at large T, and exponential resummation shows bad convergence properties in N above $\hat{\mu}_B^2 \approx 4$.

3 Conclusion

We judged the reliability of different approximation schemes by comparing them with direct non-zero chemical potential results in the range $0 \leq \hat{\mu}_B \leq 3$. While this gives a practical answer to the question of which approximation one can trust, a theoretical understanding of the reasons would also be welcome. For the schemes defined purely in terms of thermodynamic quantities, such as the Taylor expansion or the resummations based on shifting sigmoids, this requires knowledge of the position of partition function (Lee-Yang) zeros in the complex μ_B plane [7, 12–15]. The exponential resummation scheme, instead, is not defined in terms of thermodynamic quantities, but comes rather from manipulating the integrand of the path integral for the partition function. Understanding its convergence region might also require better understanding of the nuances of the path integral, in addition to the thermodynamic singularities. We speculate that the limited convergence region has to do with quark determinant zeros. In fact, the effective action of the quarks in a fixed gauge field background is approximated by a finite sum, with a radius of convergence determined by the determinant zeros, which correspond to logarithmic divergences of the effective action. These are not simply related to the Lee-Yang zeros of the partition function, and may provide stronger limitations on the convergence of the expansion. An obvious challenge is to extend the range of validity of the methods studied in this paper to lower T and higher $\hat{\mu}_B$, so that the transition line [16–20], and the location of the conjectured critical endpoint [21–25] can be studied with first principle lattice calculations. Of course, the continuum and infinite volume limits will also have to be taken eventually.

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