# Infrared subtractions and numerical integration

with George Sterman

And

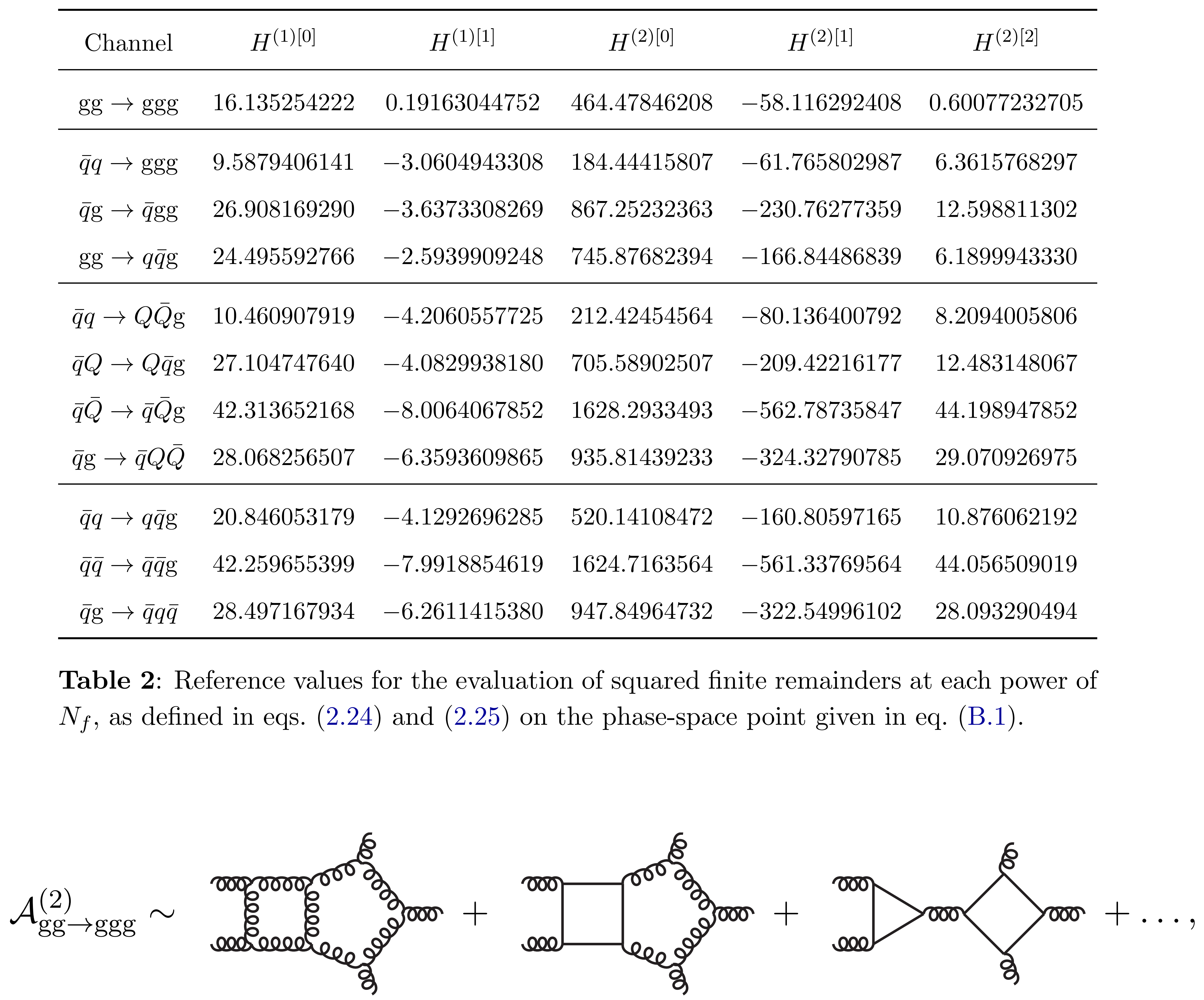
R. Haindl, G. Sterman, Z. Yang, M. Zeng in *JHEP 04 (2021) 222*
G. Sterman in *JHEP 07 (2019) 056*

Thanks to: Zeno Capatti, Dario Kermanschah, Valentin Hirschi, Andrea Pelloni, Ben Ruijl

Nikos Kalntis , Armin Schweizer

# The challenge of QCD two-loop amplitudes for processes with many particles in the final state.

- A dream that is tangible.
- Significant efforts and amazing new results.
- Challenge: “master” integrals with many scales in dimensional regularisation.
- Even more powerful methods are needed (Wjj, diboson+jet, WWZ, ttH, … )
- Understand  the singularity structure



| Channel | $H^{(1)[0]}$ | $H^{(1)[1]}$ | $H^{(2)[0]}$ | $H^{(2)[1]}$ | $H^{(2)[2]}$ |
|---|---|---|---|---|---|
| $\mathrm{gg} \to \mathrm{ggg}$ | 16.135254222 | 0.19163044752 | 464.47846208 | −58.116292408 | 0.60077232705 |
| $\bar{q}q \to \mathrm{ggg}$ | 9.5879406141 | −3.0604943308 | 184.44415807 | −61.765802987 | 6.3615768297 |
| $\bar{q}\mathrm{g} \to \bar{q}\mathrm{gg}$ | 26.908169290 | −3.6373308269 | 867.25232363 | −230.76277359 | 12.598811302 |
| $\mathrm{gg} \to q\bar{q}\mathrm{g}$ | 24.495592766 | −2.5939909248 | 745.87682394 | −166.84486839 | 6.1899943330 |
| $\bar{q}q \to Q\bar{Q}\mathrm{g}$ | 10.460907919 | −4.2060557725 | 212.42454564 | −80.136400792 | 8.2094005806 |
| $\bar{q}Q \to Q\bar{q}\mathrm{g}$ | 27.104747640 | −4.0829938180 | 705.58902507 | −209.42216177 | 12.483148067 |
| $\bar{q}\bar{Q} \to \bar{q}\bar{Q}\mathrm{g}$ | 42.313652168 | −8.0064067852 | 1628.2933493 | −562.78735847 | 44.198947852 |
| $\bar{q}\mathrm{g} \to \bar{q}Q\bar{Q}$ | 28.068256507 | −6.3593609865 | 935.81439233 | −324.32790785 | 29.070926975 |
| $\bar{q}q \to q\bar{q}\mathrm{g}$ | 20.846053179 | −4.1292696285 | 520.14108472 | −160.80597165 | 10.876062192 |
| $\bar{q}\bar{q} \to \bar{q}\bar{q}\mathrm{g}$ | 42.259655399 | −7.9918854619 | 1624.7163564 | −561.33769564 | 44.056509019 |
| $\bar{q}\mathrm{g} \to \bar{q}q\bar{q}$ | 28.497167934 | −6.2611415380 | 947.84964732 | −322.54996102 | 28.093290494 |

**Table 2**: Reference values for the evaluation of squared finite remainders at each power of $N_f$, as defined in eqs. (2.24) and (2.25) on the phase-space point given in eq. (B.1).

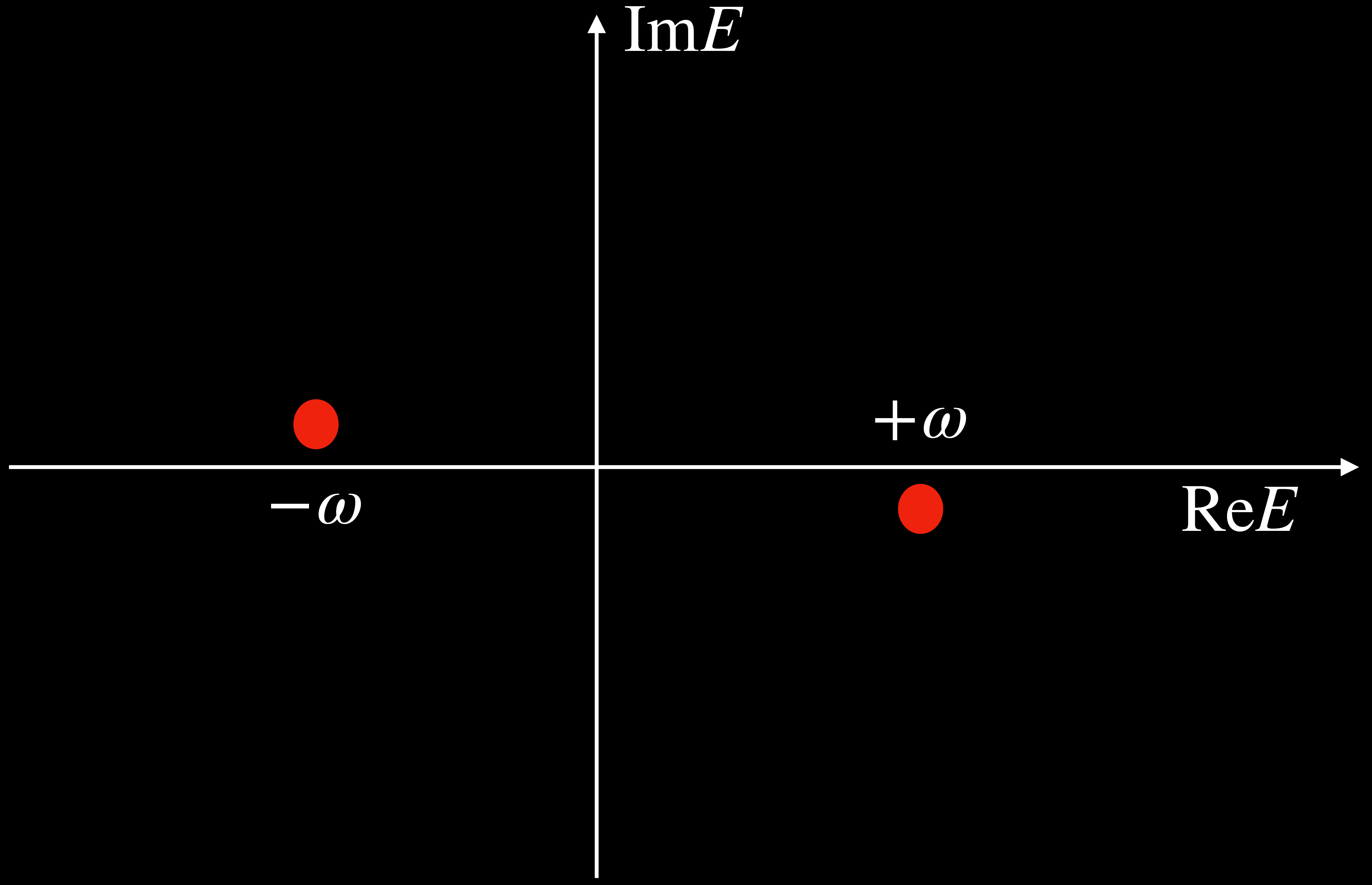
$$\mathcal{A}^{(2)}_{\mathrm{gg}\to\mathrm{ggg}} \sim \quad + \quad + \quad + \ldots,$$

**Abreu, Cordero, Ita, Page, Sotnikov**

# Singularities of scattering amplitudes

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega^2 + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left( \frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles can lie inside the domain of integration.

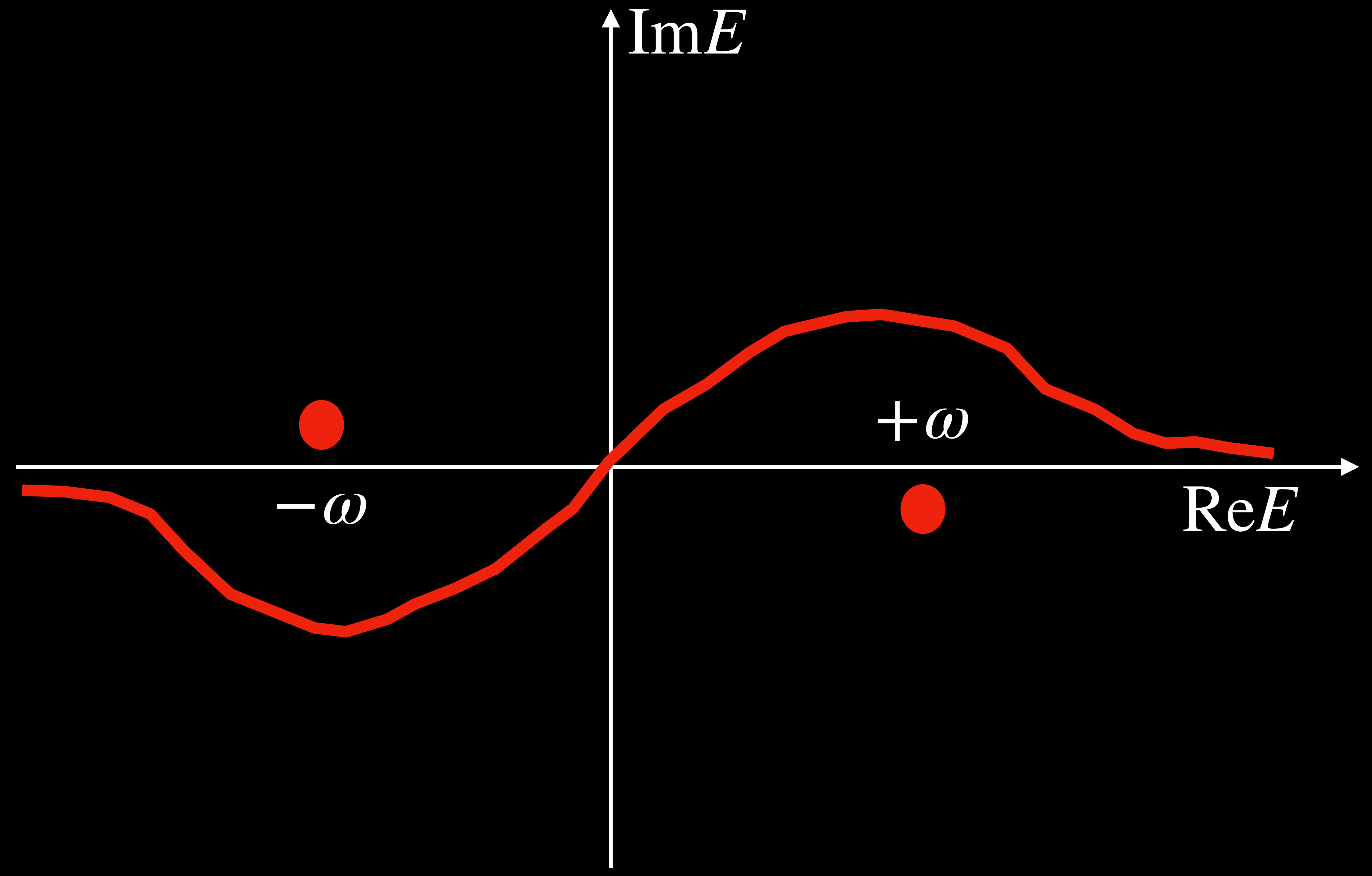


$\omega \to \omega - i\delta$ with $\delta \to 0$

# Singularities

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2-\omega^2+i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega}\left(\frac{1}{E-\omega+i\delta}-\frac{1}{E+\omega-i\delta}\right)$$

- The poles can lie inside the domain of integration.
- If we can deform the path of integration away from the poles, then they lead to no singularities

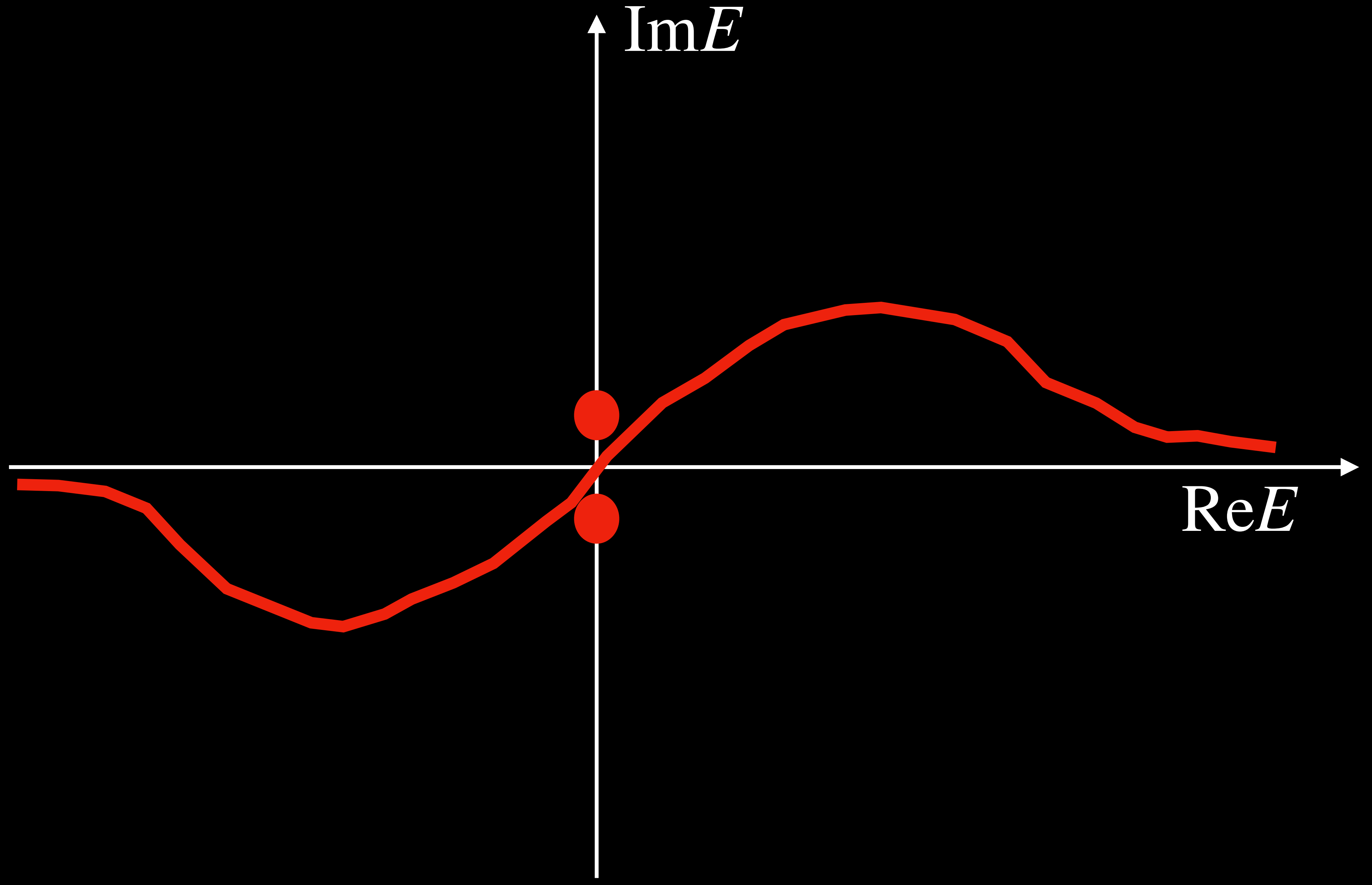


$\omega \to \omega - i\delta$ with $\delta \to 0$

# Soft massless particles

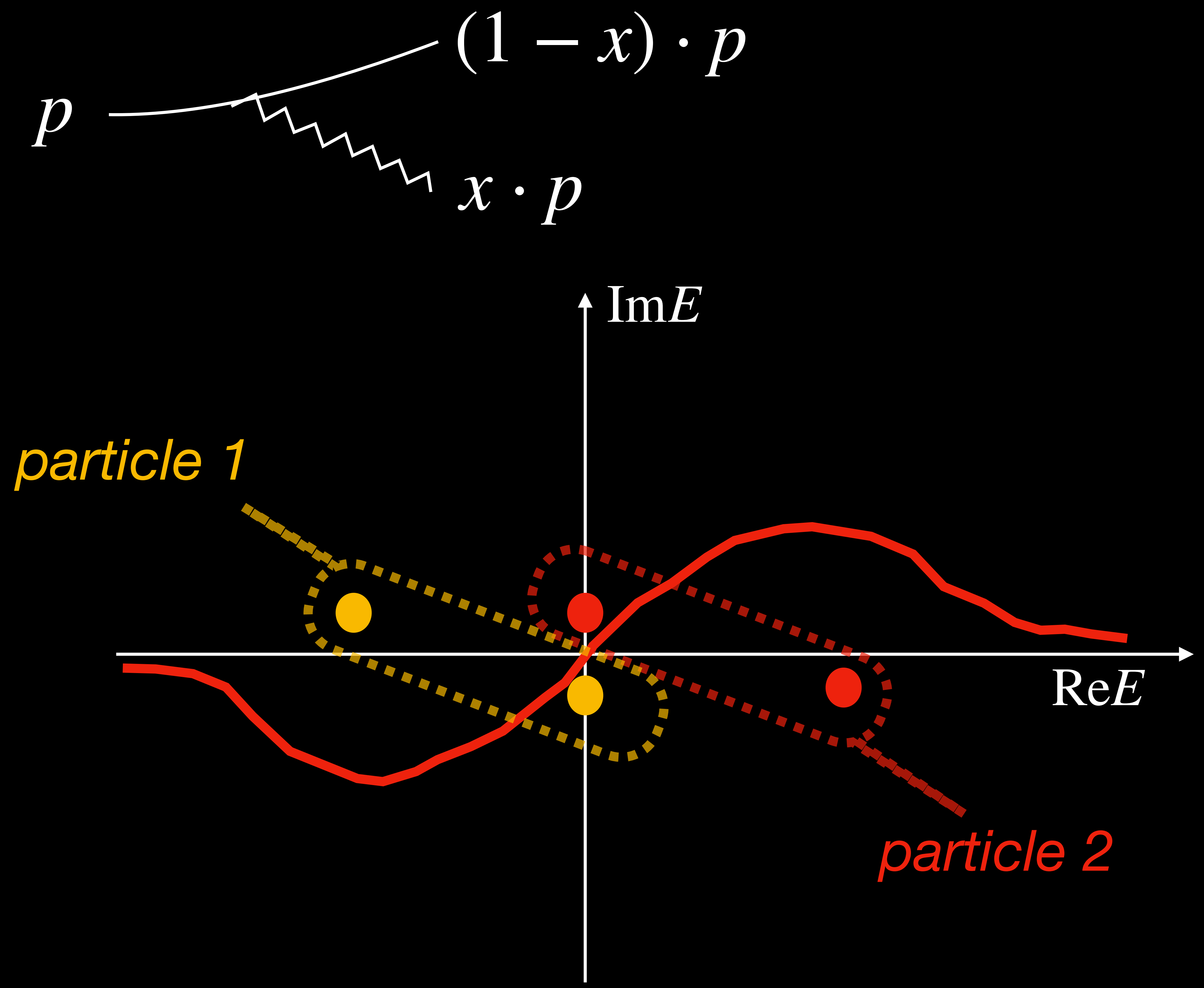
$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{(E + i\delta)(E - i\delta)}$$

- Poles due to soft massless particles.
- These singularities pinch the integration path from both sides.
- Condition for a TRUE INFINITY



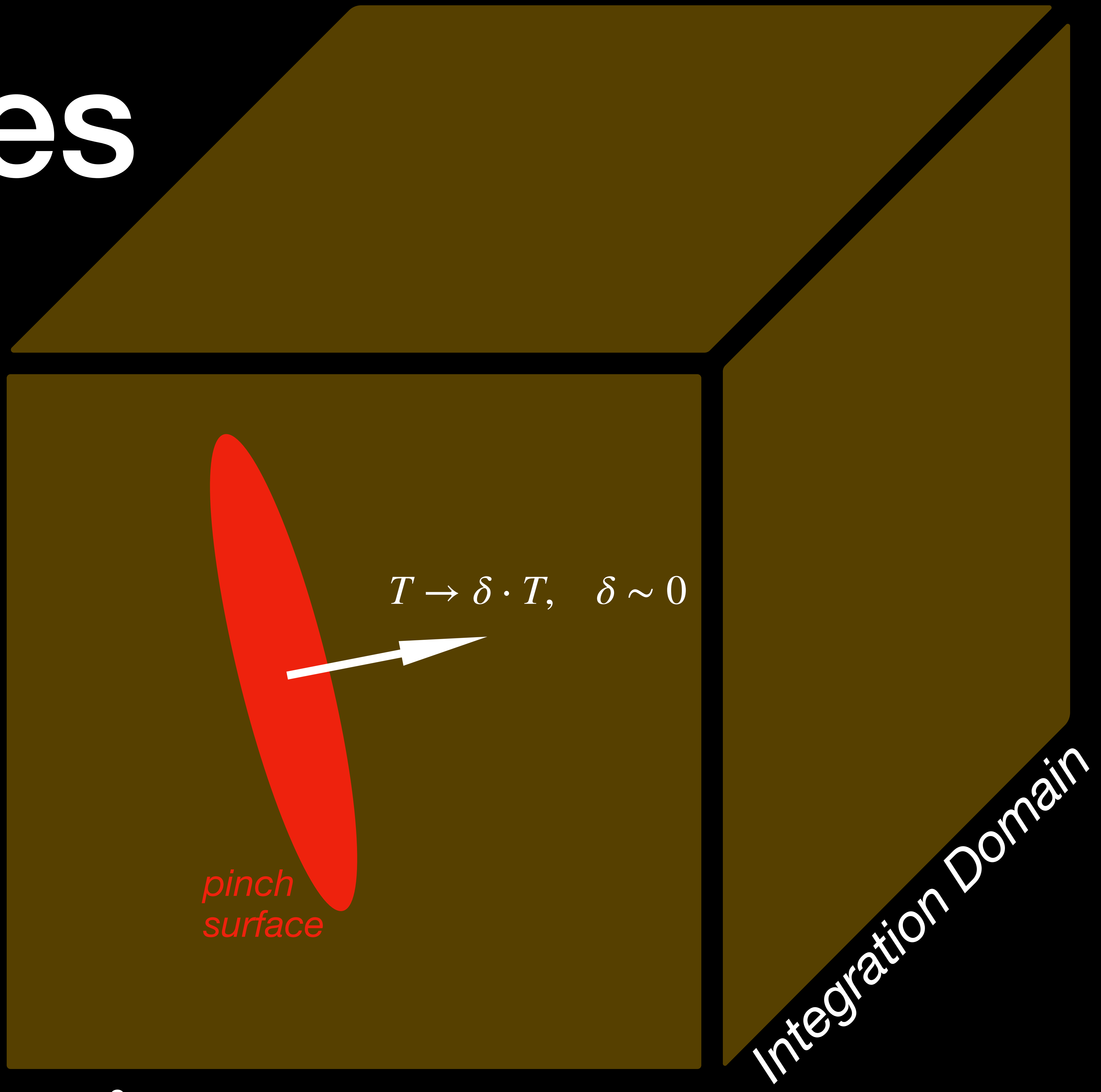
# Collinear massless particles

- A second source of infinities due to massless collinear particles.
- A singularity of one particle in the lower half-plane lines up with the singularity of a collinear particle in the higher half-pane.
- The singularities pinch the integration path from both sides.
- We cannot deform the path, a condition for a TRUE INFINITY!



# Pinch singularities

- To know if a singularity develops, we need to study the behaviour of the integral in the vicinity of the pinch surface.
- We can calculate a degree of divergence.
- Scale variables which are perpendicular to the pinched surface with a small parameter and calculate the scaling of the integrand as the parameter is driven to zero.



**Soft** $k^\mu \sim \delta Q, \quad d^4k \sim \delta^4$

Libby, Sterman

**Collinear** $k = xp + \alpha\eta + \beta p_\perp, \quad x \sim \delta^0, \alpha \sim \delta, \beta \sim \delta^{\frac{1}{2}} \quad d^4k \sim \delta^2$

**Integrand:** $d^4k\mathcal{I}(k) \sim \delta^n$

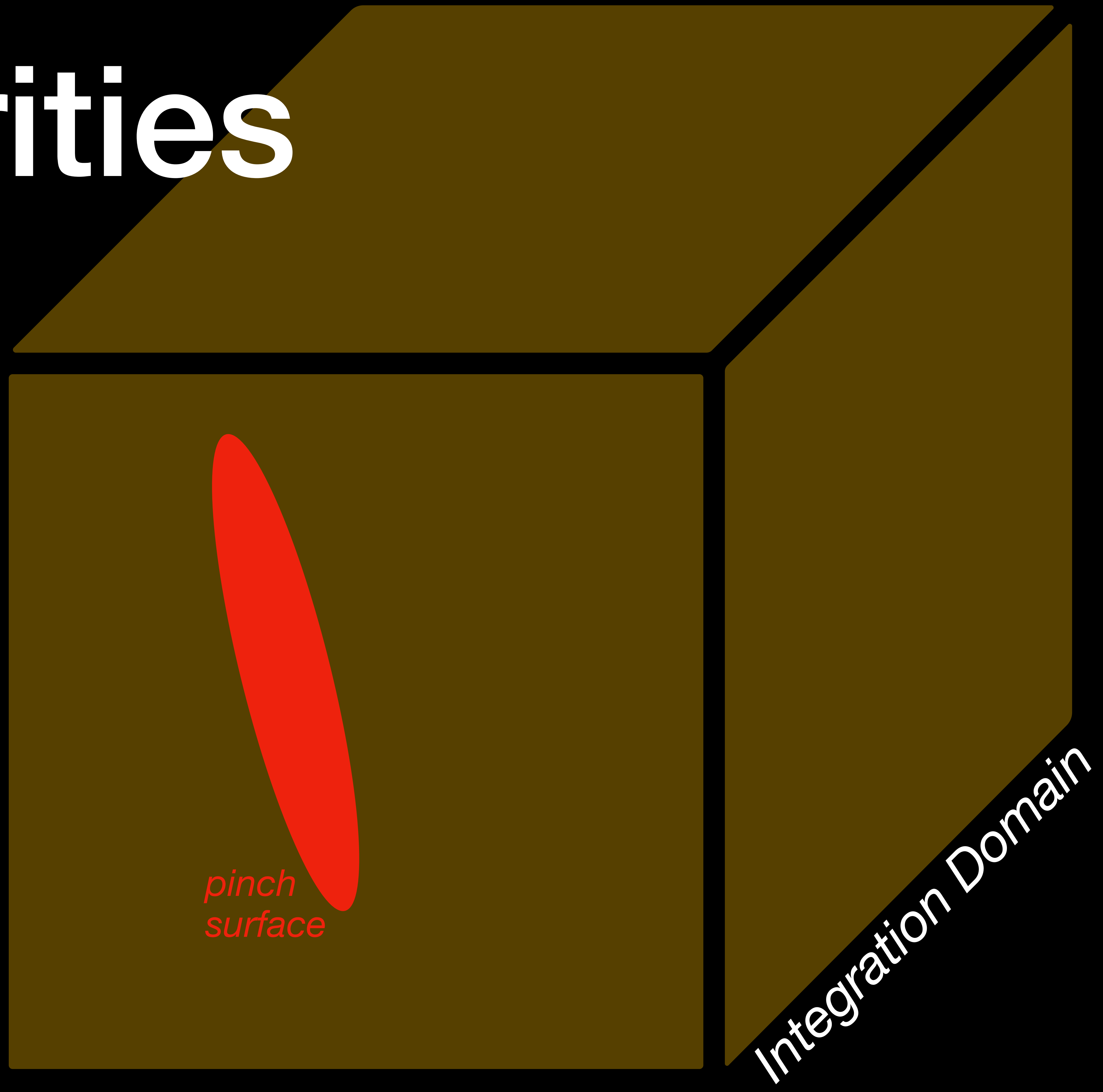
**Divergent:** $n \leq 0$

**Convergent:** $n > 0$

# Removing singularities

- Once a pinch surface which yields a singularity is identified, then we can remove the singularity with a subtraction.

$$A = \int [dk] \mathscr{F}(k)$$



# Removing singularities

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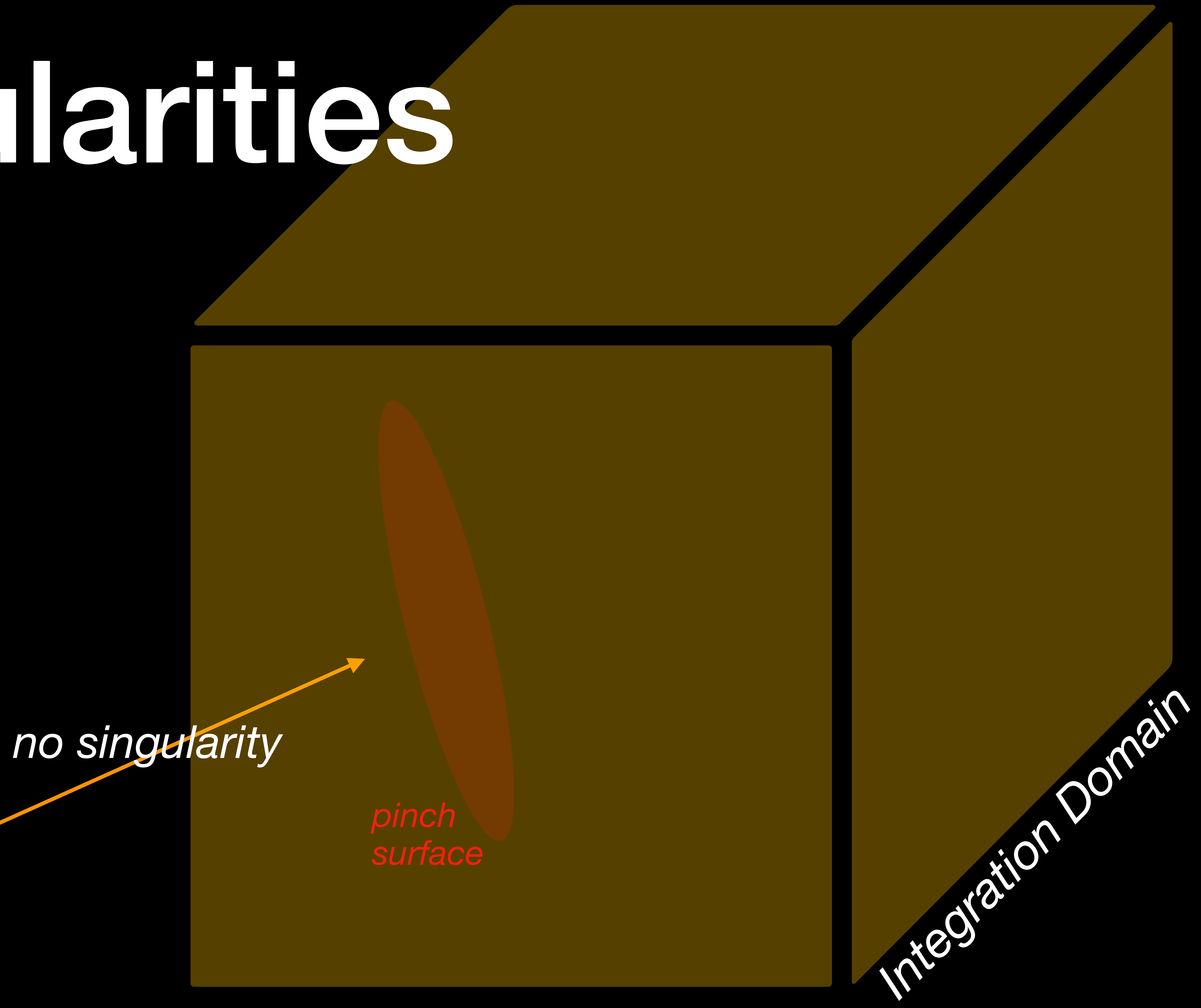
$$A = \int [dk]\ \mathscr{F}(k)$$

$$\to \int [dk]\ \left[\mathscr{F}(k) - t\mathscr{F}(k)\right]$$

*integrand*

*approximation of integrand on singular surface*



# Removing singularities

- Once a pinch surface which yields a singularity is identified, then we can remove the singularity with a subtraction.

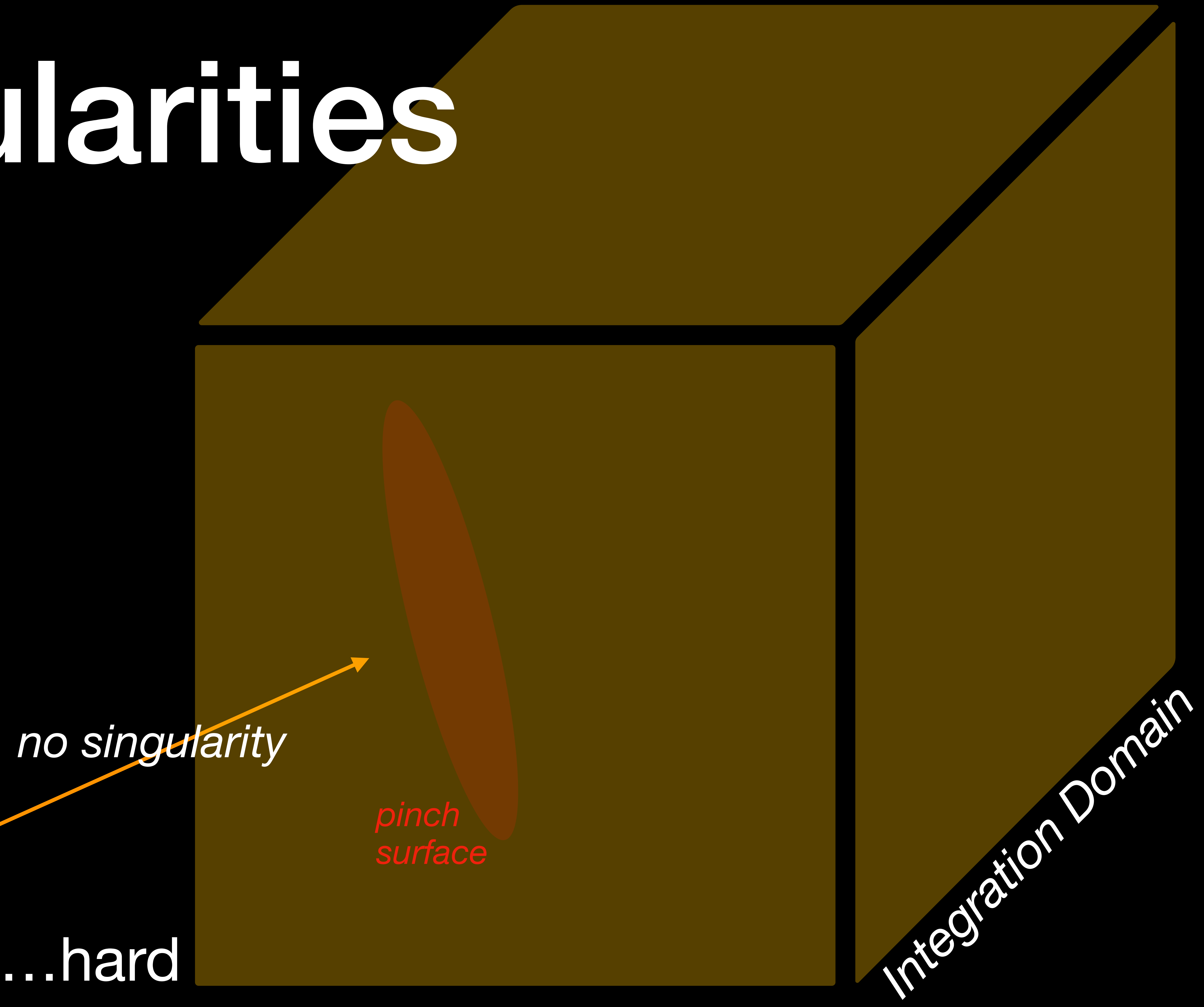
$$A = \int [dk]\ \mathscr{F}(k)$$

$$= \int [dk]\ \left[\mathscr{F}(k) - t\mathscr{F}(k)\right] \longrightarrow$$

…hard

$$+ \int [dk]\ t\mathscr{F}(k) \longrightarrow$$

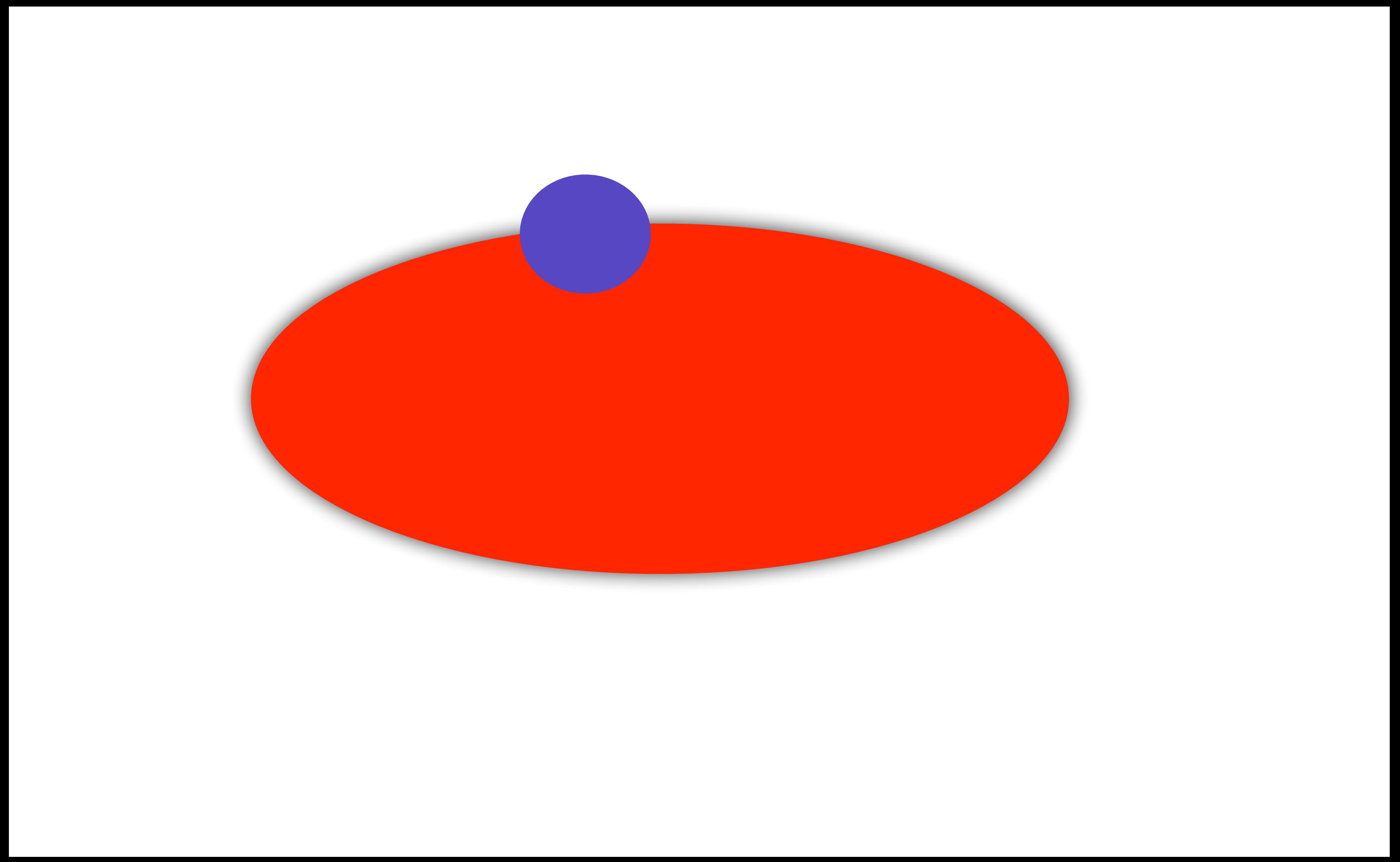
…soft or jet



# Nested subtractions

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?
- Order the singular regions by their "volume"

Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman



# Nested subtractions

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- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?
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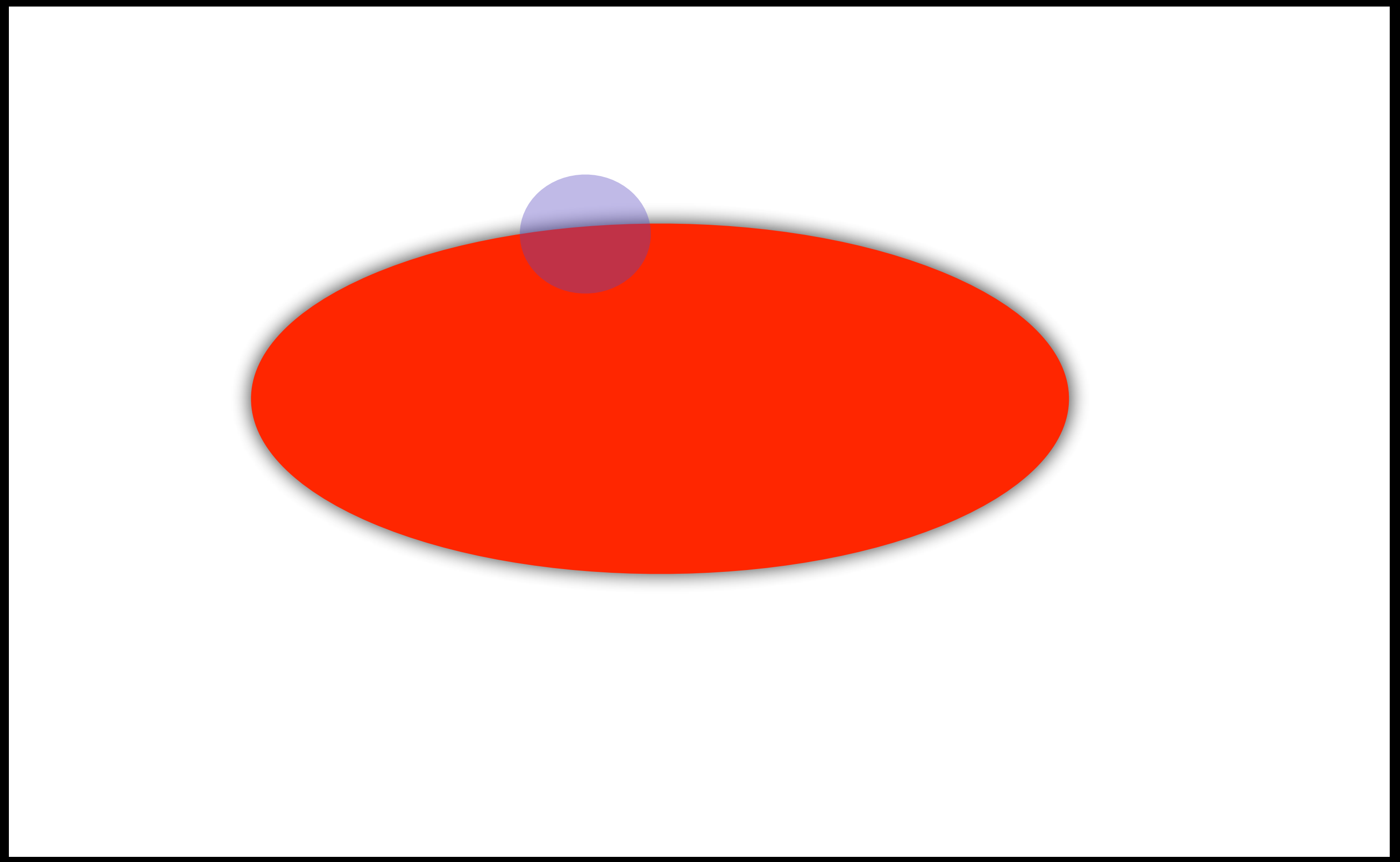
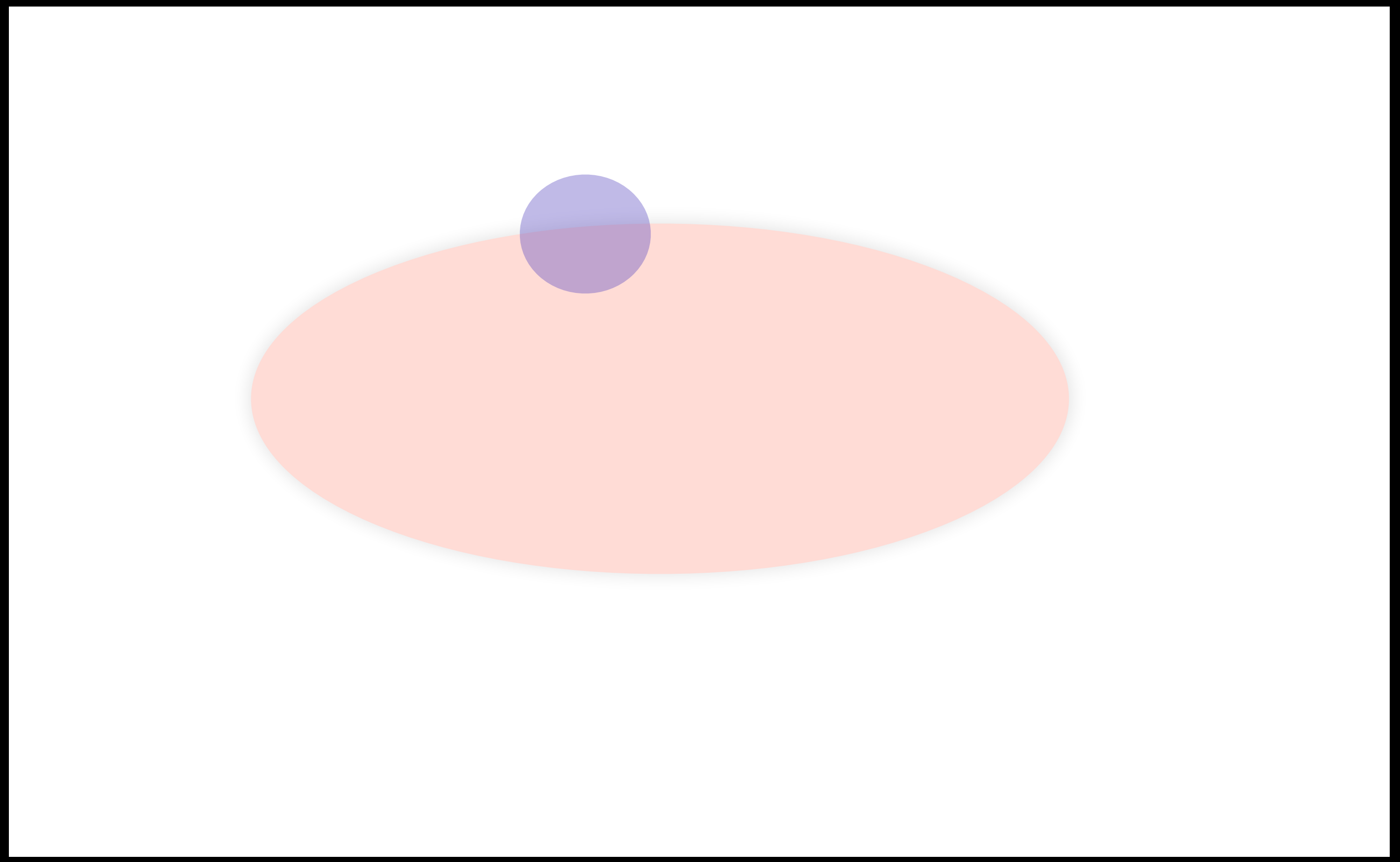
- Subtract an approximation of the integrand in the smallest volume
-



# Nested subtractions

Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?
- Order the singular regions by their “volume”
- Subtract an approximation of the integrand in the smallest volume
- Then, proceed to the next volume and repeat until there are no more singularities to remove.



# Nested subtractions

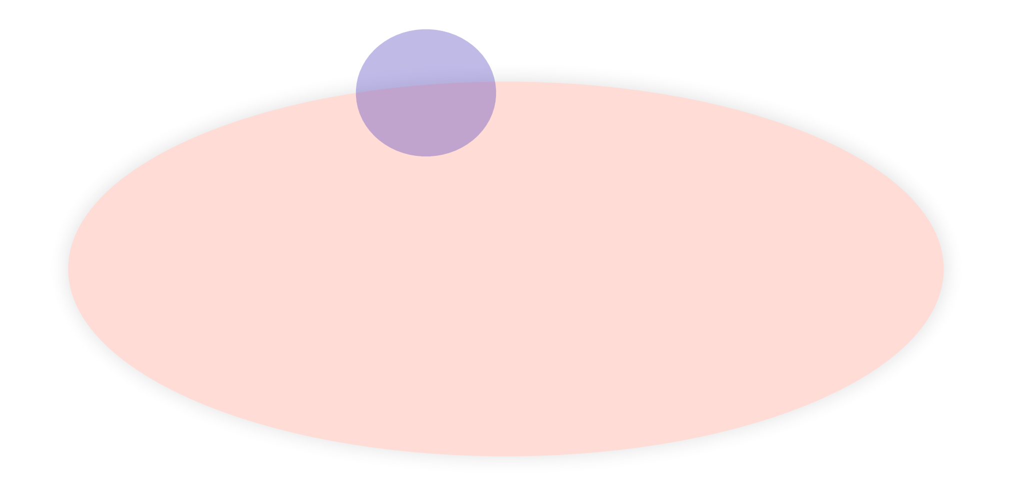
- The procedure of nested subtractions has a solution for the finite remainder at any loop order as a Forest formula (similarly to BPHZ of UV renormalzation)

- It is valid term by term in an amplitude or a Feynman diagram.

- This forest formula structure combined with gauge symmetry, gives rise to the factorization of gauge theory amplitudes in terms of Jets, Soft and Hard fucntions.

Ma; Erdogan, Sterman; Collins;
Collins, Soper, Sterman

$$R^{(n)}\,\gamma^{(n)} \;=\; \gamma^{(n)} + \sum_{N\in\mathcal{N}[\gamma^{(n)}]} \prod_{\rho\in N} (-t_\rho)\,\gamma^{(n)}\,,$$



# A concept that can bring further progress
# INFRARED FACTORIZATION

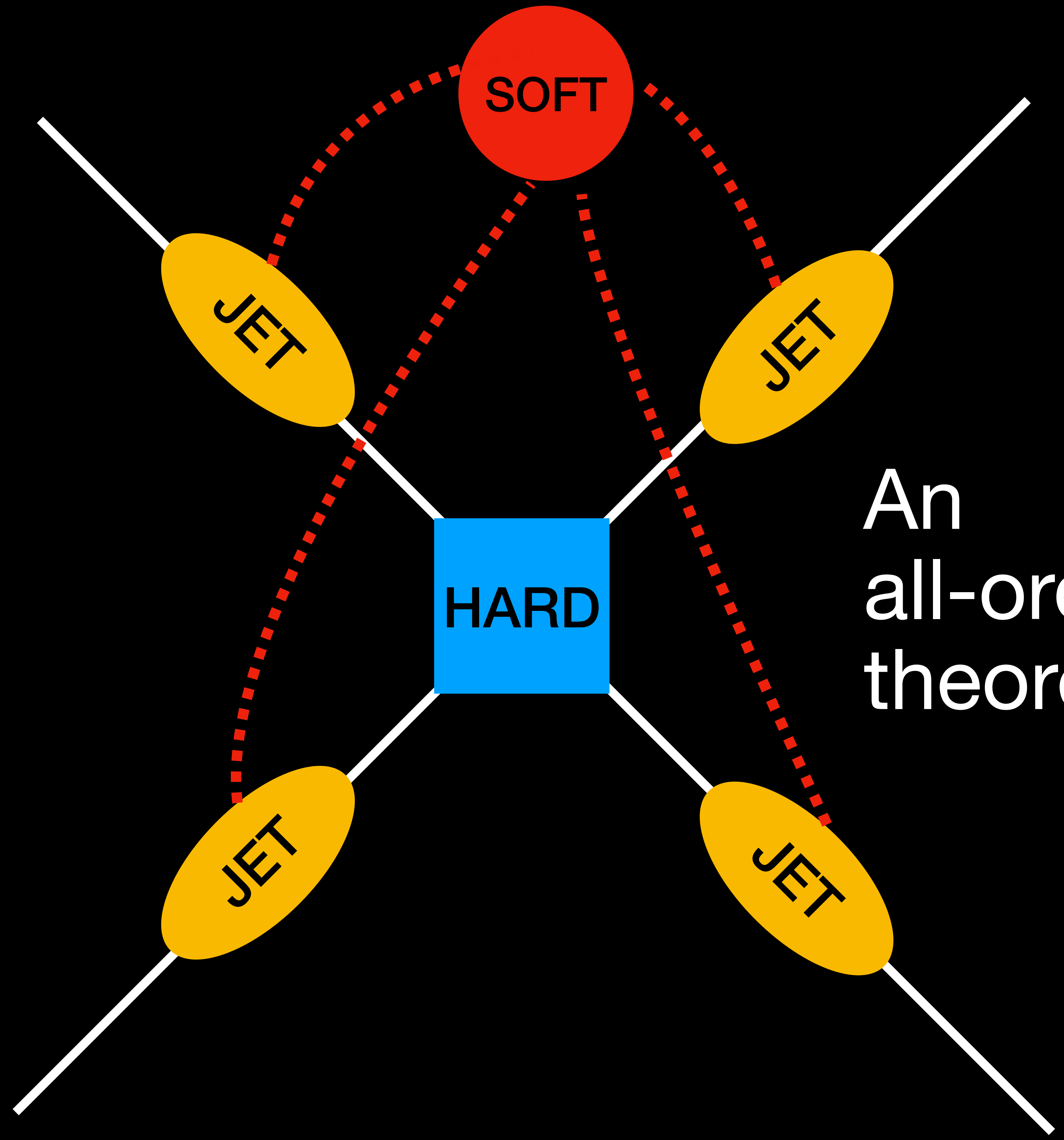
- UV Renormalized scattering amplitudes for well-separated final-states take a simple factorized form

$$Amplitude = \mathrm{hard} \cdot \mathrm{soft} \cdot \prod_i \mathrm{jet}_i \, .$$

  - “soft” and “jet” functions contain all divergences.

- These are universal functions. For any new process we should need to compute only the “hard” function.

- So far, we do not have a way to compute the “hard” function directly



An all-orders theorem

*Ma; Erdogan, Sterman; Schwartz; Collins*

# How would we like to use factorization?

$$Amplitude = \cancel{\sum_j c_j\, Master_j = \sum_k d_k Polylogs_k(momenta)} = Numbers$$

$$= \int [dk]\, \mathscr{A}(k) = \int \mathscr{S} \cdot \int \prod_i \mathscr{J}_i \cdot \int [dk]\, \mathscr{A}(k) \cdot \mathscr{S}^{-1}(k) \cdot \prod_i \mathscr{J}_i^{-1}(k)$$

Analytic Integration

Universal

Numerical process-dependent integration

*integrand of hard function*

From factorisation we could identify, remove and integrate separately the singular parts of amplitudes order by order in perturbation theory:

$$\mathscr{H}^{(0)} = \mathscr{A}^{(0)} \quad \Big| \quad \mathscr{H}^{(1)} = \mathscr{A}^{(1)} - \mathscr{J}^{(1)}\mathscr{H}^{(0)} - \mathscr{S}^{(1)}\mathscr{H}^{(0)} \quad \Big| \quad \mathscr{H}^{(2)} = \mathscr{A}^{(2)} - \mathscr{J}^{(1)}\mathscr{H}^{(1)} - \mathscr{S}^{(1)}\mathscr{H}^{(1)} - \mathscr{J}^{(2)}\mathscr{H}^{(0)} - \mathscr{S}^{(2)}\mathscr{H}^{(0)} + \mathscr{J}^{(1)}\mathscr{S}^{(1)}\mathscr{H}^{(0)} \quad \ldots$$

This procedure is universal…can be applied to any process, irrespectively of the complexity of its final state.

# How would we like to use factorization?

$$\text{Amplitude} = \int [dk]\, \mathscr{A}(k) = \left( \int \mathscr{S} \cdot \int \prod_i \mathscr{J}_i \right) \cdot \int [dk]\, \mathscr{A}(k) \cdot \mathscr{S}^{-1}(k) \cdot \prod_i \mathscr{J}_i^{-1}(k)$$

This approach is not unique in its design…

$$\text{Amplitude} = \int [dk]\, \overline{\mathscr{A}(k)} = \left( \int \overline{\mathscr{S}} \cdot \int \prod_i \overline{\mathscr{J}}_i \right) \cdot \int [dk]\, \overline{\mathscr{A}(k)} \cdot \overline{\mathscr{S}}^{-1}(k) \cdot \prod_i \overline{\mathscr{J}_i}^{-1}(k)$$

Use Amplitudes of simpler processes in lieu of soft and jet functions

$$\text{Amplitude} = \int [dk]\, \overline{\mathscr{A}(k)} = \left( \int \prod \mathscr{A}_{\text{IR}-\text{approx.}} \right) \cdot \int [dk]\, \overline{\mathscr{A}(k)} \cdot \prod \mathscr{A}^{-1}_{IR-approx.}(k)$$

# How would we like to use factorization?

Goal:

$$\left( \int \prod \mathscr{A}_{\mathrm{IR-approx.}} \right)$$

The analytic integration over the infrared approximation of the amplitude should be feasible

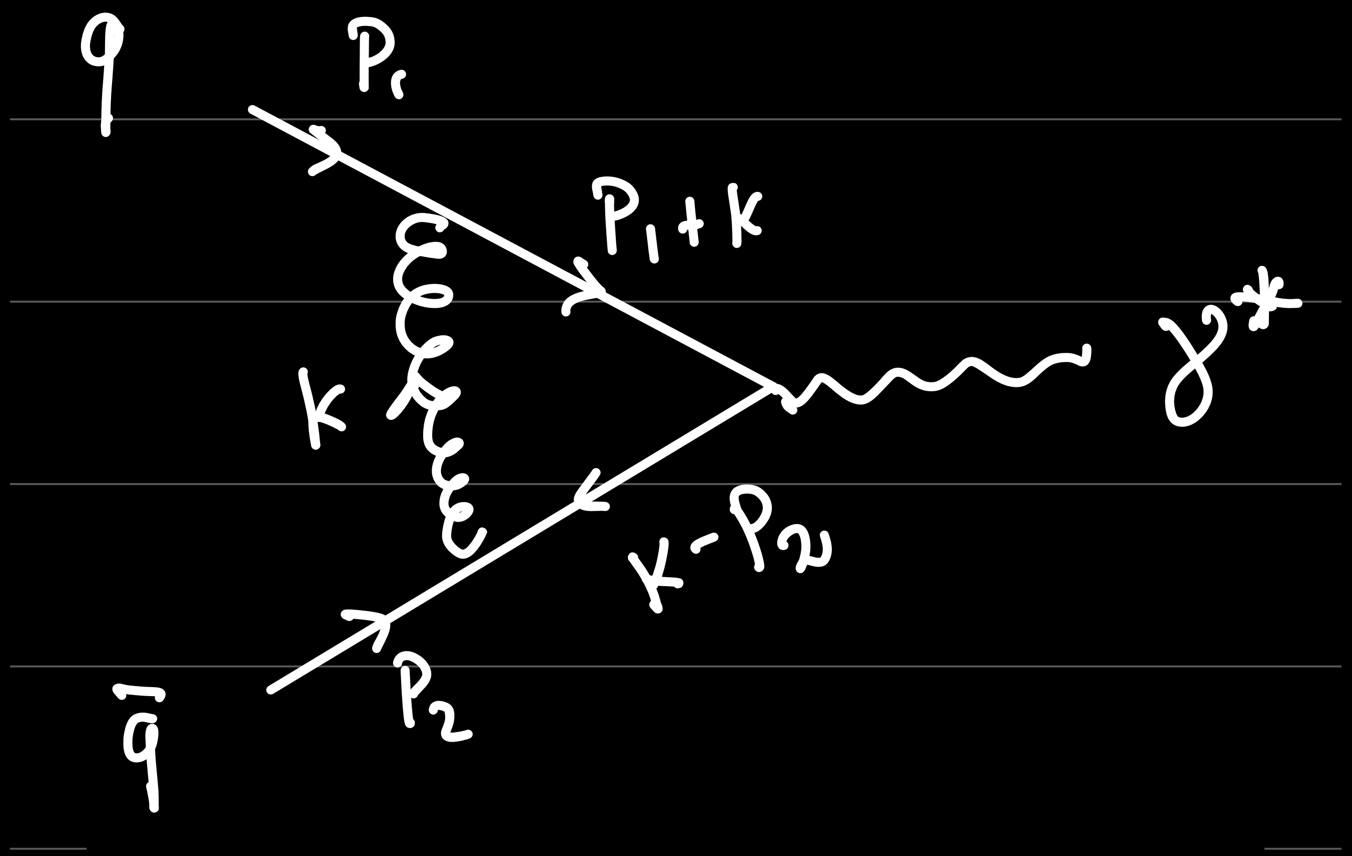
Goal:

$$\int [dk] \, \overline{\mathscr{A}(k)} \cdot \prod \mathscr{A}^{-1}_{IR-approx.}(k)$$

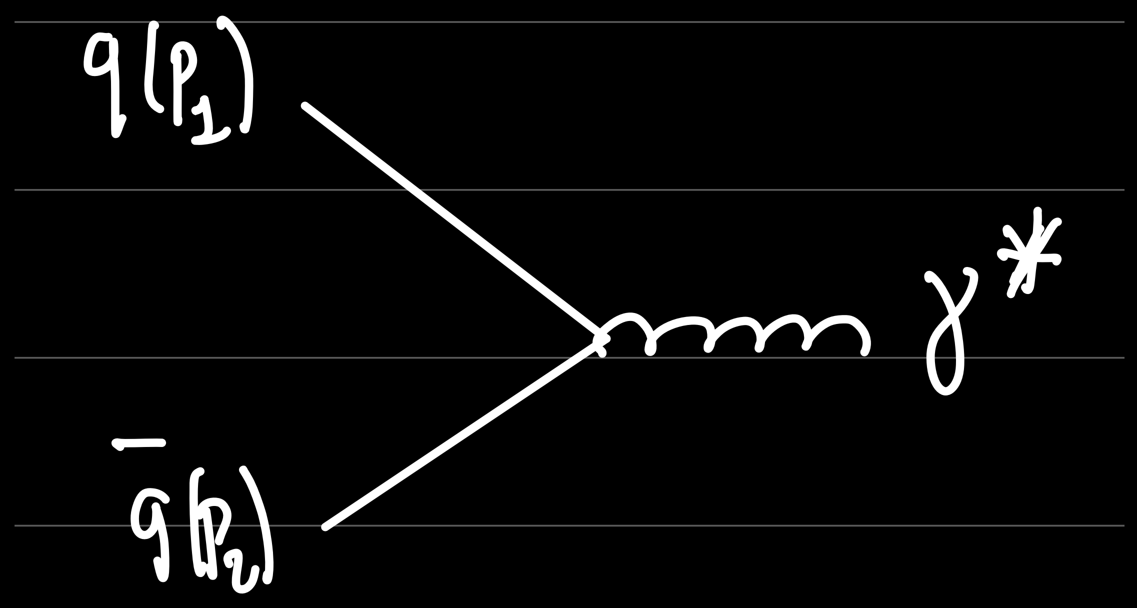
The integration of the hard amplitude remainder should be possible in exactly D=4, numerically.

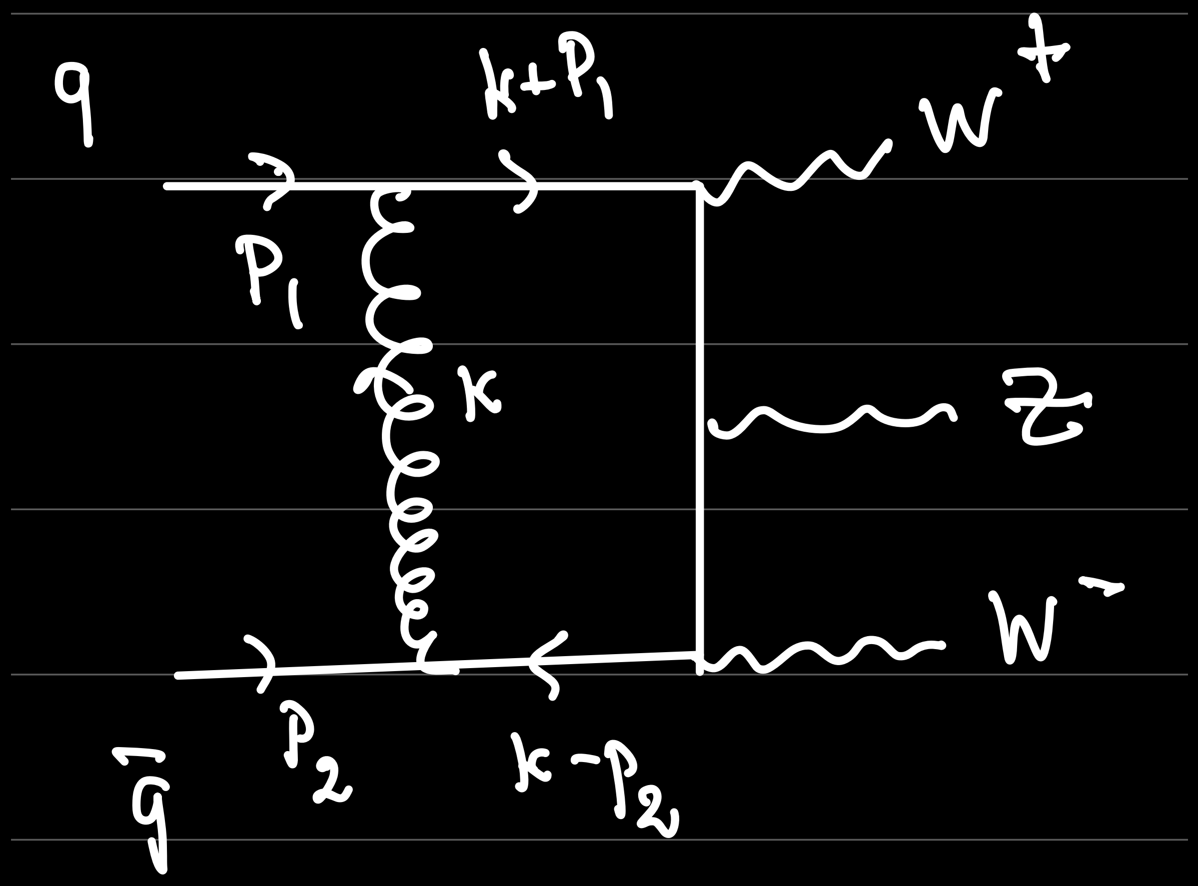
The integrand must be free of infrared and ultraviolet divergences locally.

# Factorization and subtraction of soft singularities



$$\xrightarrow{k \to 0} i g_s^2 \frac{(p_1 \cdot p_2)\, T_q \cdot T_{\bar{q}}}{k^2(-k \cdot p_2)(k \cdot p_1)}$$





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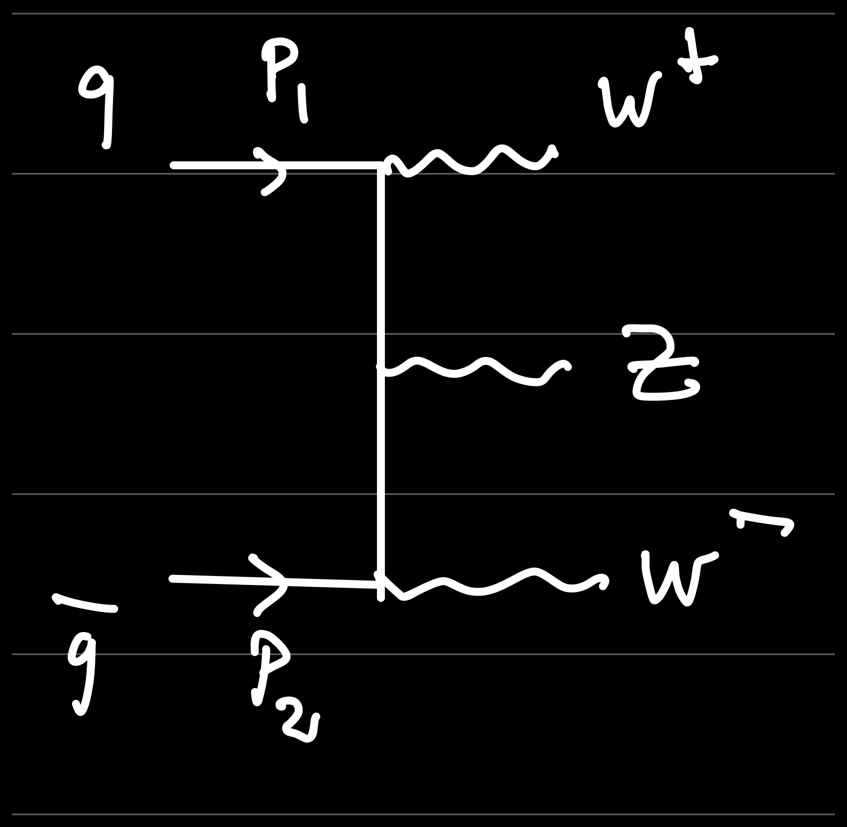
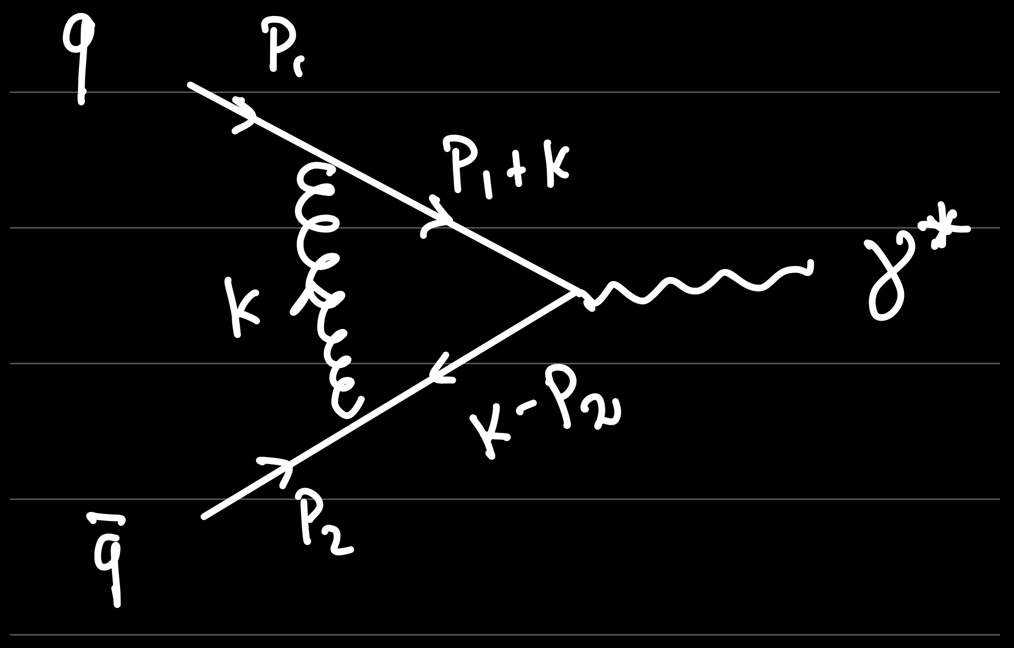
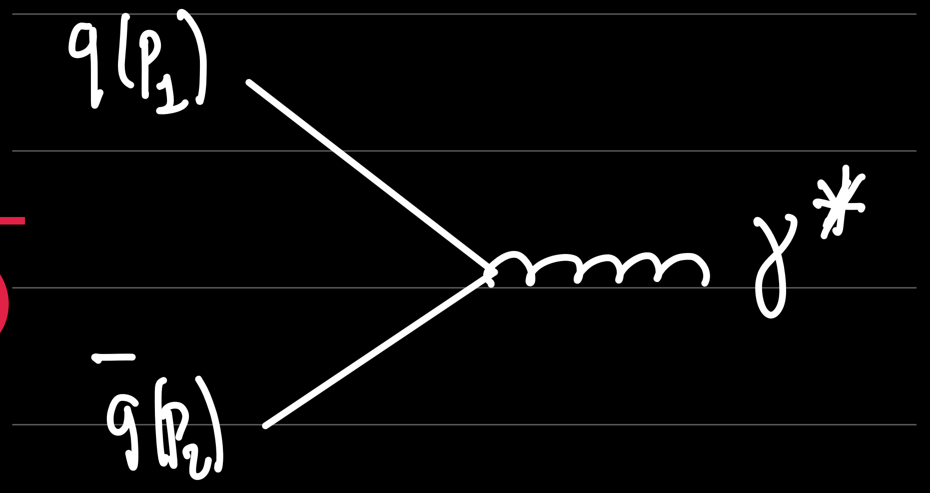


# Factorization and subtraction of soft singularities

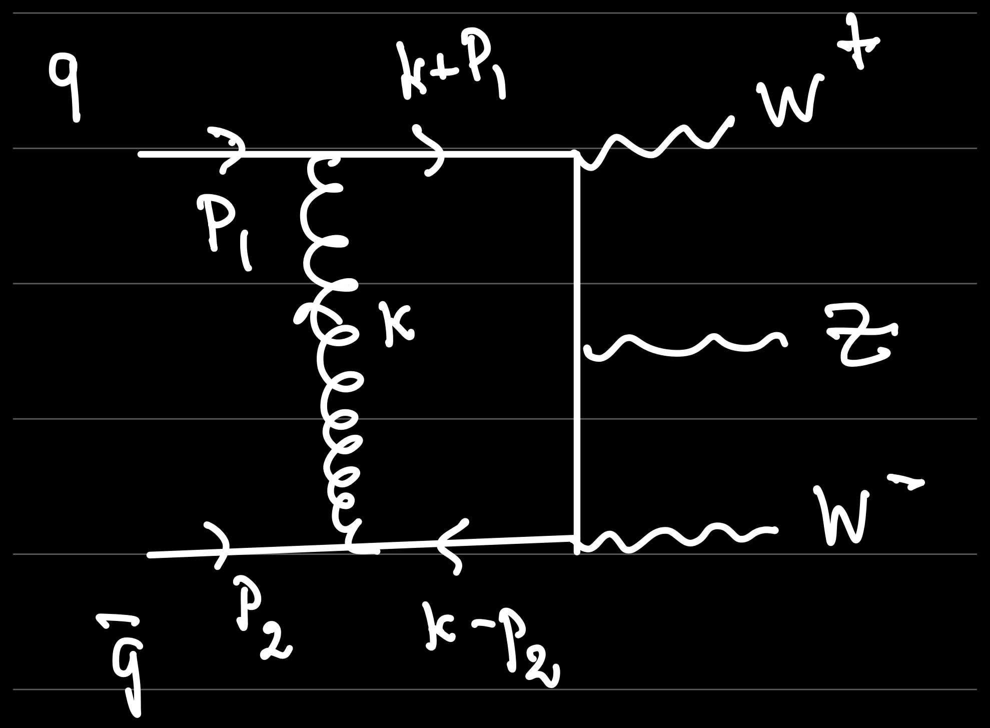


$$- \; igs^2 \frac{(p_1 \cdot p_2)\, T_q \cdot T_{\bar{q}}}{k^2(-k \cdot p_2)(k \cdot p_1)}$$

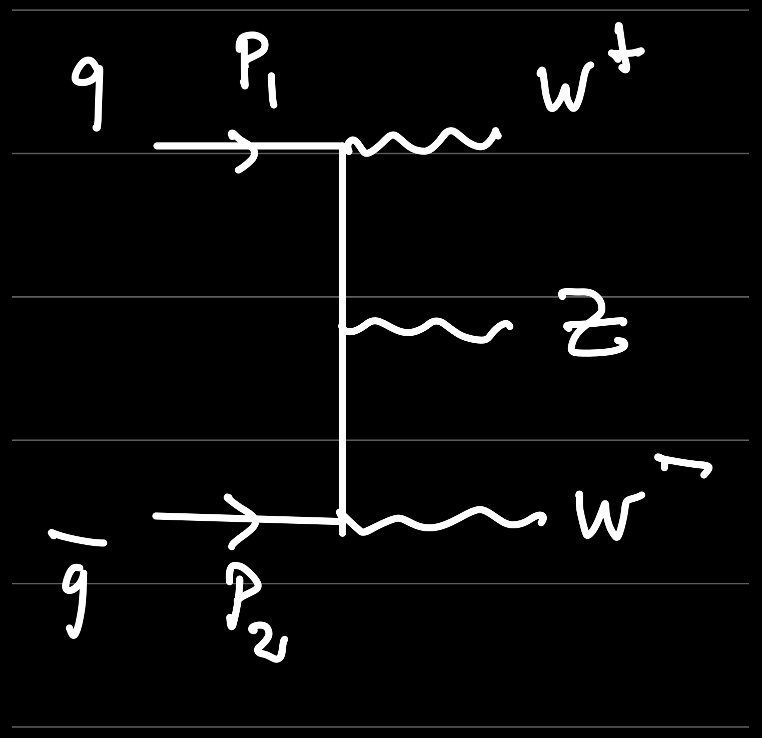


$$\xrightarrow{k \to 0}$$

Finite



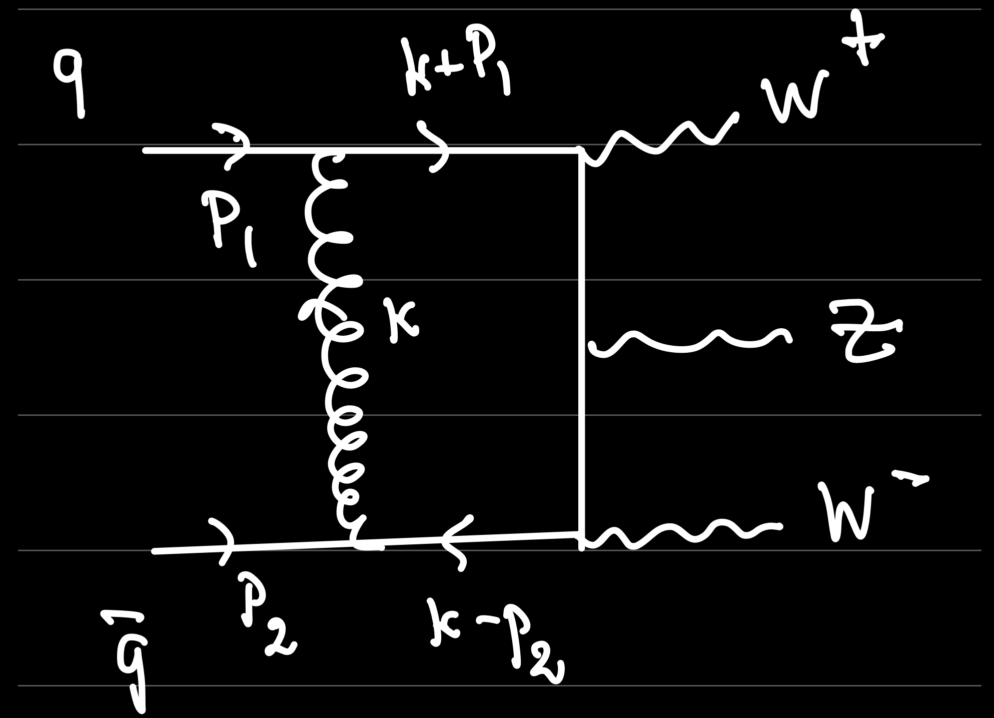
$$- \; igs^2 \frac{(p_1 \cdot p_2)\, T_q \cdot T_{\bar{q}}}{k^2(-k \cdot p_2)(k \cdot p_1)}$$



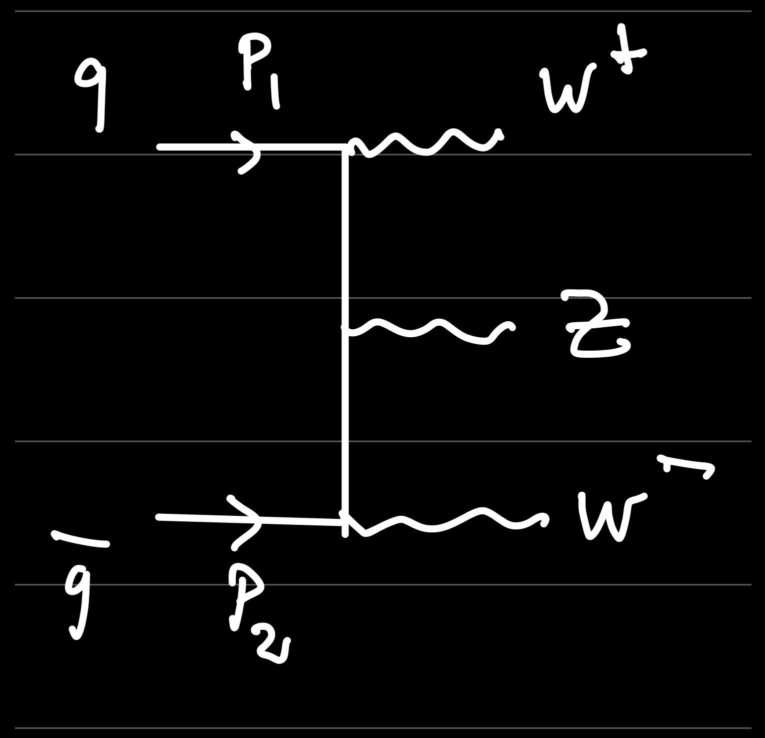
$$\xrightarrow{k \to 0}$$

Finite

# Factorization and subtraction of soft singularities



$$- igs^2 \frac{(p_1 \cdot p_2)\, T_q \cdot T_{\bar{q}}}{k^2(-k \cdot p_2)(k \cdot p_1)}$$

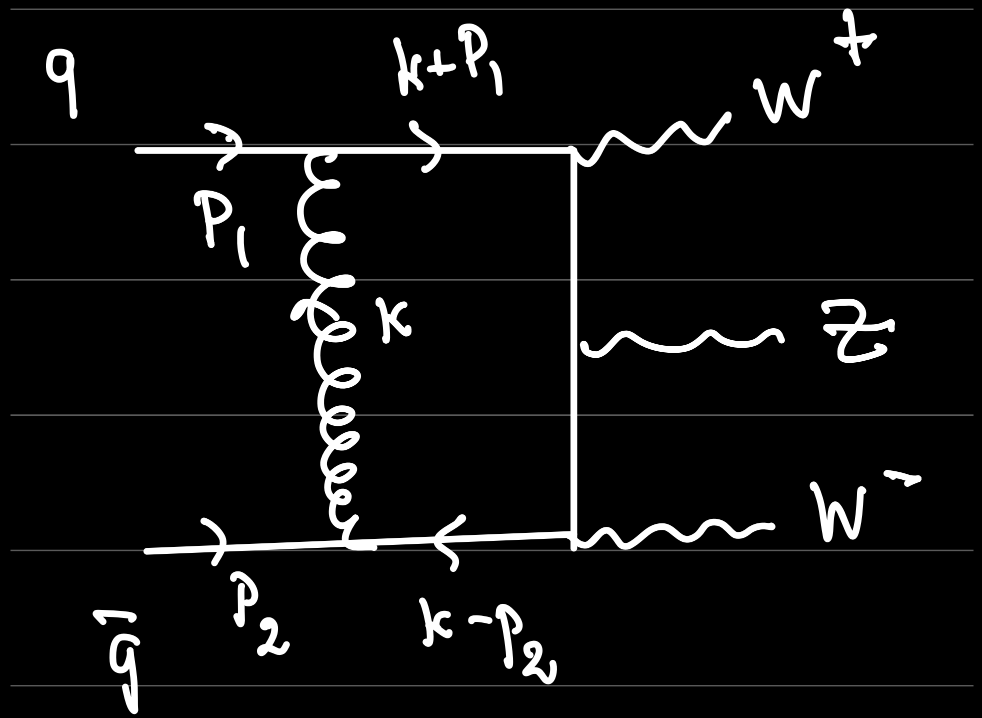


$$\xrightarrow{k \to 0}$$ Finite

Drawbacks:

- Introduces novel ultraviolet singularities that need to be taken care with further subtractions
- Changes the denominator structure (linearised propagators), which need to be taken into account in treating integrable singularities.
- Subtracting strict soft and collinear limits leads to a proliferation of subtraction terms at two-loops

# A Form factor subtraction



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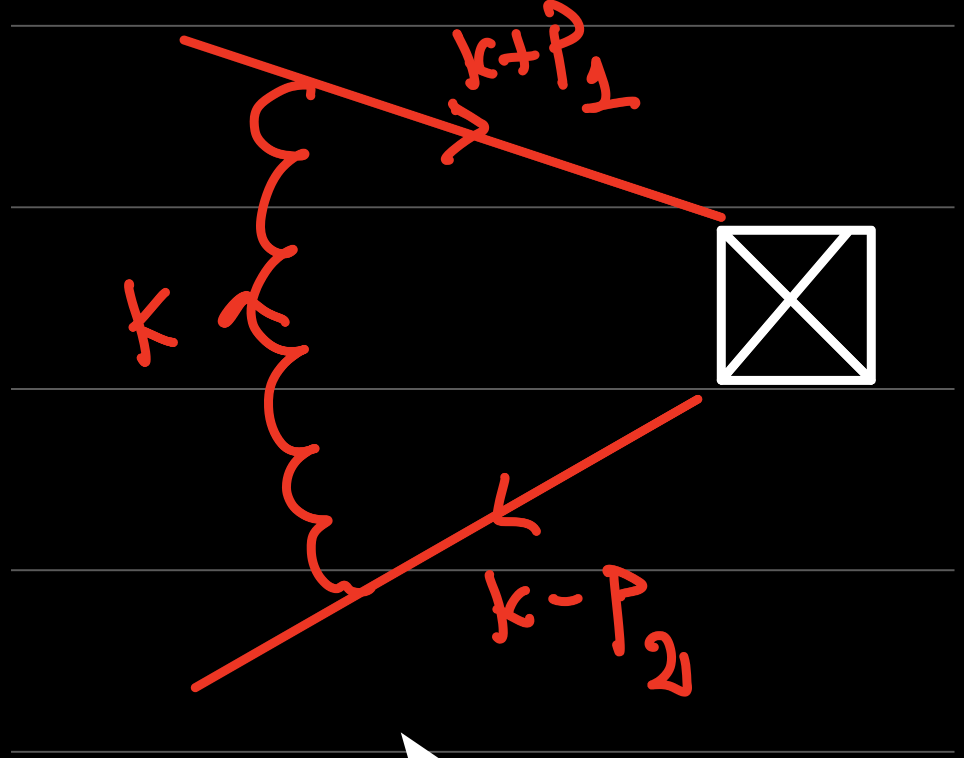
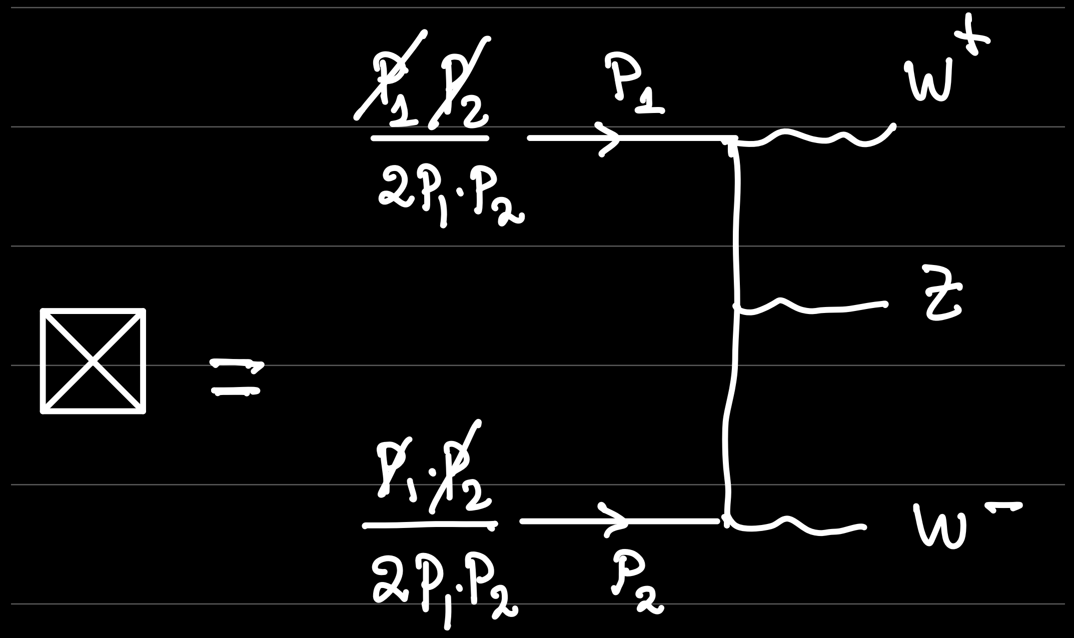


$k \to 0$

Finite

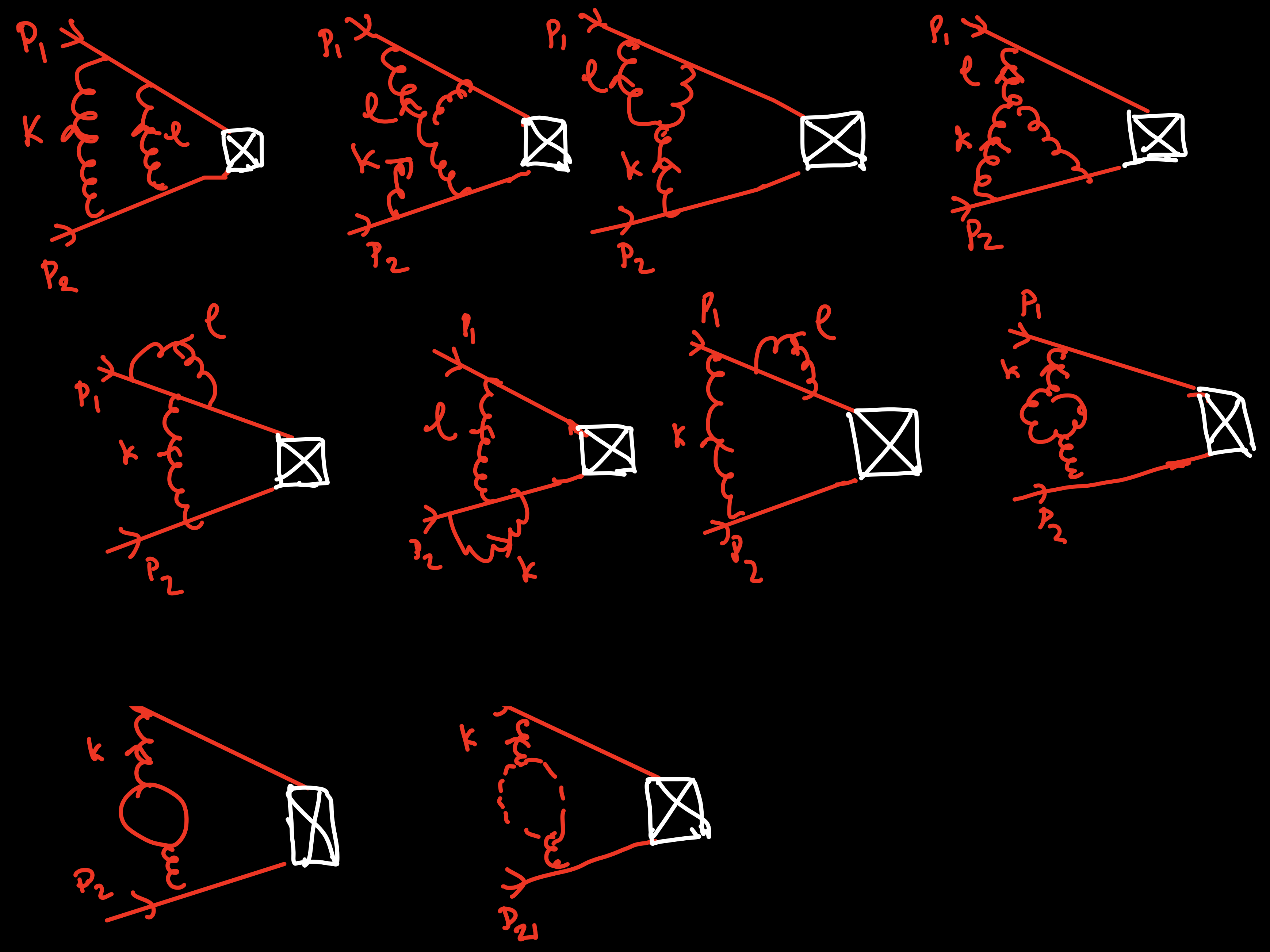


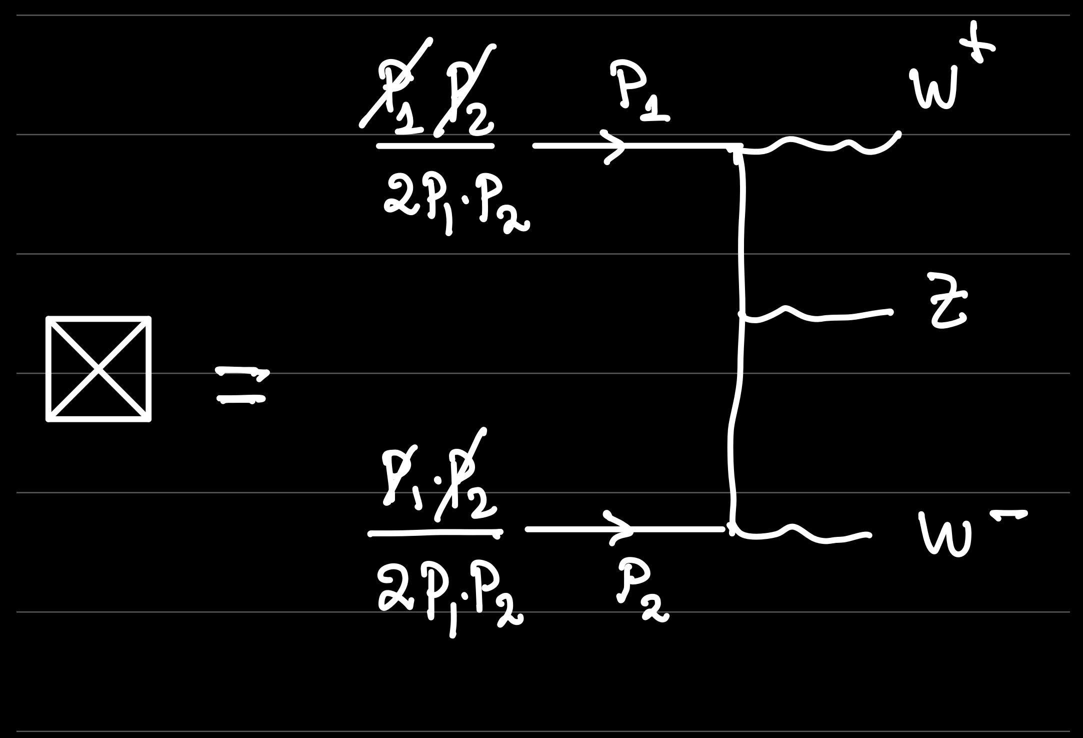
Composite vertex, the tree amplitude. More generally, the hard function of a previous loop order.

$$\mathcal{F}^{(1)}\left[\text{Tree}\right] = \bar{v}(p_2)\left(-ig_s\gamma^{\mu}t^{a}\right)\frac{i}{\not{k}-\not{p}_2}\cdot\text{Tree}\cdot\frac{i}{\not{k}-\not{p}_1}\left(-ig_s\gamma_{\mu}t^{a}\right)u(p_1)\frac{(-i)}{k^2}$$

# Generalised form factor at two-loops

$$\mathscr{F}^{(2)}\,[\text{Tree}] =$$





Composite vertex, the tree amplitude. More generally, the hard function of a previous loop order.

# A main result of this work

$q + \bar{q} \rightarrow$ any set of heavy colourless particles

$$\mathscr{H}_{1-loop}(k) = \mathscr{A}_{1-loop} - \mathscr{F}^{(1)}\left[\mathscr{A}_0\right]$$

$$\mathscr{H}_{2-loop}(k, l) = \mathscr{A}_{2-loop} - \mathscr{F}^{(2)}\left[\mathscr{A}_0\right] - \mathscr{F}^{(1)}\left[\mathscr{H}_{1-loop}\right]$$

**Free of ALL soft and collinear singularities LOCALLY!**

$$\int \mathscr{H}^{(R)}_{2-loop}(k, l)$$

Integrable in D=4 dimensions with numerical methods.

# A main result of this work

$q + \bar{q} \to$ any set of heavy colourless particles

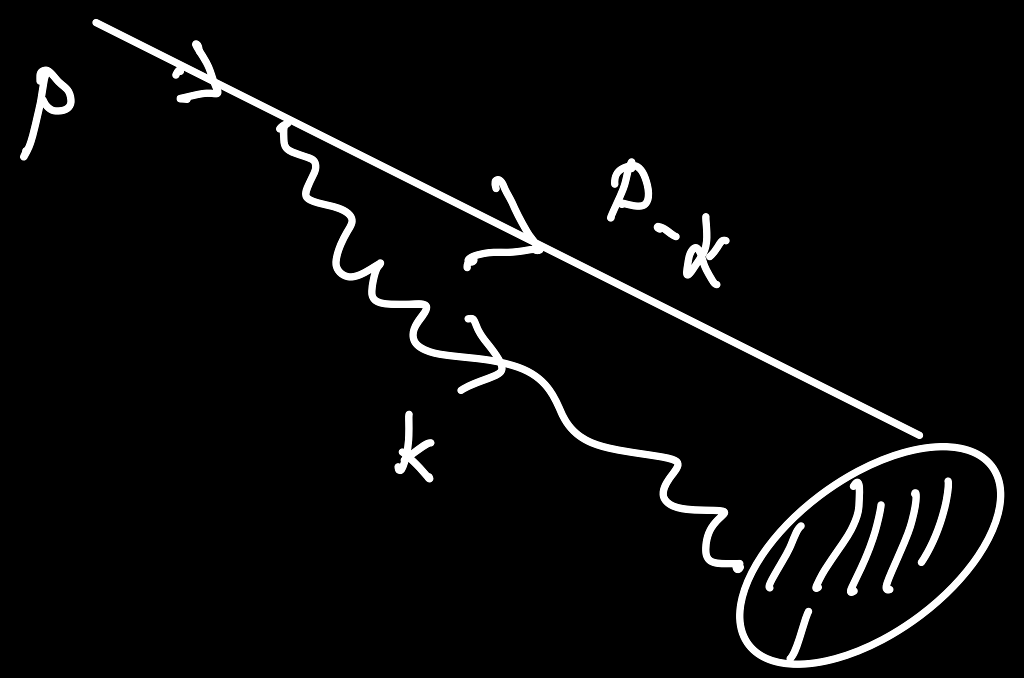
$$\mathscr{H}_{2-loop}(k,l) = \mathscr{A}_{2-loop} - \mathscr{F}^{(2)}\left[\mathscr{A}_0\right] - \mathscr{F}^{(1)}\left[\mathscr{H}_{1-loop}\right]$$

**Due to factorisation, collinear singularities also get subtracted by the form-factor subtractions. But this is not a straightforward "out-of-the-box" result.**

**We must first write a suitable representation of the two-loop amplitude and form-factor integrands which render factorisation manifest locally.**

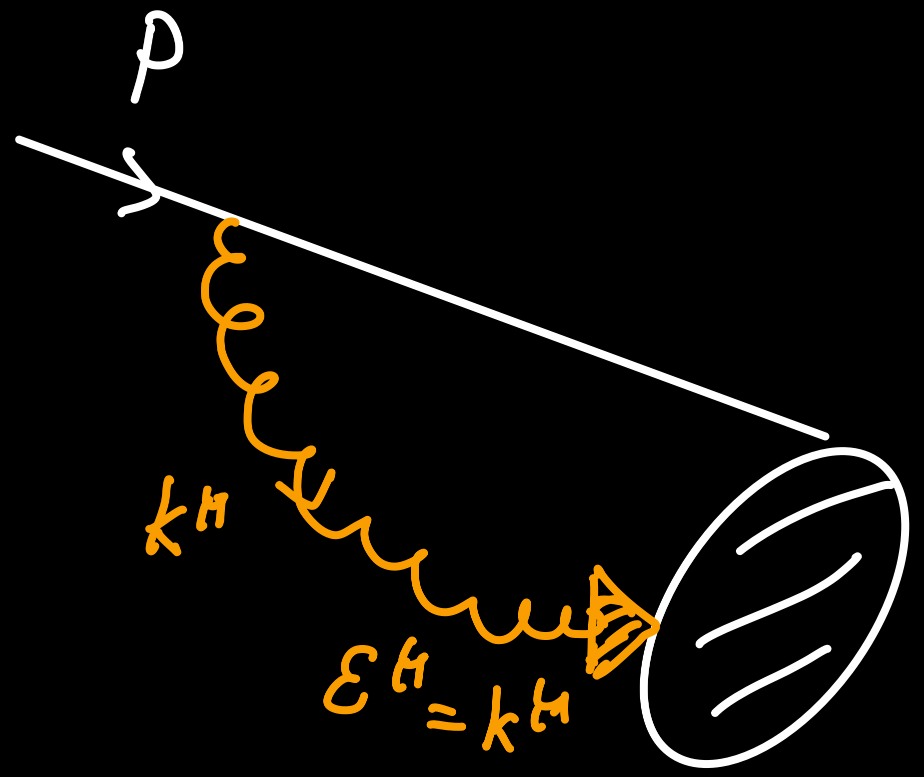
# Collinear singularities

Collinear singularities → longitudinally polarised gluons

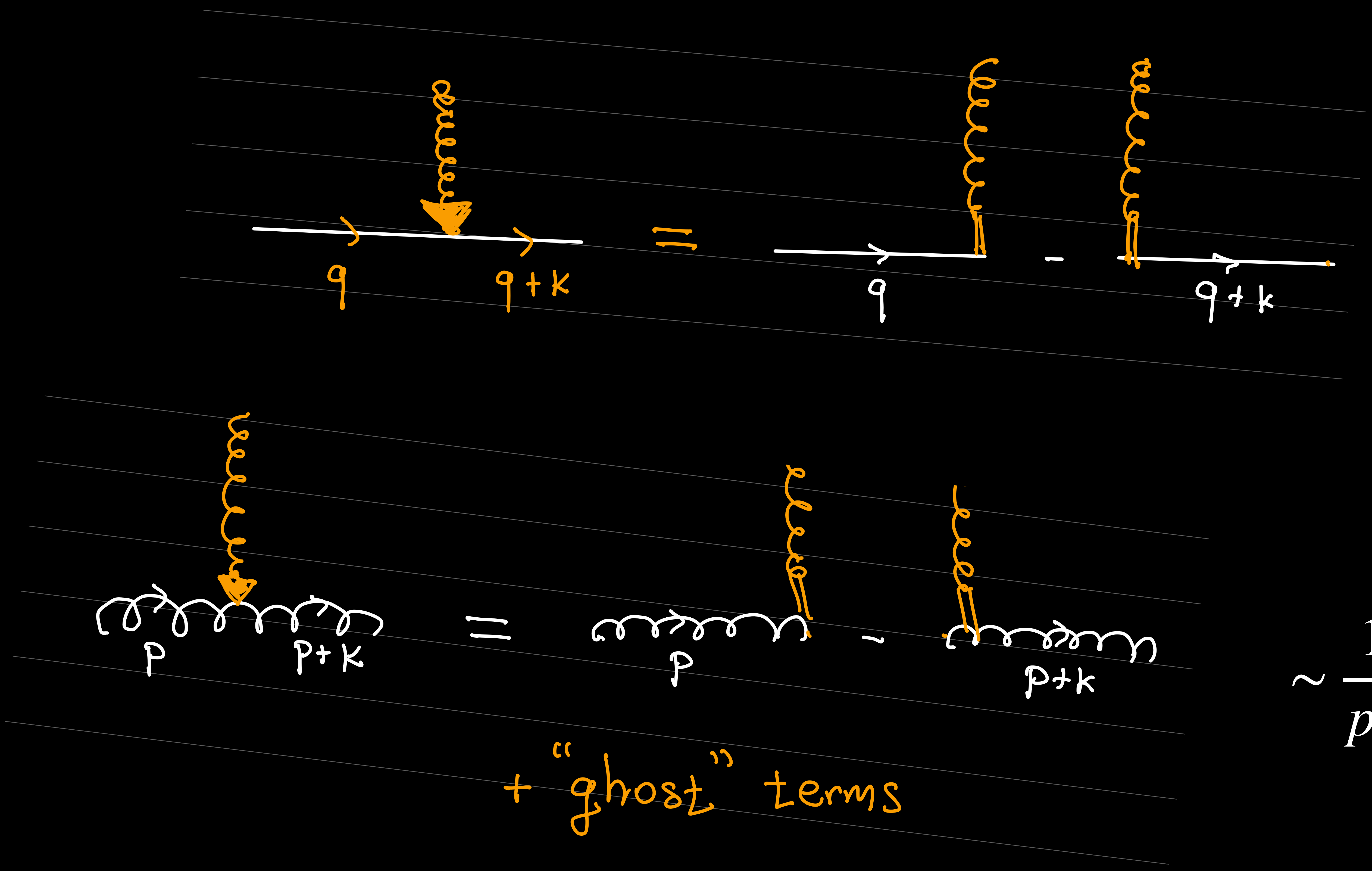


$$= \ldots \frac{(\not p - \not k)\gamma^\mu u(p)}{k^2(k+p)^2} \xrightarrow{k = xp, \quad x = \frac{k\cdot\eta}{p\cdot\eta}} \ldots u(p)\frac{2(1-x)}{x}\frac{k^\mu}{k^2(k+p)^2}$$

$\rightsquigarrow$



# Fate of longitudinal gluons



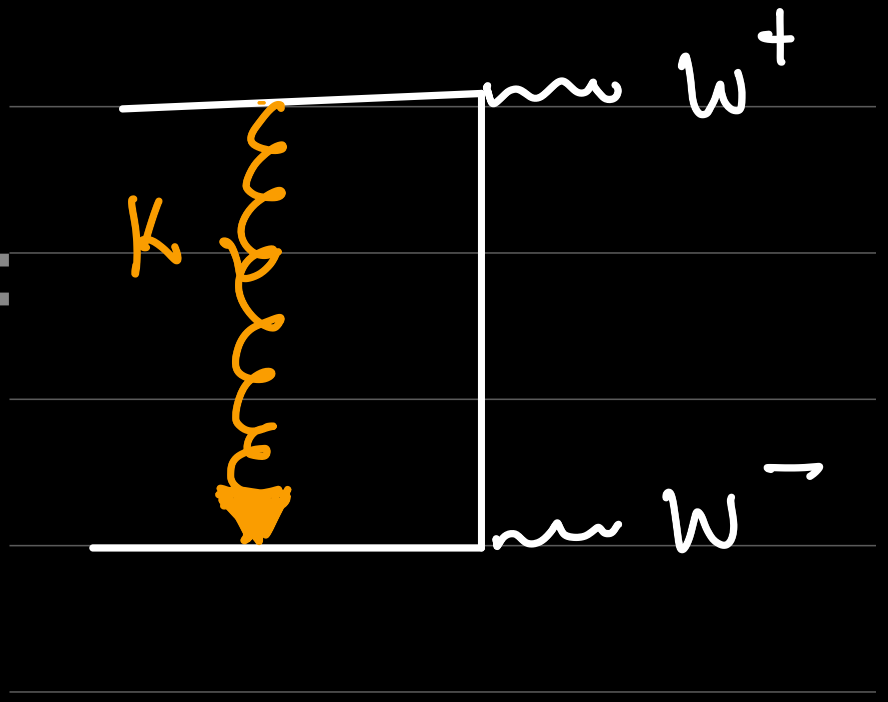
$$\sim \frac{1}{\not p} - \frac{1}{\not p + \not k}$$

$$\sim \frac{1}{p^2}\left[-\eta^{\mu\nu} + \underbrace{\frac{(p+k)^\mu (p+k)^\nu}{(p+k)^2}}_{\text{ghost term}}\right] - \frac{1}{(p+k)^2}\left[-\eta^{\mu\nu} + \underbrace{\frac{p^\mu p^\nu}{p^2}}_{\text{ghost term}}\right]$$

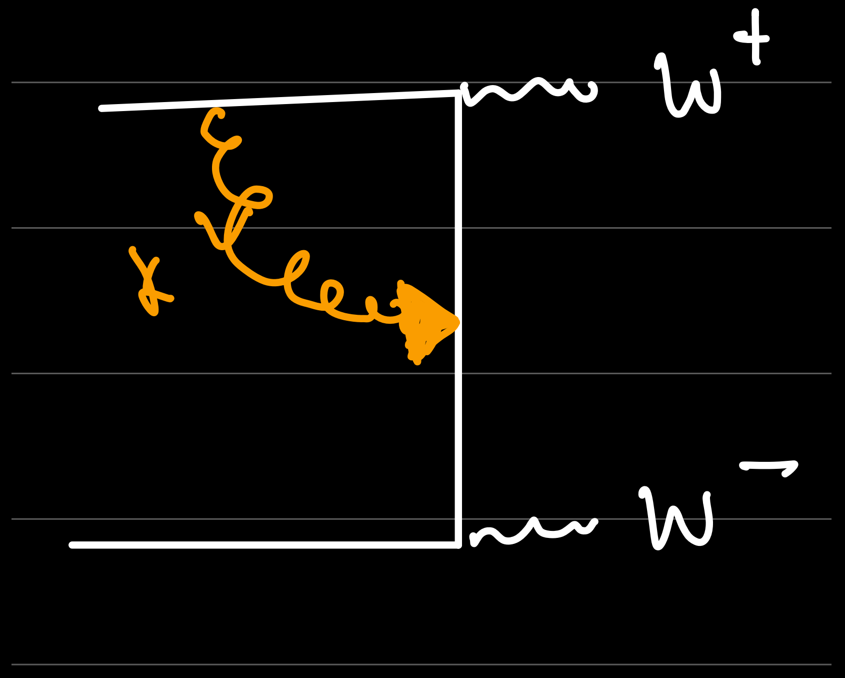
Gauge symmetry: Ward identities at play, lead to cancellations and factorisation

# Cancellation of collinear singularities at one-loop

$$\lim_{k\|p_1} \mathscr{F}^{(1)} =$$



$+$



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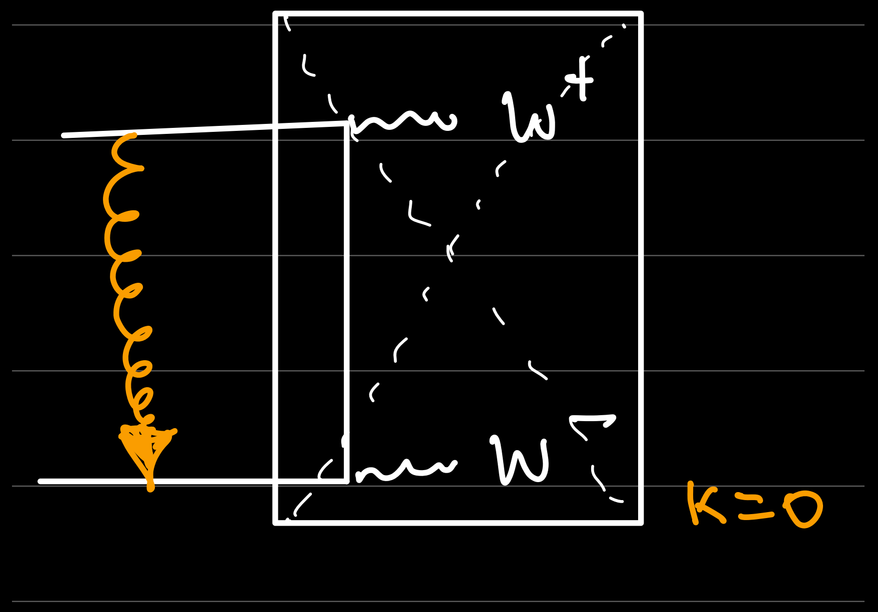

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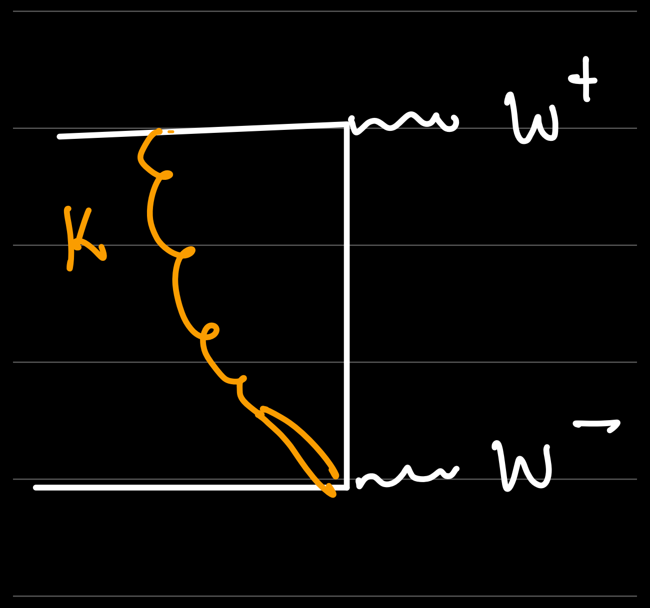

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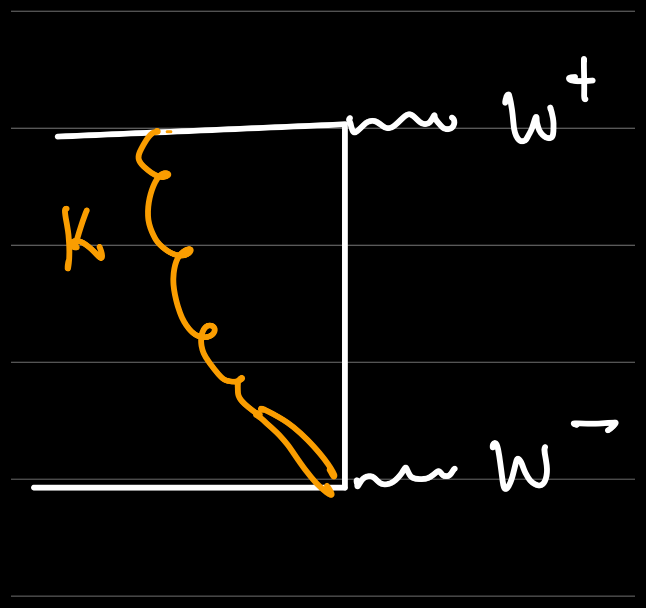

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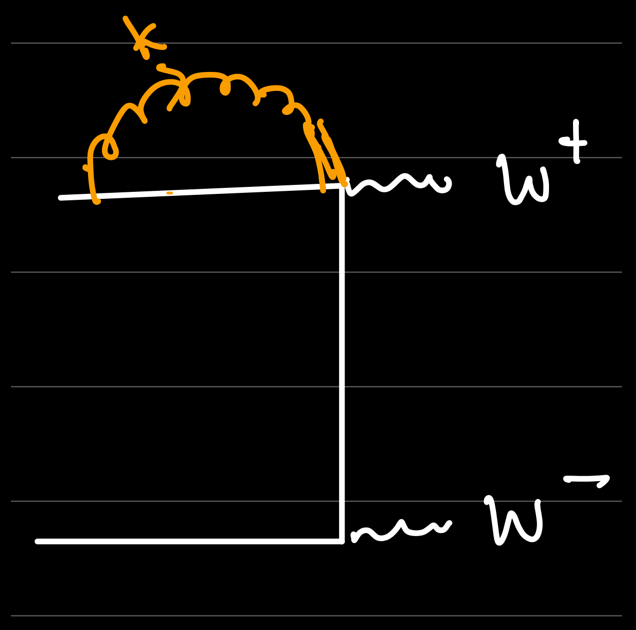


$+$

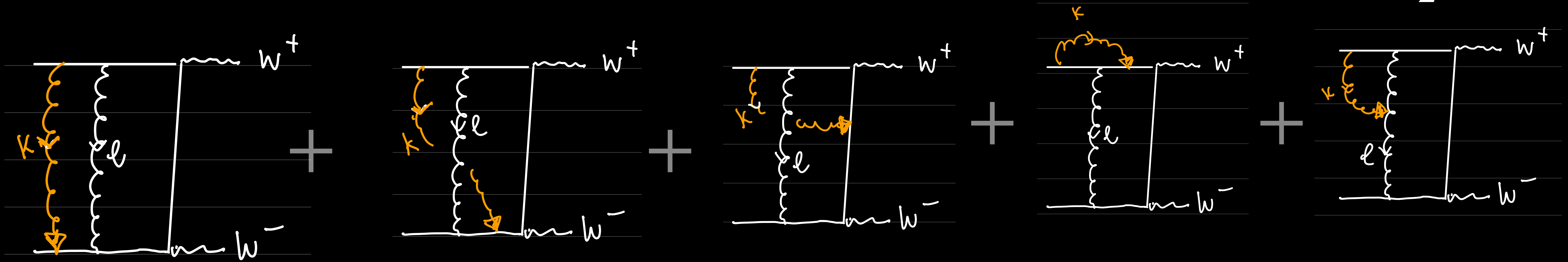
$$= 0.$$

# Cancellation of collinear singularities at two-loops

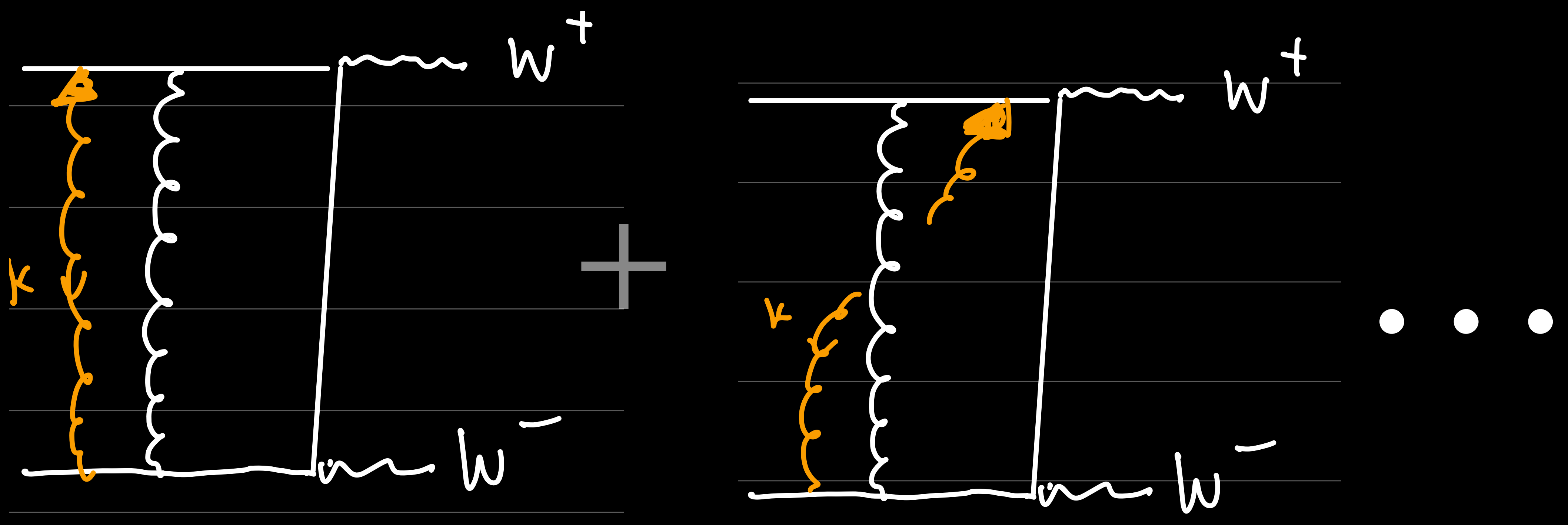
- Ward identities leave remnants which cancel with shifts of momenta
- Collinear gluons emerging from one-loop triangle subgraphs are not purely longitudinal. Their polarisation can be random (loop polarisation).
- Self-eergy corrections lead to power rather than logarithmic singularities.

# Loop momenta symmetrisation

Loop Momentum routing which is necessary for factorising the $k \parallel p_2$ singularity



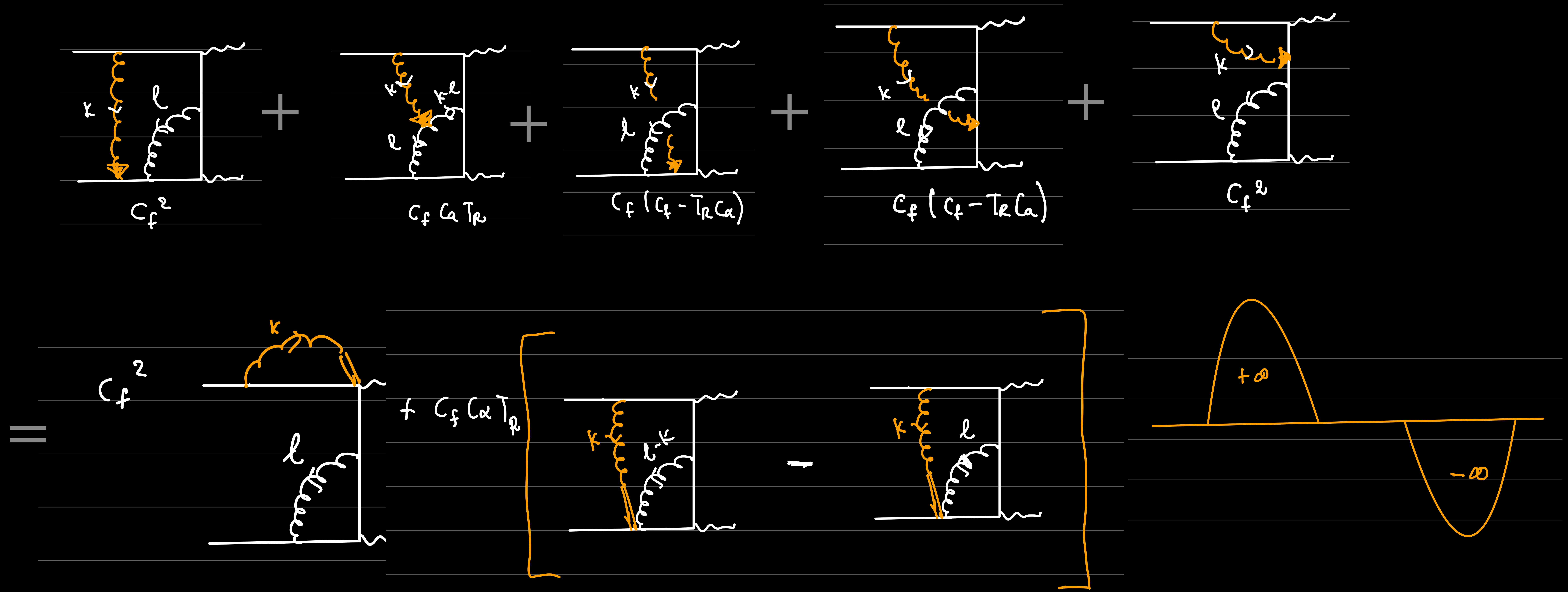
Incompatible for factorising the $k \parallel p_1$ singularity



SOLUTION: We symmetrise the two-loop amplitude in the momenta of the virtual gluons

$$\overline{\mathscr{A}^{(2)}}(k,l) = \frac{1}{2}\mathscr{A}^{(2)}(k,l) + \frac{1}{2}\mathscr{A}^{(2)}(l,k)$$

# Non-local cancellations requiring loop momenta shifts



# Localising shift cancellations with non-abelian planar copies

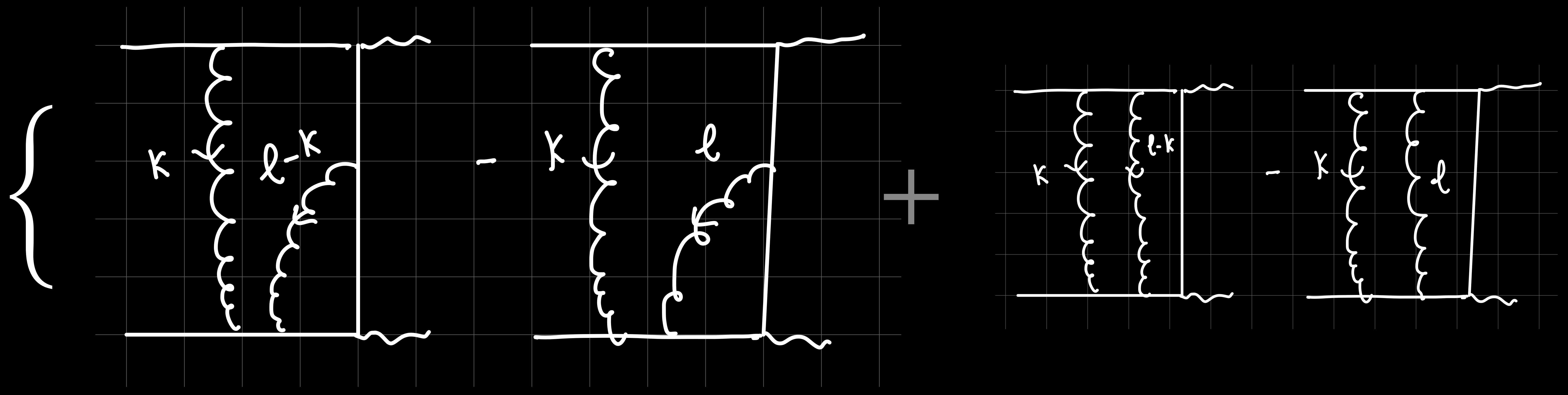
Add a suitably engineered “zero” to the amplitude

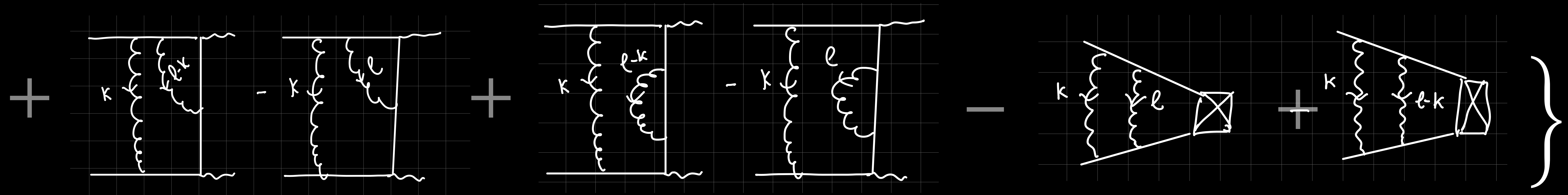
$$\mathscr{A}^{(2)} \to \mathscr{A}^{(2)} + f(k, l)$$

**with** $\int d^dk d^dl\, f(k, l)$

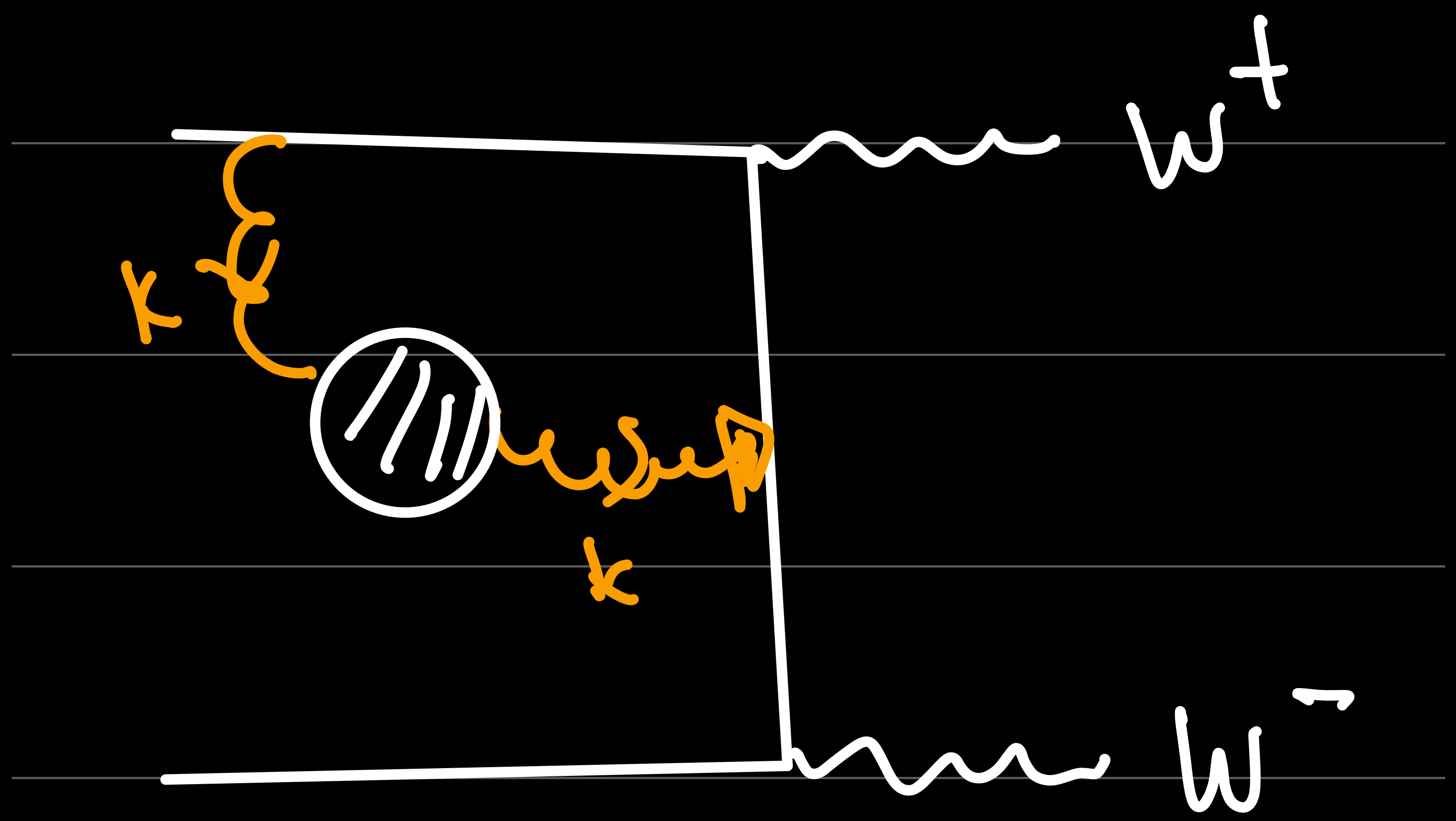
# Localising shift cancellations with non-abelian planar copies

**and** $f(k, l) = \dfrac{C_f C_a T_R}{C_f^2}$





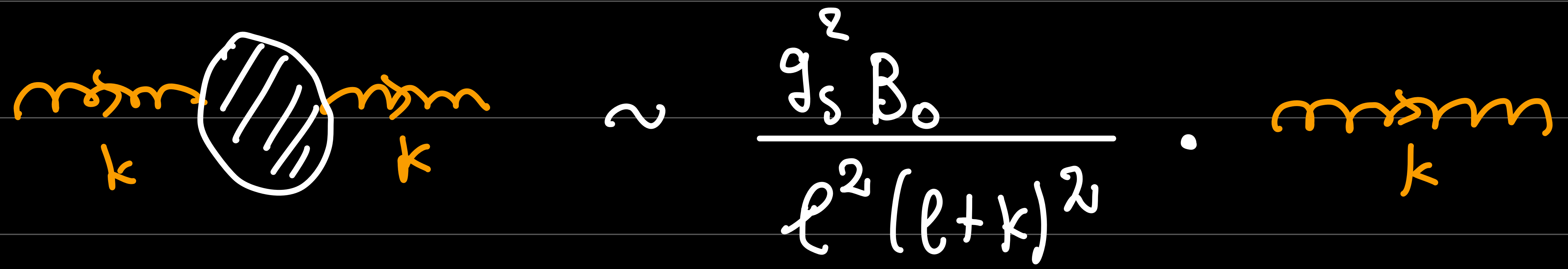
# Eliminating local power singularities



Vacuum polarization diagrams contain a "doubled" propagator. This leads to power singularities:

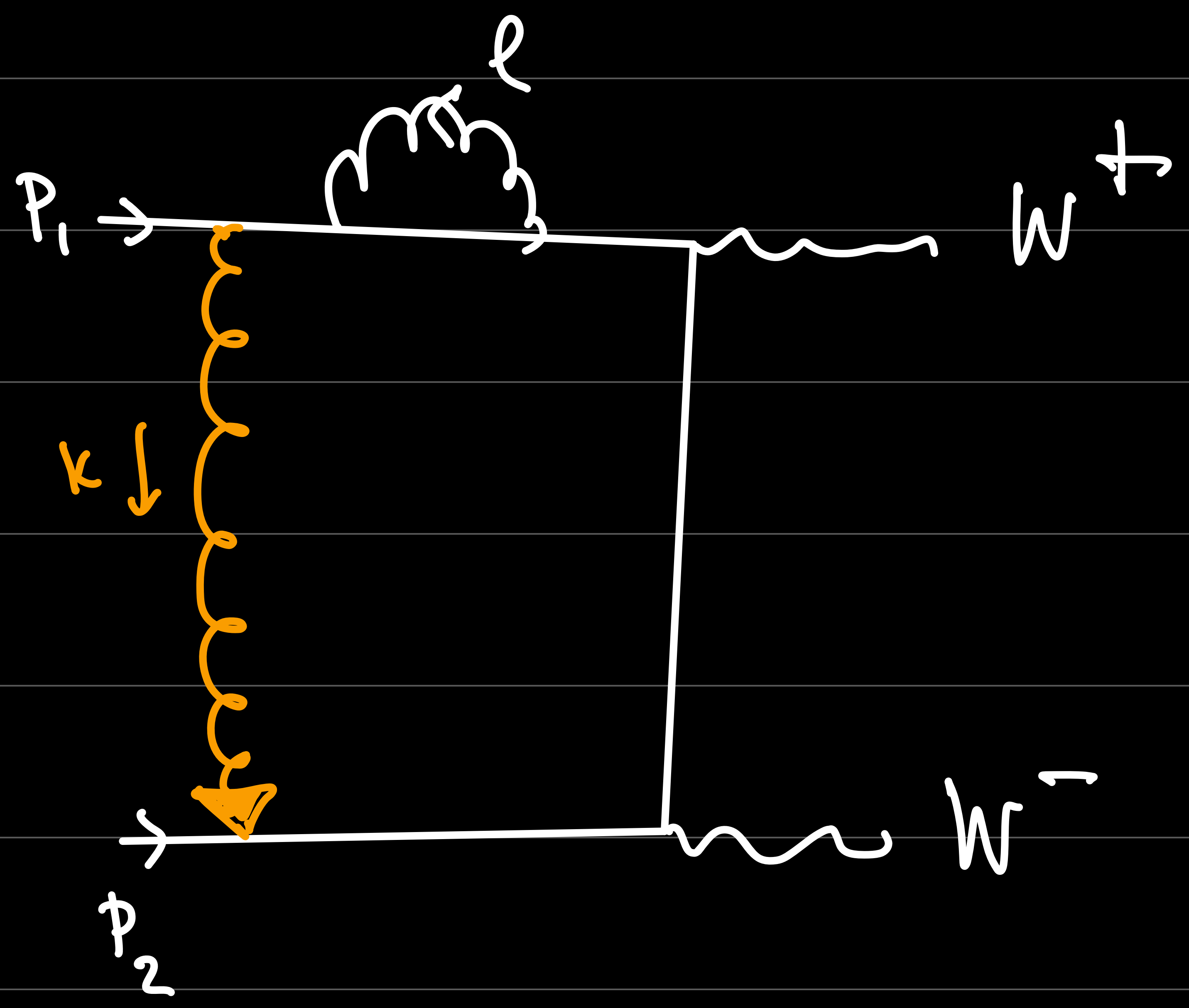
$$\frac{d^4k}{\left(k^2\right)^2(k+p_1)^2} \xrightarrow{k = x\,p_1 + \delta\,\beta\eta + \sqrt{\delta}k_\perp} \frac{1}{\delta}$$

We can replace vacuum polarisation integrands with equivalent (tensor reduced) integrands.



IR singlularity structure becomes identical to the one of the one-loop amplitude.

# Eliminating local power singularities



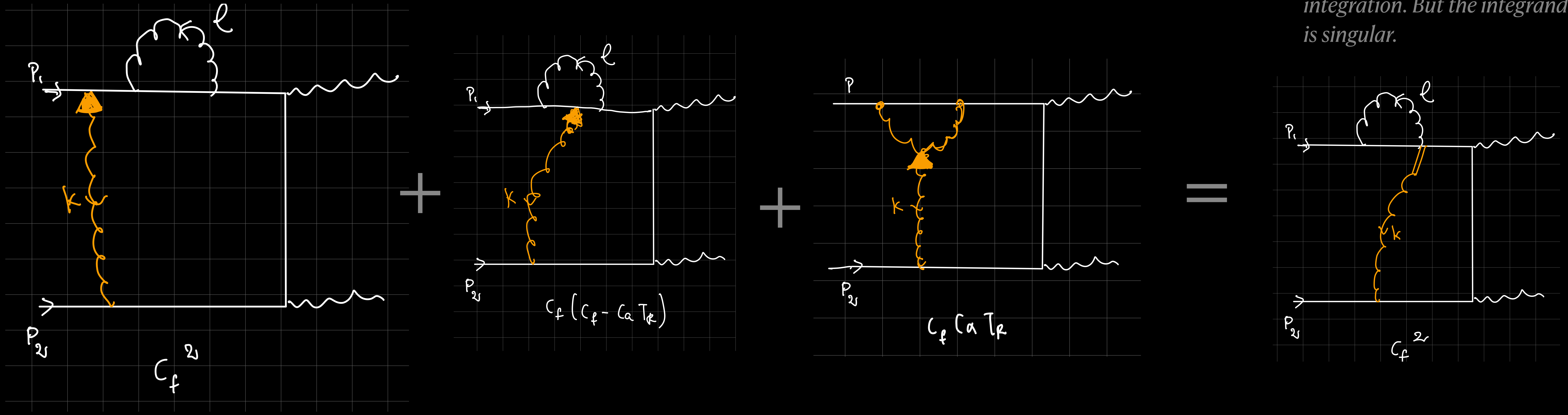
Self-energy corrections on quark lines adjacent to external legs exhibit power singularities too. We treat it in the same way replacing the integrand with an equivalent term (tensor reduction).

IR singlularity structure simplifies in the limit $k \parallel p_1$. But becomes more intricate in another limit, $k \parallel p_2$

# Have we spoiled factorisation in another way?

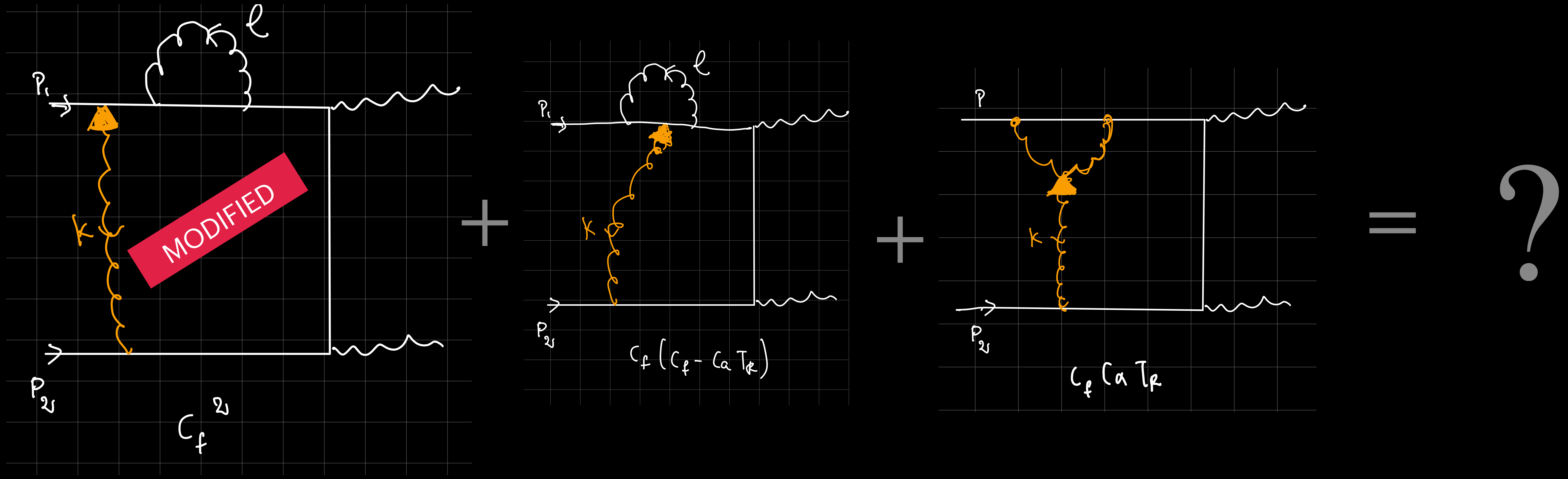
Factorization in this limit is not local. We have been already in trouble, even before changing the integrand of self-energies.

*Self-energy correction to external leg. Vanishes upon integration. But the integrand is singular.*



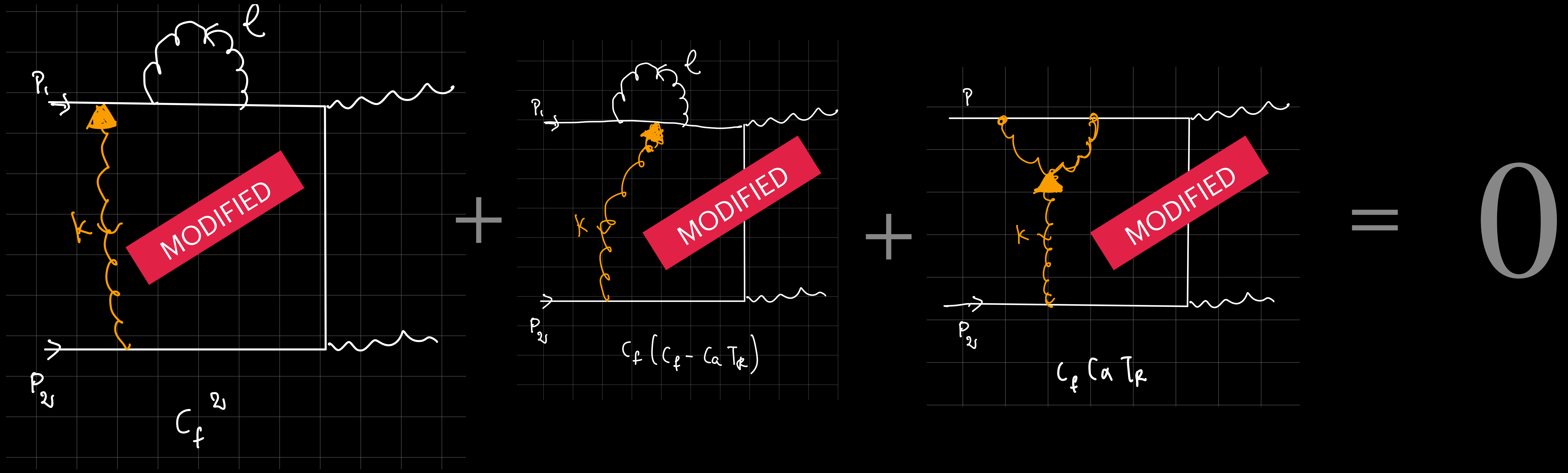
# Have we spoiled factorisation in another way?

Factorization in this limit is not local. We have been already in trouble, even before changing the integrand of self-energies.



# Have we spoiled factorisation in another way?

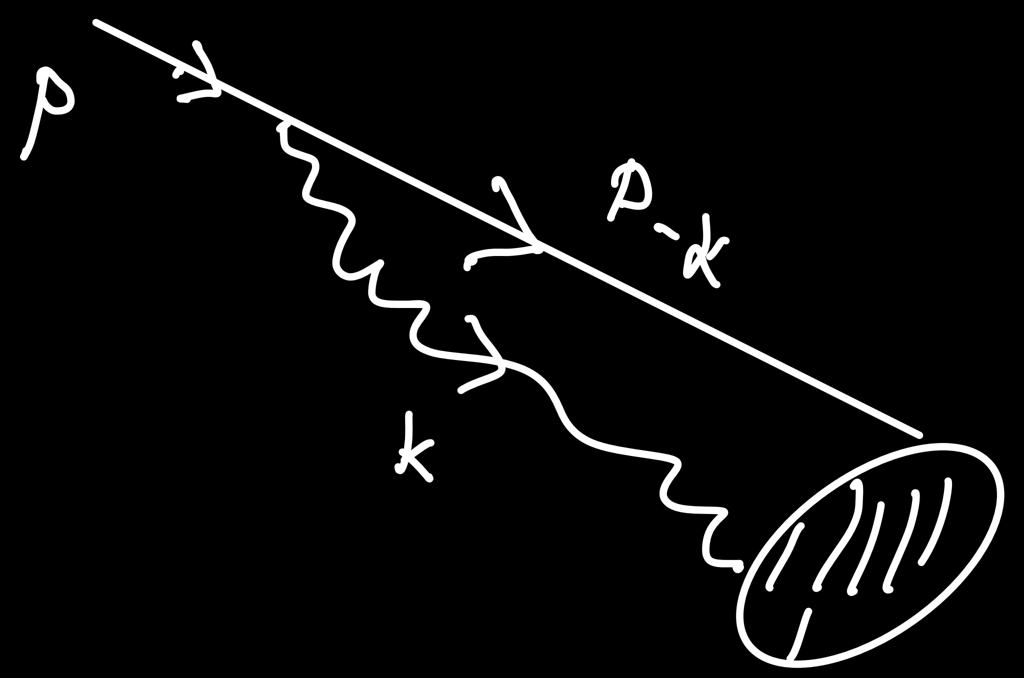
The diagrams with one-loop vertex corrections have their own problems too. We can solve all problems simultaneously if we modify the vertices too and impose a reflection symmetry on loop momenta  transverse to the incoming momenta.

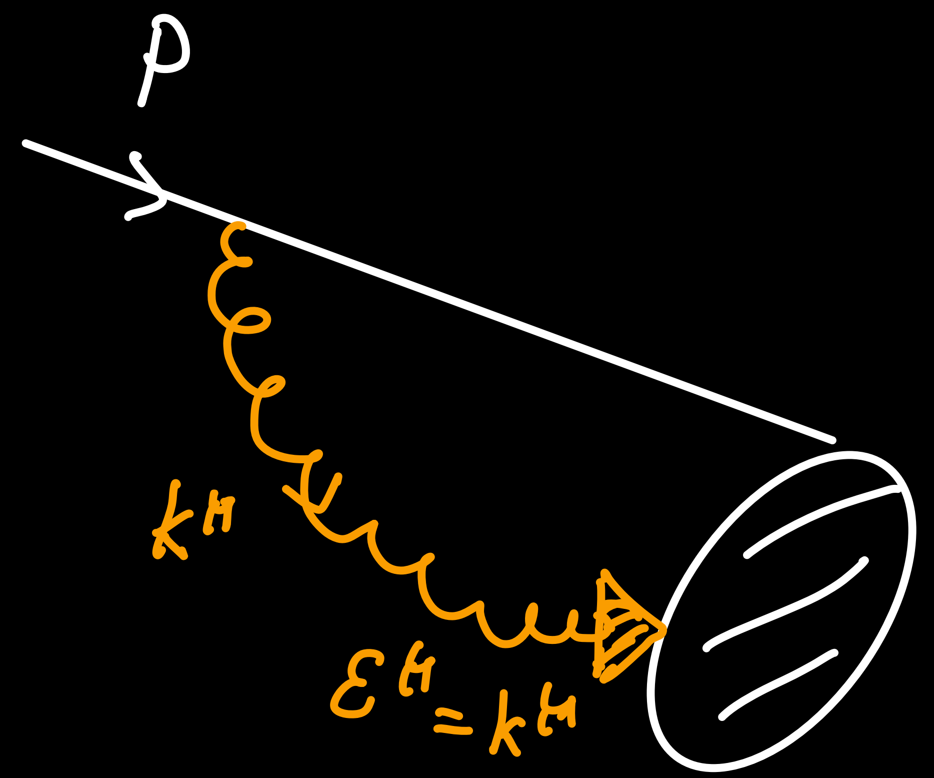


$$l_\perp \leftrightarrow -l_\perp$$

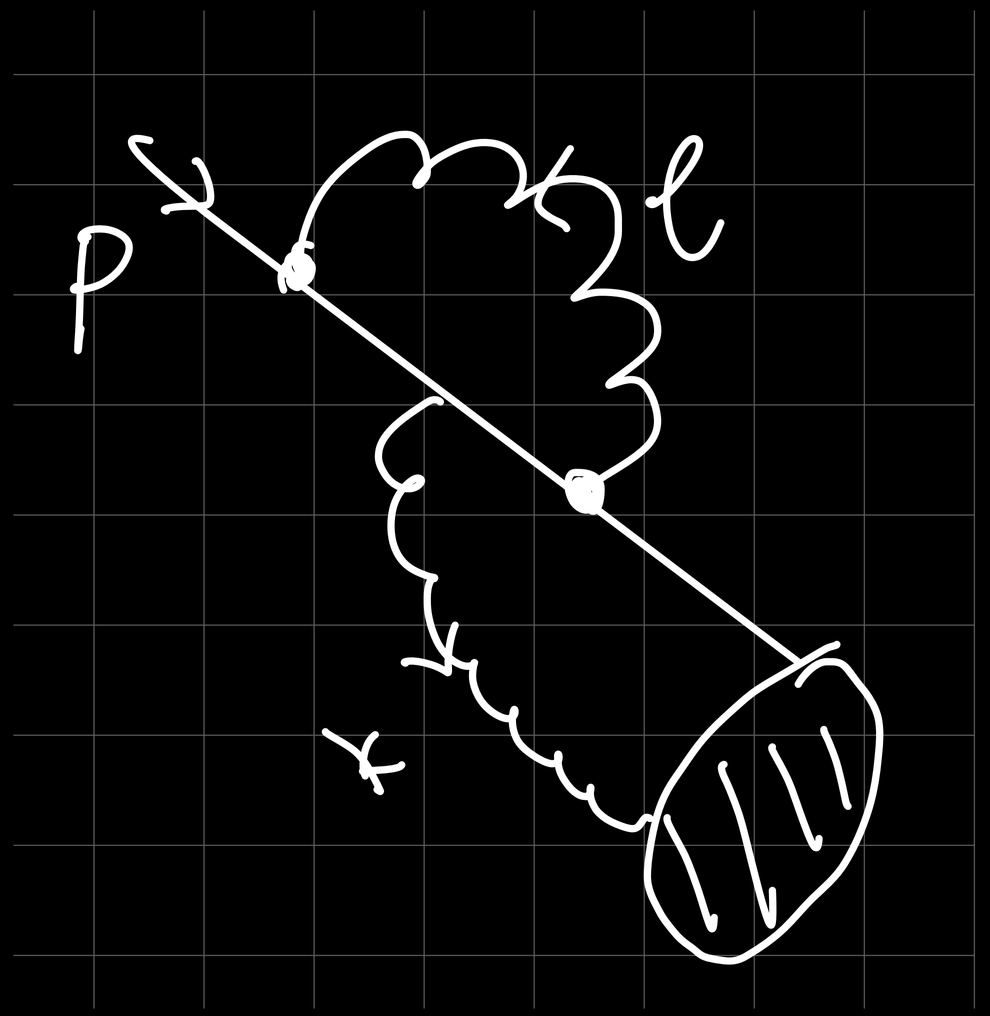
# Collinear singularities from gluons emitted off one-loop vertices

Collinear singularities → longitudinally and “**loop**” polarised gluons



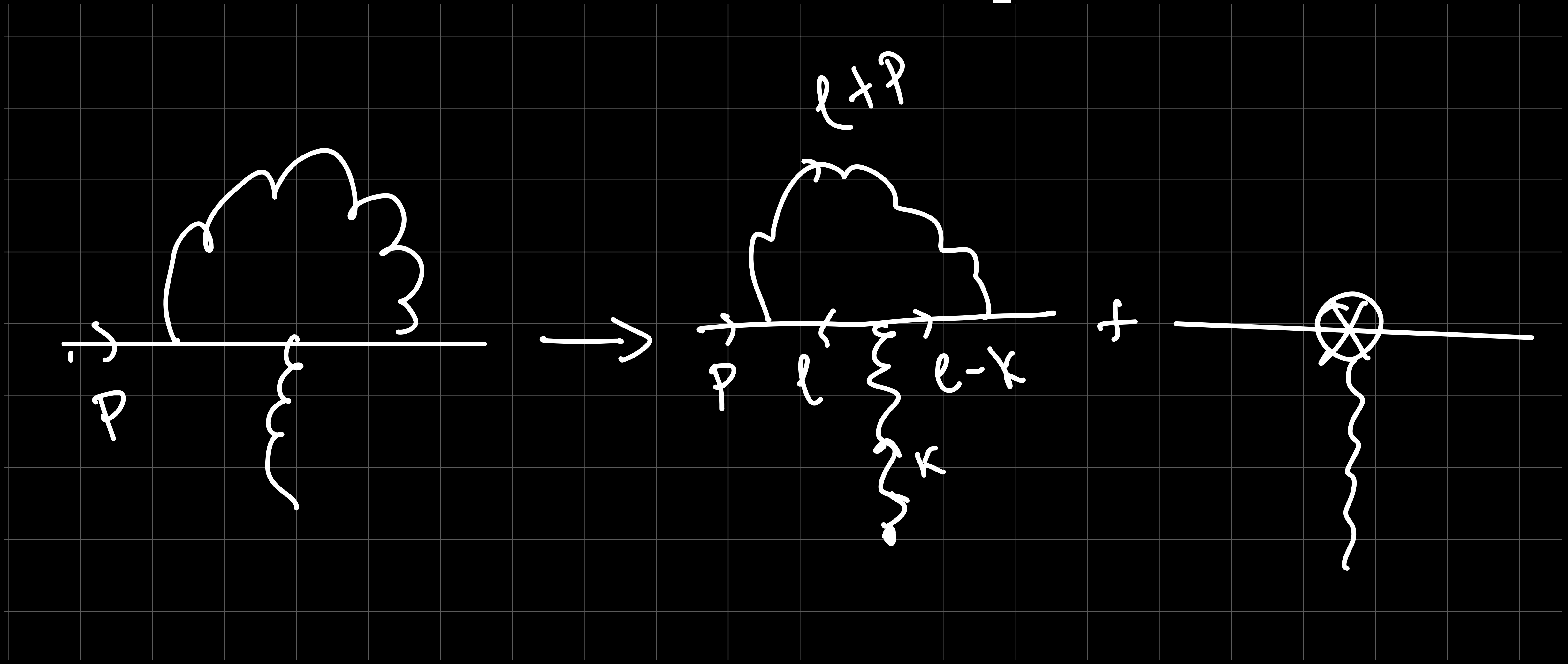


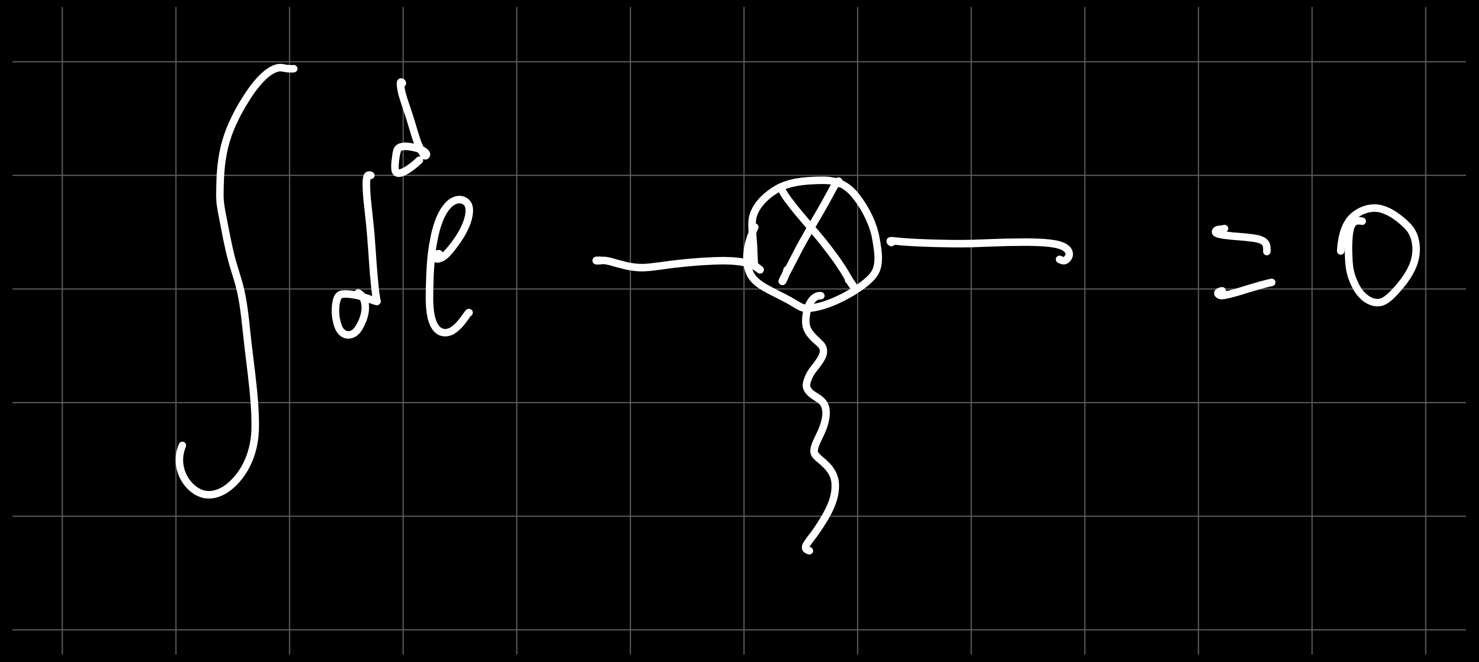
$$\ldots u(p)\frac{2(1-x)}{x}\frac{k^\mu}{k^2(k+p)^2}$$

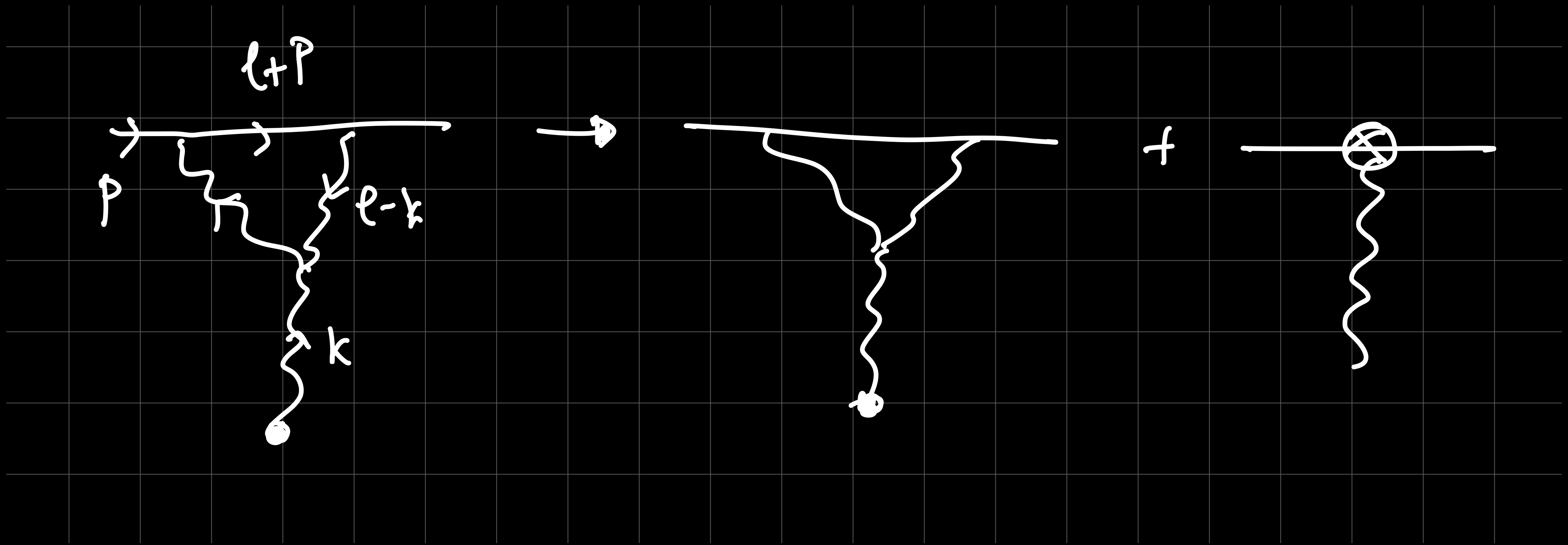


$$\frac{\ldots k^\mu + \ldots l^\mu}{k^2(k+p)^2 l^2(l+p)^2(k+l+p)^2}$$

# Eliminating Loop polarisations







$$2(1-\epsilon)\frac{\not{p}_2}{2p_1\cdot p_2}\frac{1}{(l-k)^2}\left[\frac{2l^\mu-k^\mu}{l^2}-\frac{2l^\mu-k^\mu+p_1^\mu}{(l+p_1)^2}\right]$$

# A main result of this work

$q + \bar{q} \rightarrow$ any set of heavy colourless particles

$$\mathscr{H}_{1-loop}(k) = \mathscr{A}_{1-loop} - \mathscr{F}^{(1)}\left[\mathscr{A}_0\right]$$

$$\mathscr{H}_{2-loop}(k,l) = \mathscr{A}_{2-loop} - \mathscr{F}^{(2)}\left[\mathscr{A}_0\right] - \mathscr{F}^{(1)}\left[\mathscr{H}_{1-loop}\right]$$

**Free of ALL soft and collinear singularities LOCALLY!**

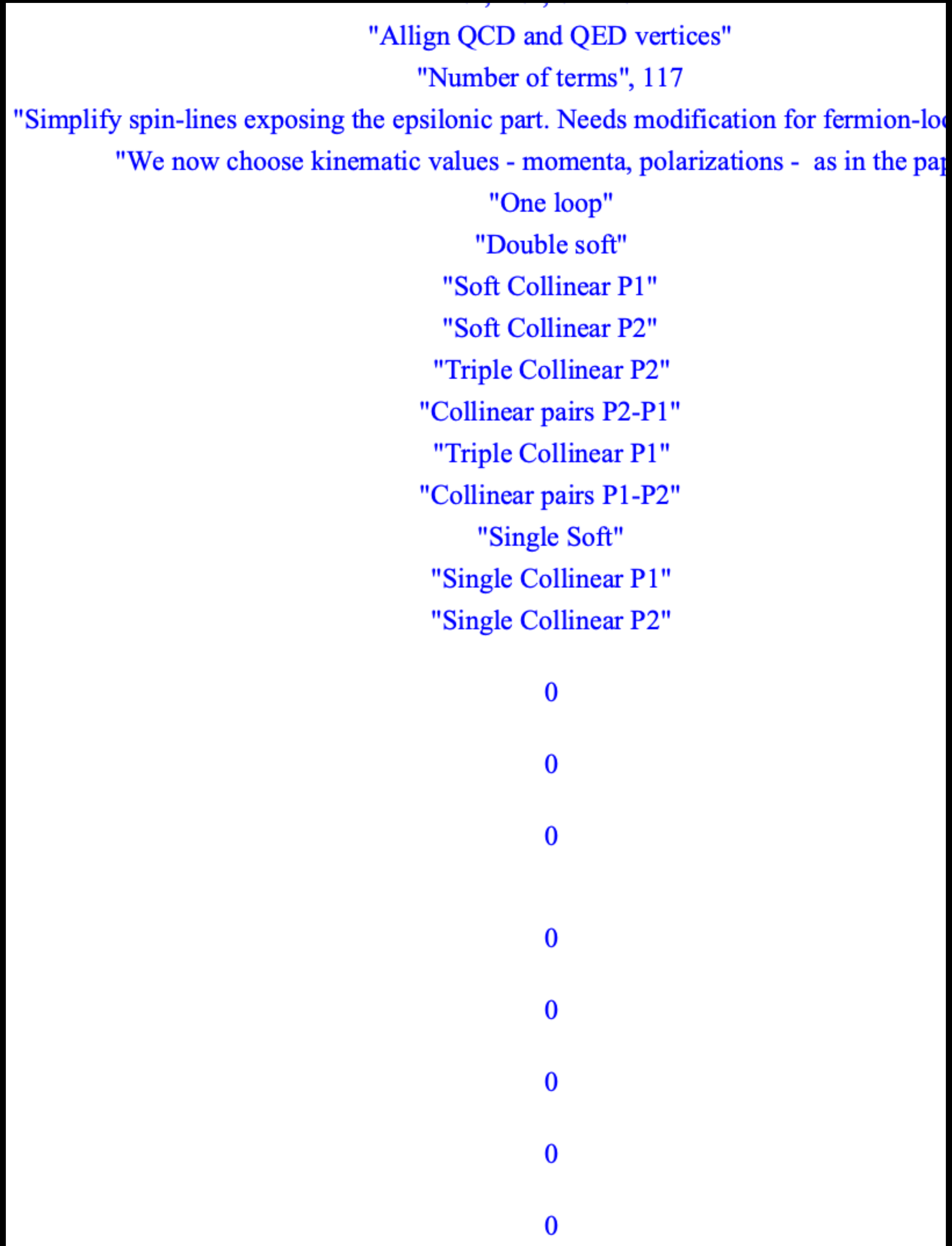
$$\int \mathscr{H}^{(R)}_{2-loop}(k,l)$$

Integrable in D=4 dimensions with numerical methods.

# Check

$$q + \bar{q} \to W^+W^-$$

- Generation of amplitudes with QGRAF
- Automated application of Feynman rules
- Automated cloning of planar diagrams, proper routing of momenta and introduction of counterterms for loop polarisations.
- Automated numerical evaluation of amplitude with exact arithmetics at all infrared singular limits.



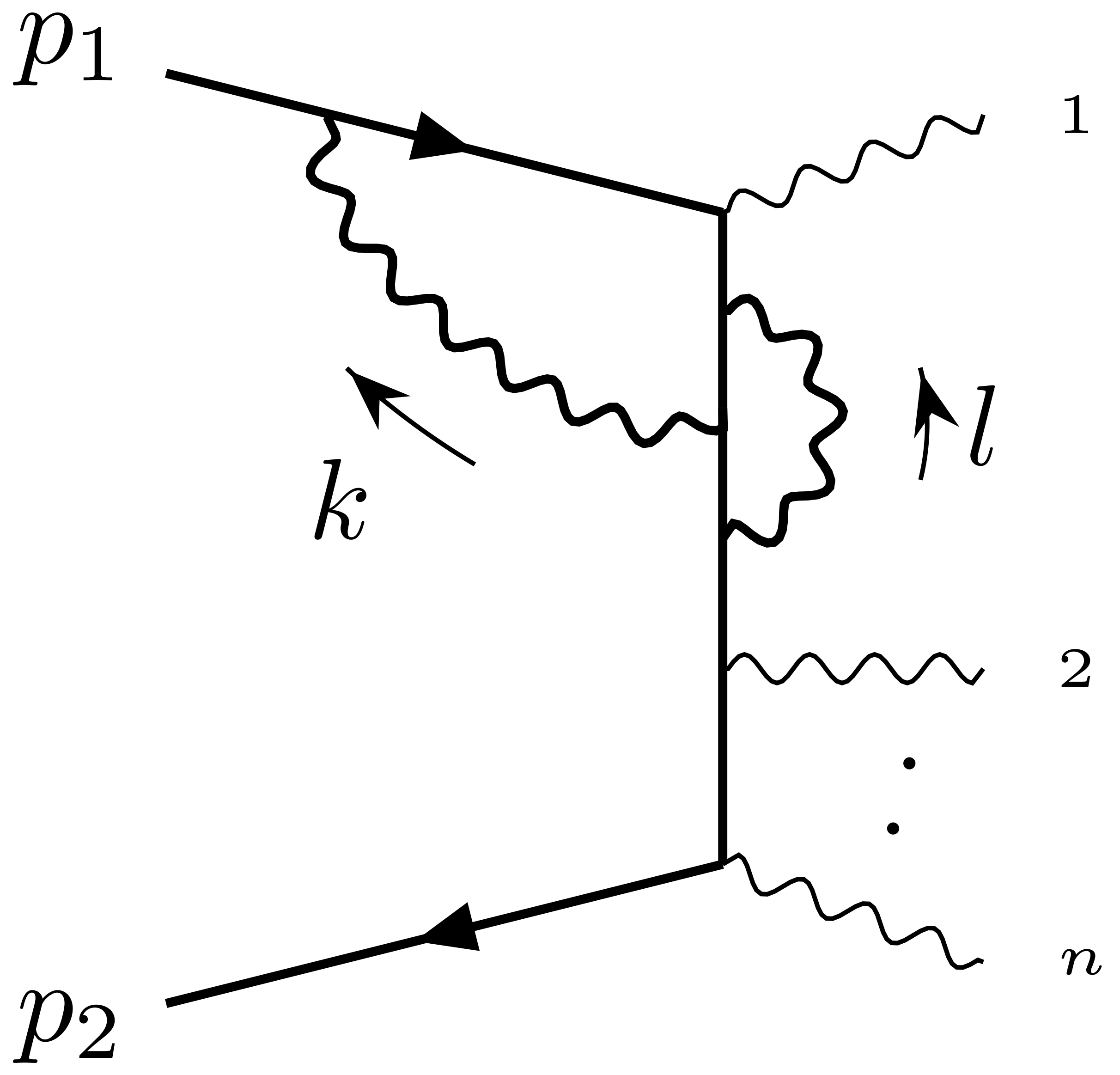
```
"Allign QCD and QED vertices"
"Number of terms", 117
"Simplify spin-lines exposing the epsilonic part. Needs modification for fermion-lo
"We now choose kinematic values - momenta, polarizations -  as in the pap
"One loop"
"Double soft"
"Soft Collinear P1"
"Soft Collinear P2"
"Triple Collinear P2"
"Collinear pairs P2-P1"
"Triple Collinear P1"
"Collinear pairs P1-P2"
"Single Soft"
"Single Collinear P1"
"Single Collinear P2"

0

0

0


0

0

0

0

0
```

# Ultraviolet Counterterms

- Ultraviolet divergences can be subtracted locally with a usual BPHZ-type of procedure.
- Constructing approximations of the integrand has a simple step, which is power counting
- And a conceptual complication beyond one-loop… mixed UV and IR singularities… we want to subtract UV singularities without spoiling IR factorization.

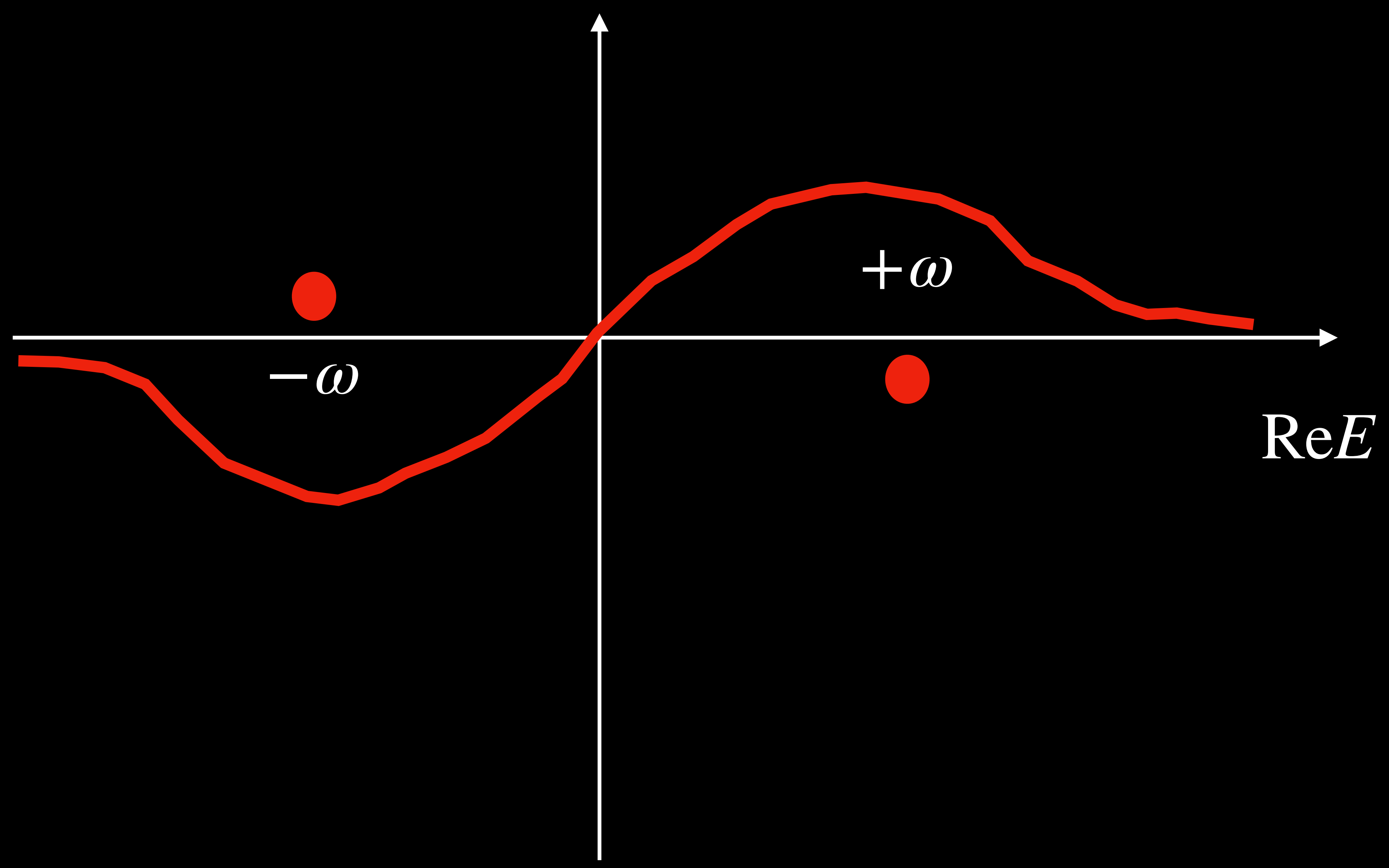
$$\mathscr{M}_1 = \mathscr{M}_{1-finite} + T_{l\to\infty}\mathscr{M}_1$$

$$\mathscr{M}_2 = \mathscr{M}_{2-finite} + T_{k\to\infty}\mathscr{M}_2 + T_{l\to\infty}\mathscr{M}_2$$
$$+T_{k,l\to\infty}\mathscr{M}_2 - T_{k,l\to\infty}\left(T_{k\to\infty}\mathscr{M}_2 + T_{l\to\infty}\mathscr{M}_2\right)$$



# Numerical integration

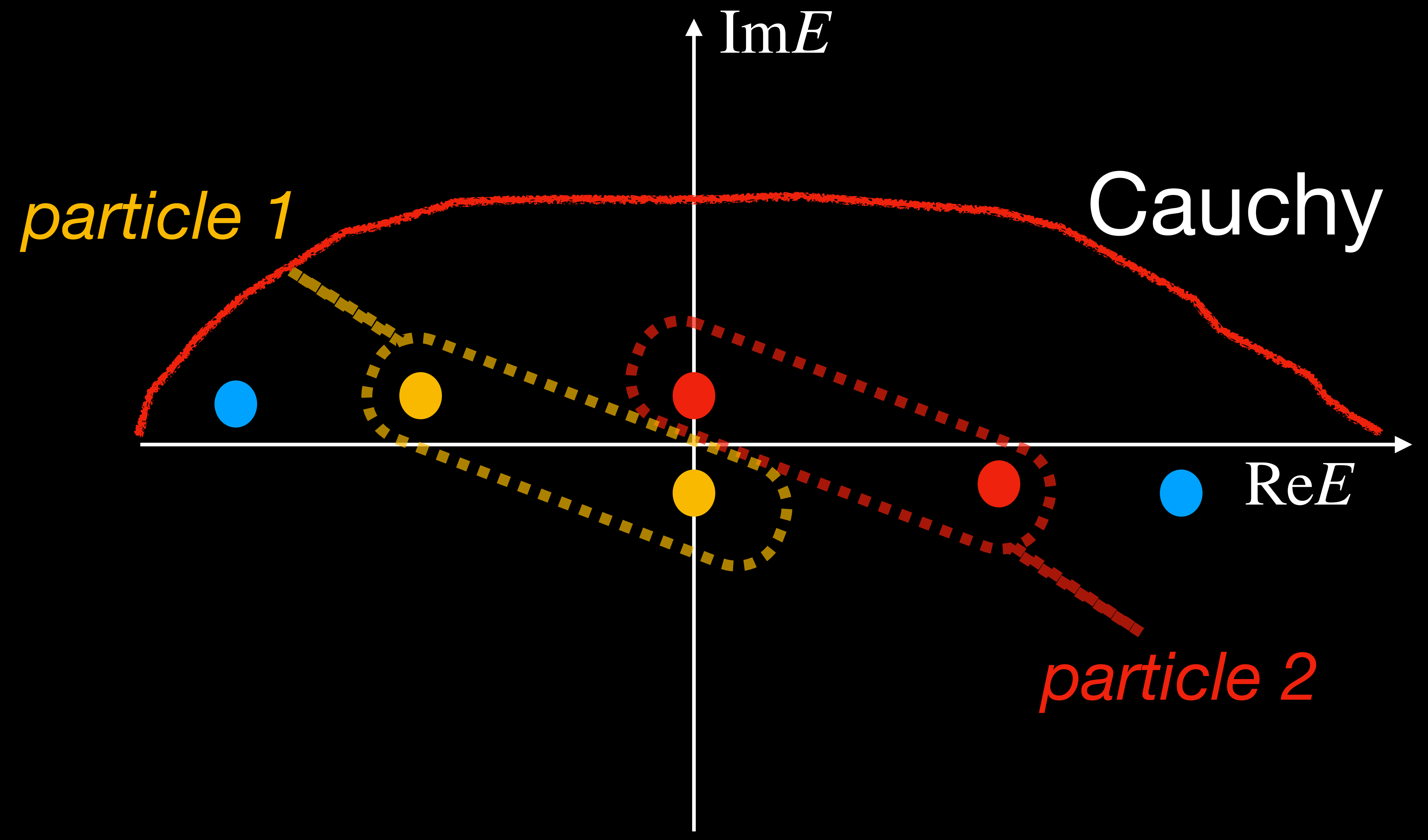
- Can such subtractions be used for evaluating loop amplitudes numerically?
- They are an important ingredient! They remove “pinch” singularities.
- Other singularities which can be avoided with appropriate contour-deformations are equally important.



# Numerical integration

- A breakthrough in numerical integration has been achieved recently
- First integrate over the energy component of all loop momenta using Cauchy’s theorem [Loop-Tree duality]
- This reduces the number of integrations.

Catani,Gleisberg,Krauss,Rodrigo, Winter; Bierenbaum, Catani, Draggiotis, Rodrigo; Capatti,Hirschi, Kermanschah, Ruijl; Aguilera-Verdugo, Driencourt-Mangin, Plenter, Ramırez-Uribe, Rodrigo, Sborlini,Torres Bobadilla, Tracz; Runkel, Szőr, Vesga, Weinzierl;…



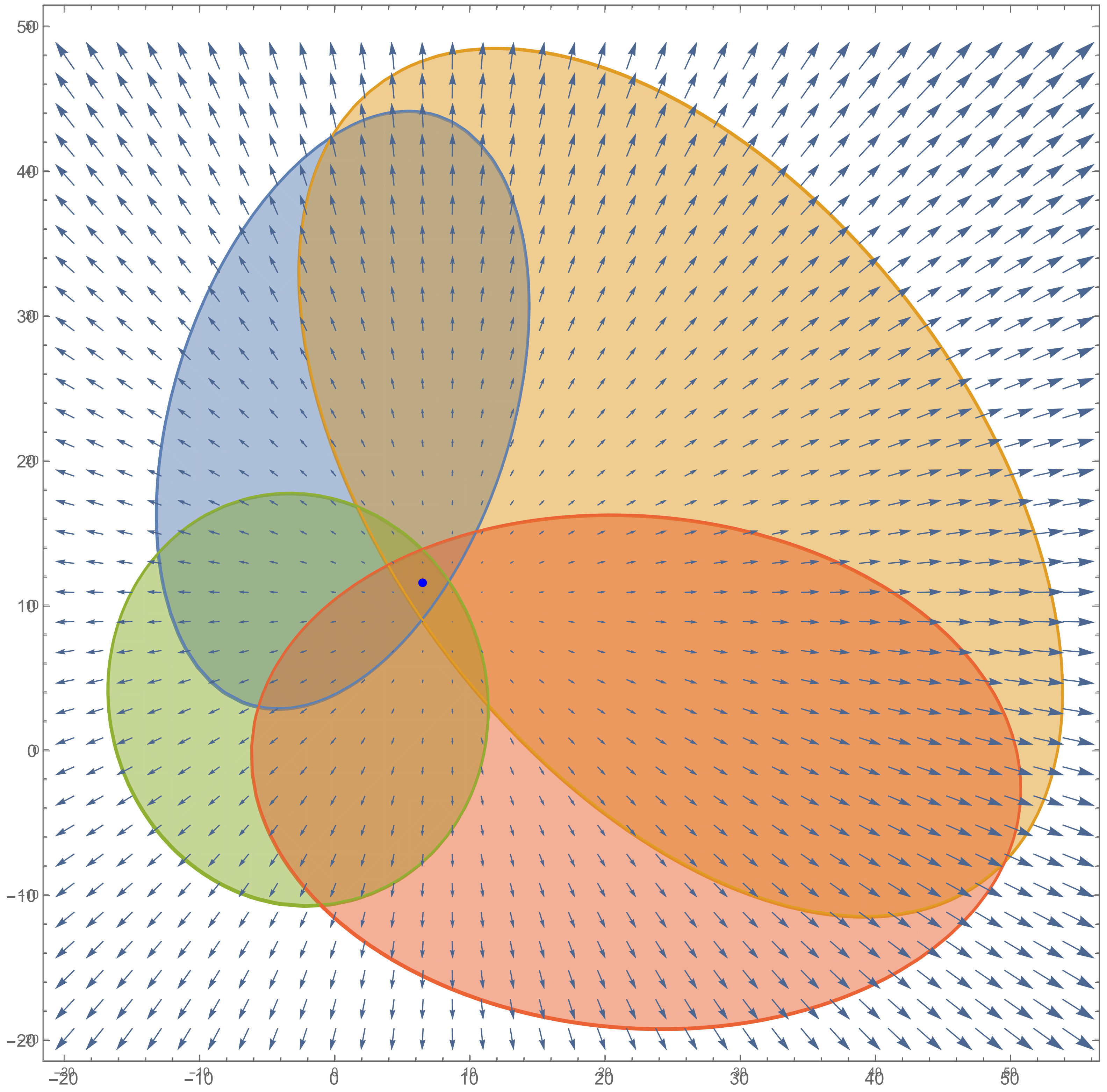
$$\int d^4k \to \int \frac{d^3\overrightarrow{k}}{|\overrightarrow{k}|} = \int d^4k\delta(k^2)\Theta(k^0)$$

*Capatti, Hirschi, Kermanschah, Ruijl*

# Numerical integration

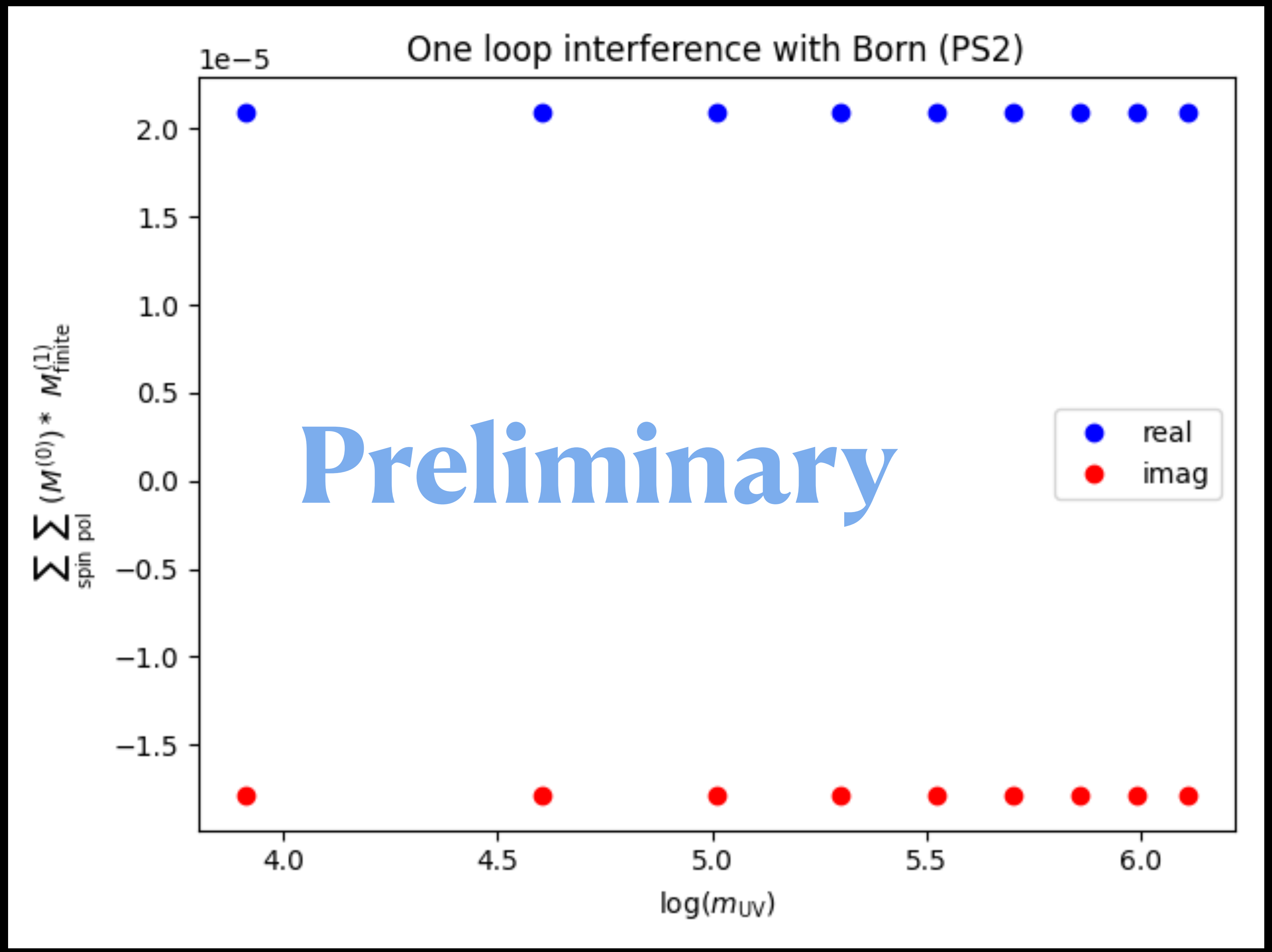
- A breakthrough in numerical integration has been achieved recently
- First integrate over the energy component of all loop momenta using Cauchy [Loop-Tree duality]
- This reduces the number of integrations.
- Then devise an algorithm to move the contour of remaining integrations away from non-pinched singularities.

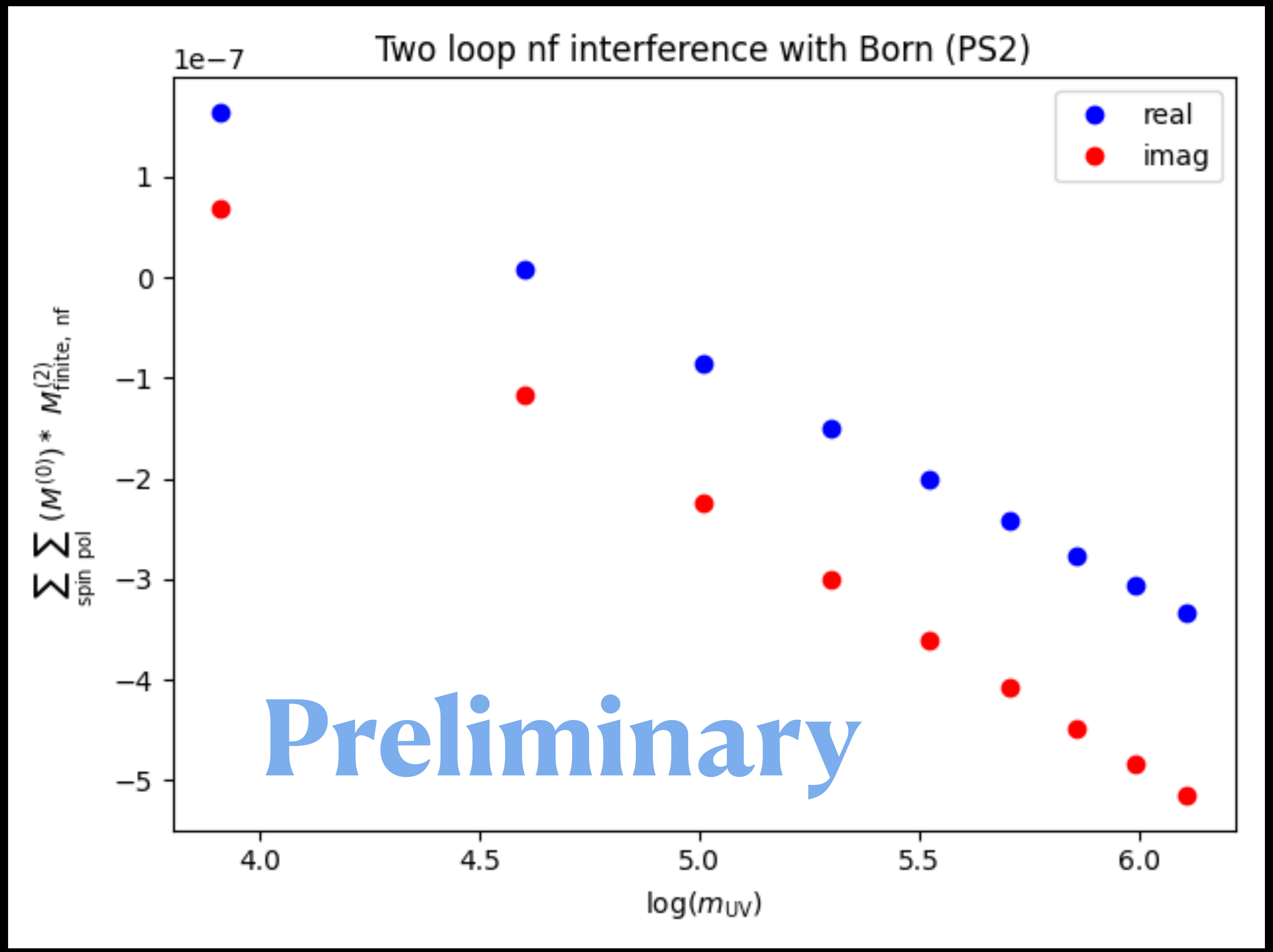
*Capatti, Hirschi, Kermanschah, Pelloni, Ruijl*



A **radial field** centered in the inside of all ellipsoids!

# Integrated Hard Functions for $q\bar{q} \to WWZ$: the $\beta_0 -$terms





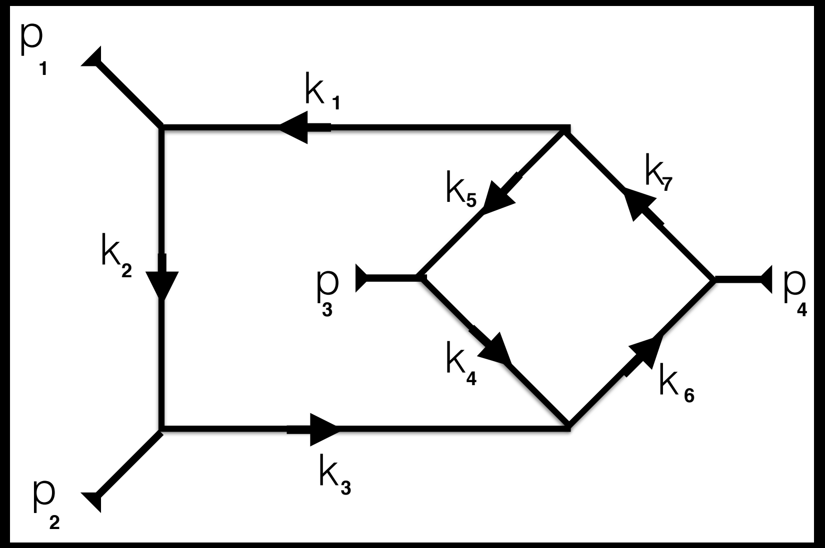
**Dario Kermanschah, Zeno Capatti**
**interfacing/adapting the framework of "Local Unitarity"**

*Capatti, Hirschi, Kermanschah, Pelloni, Ruijl*

# Local subtractions may be complementary to other approaches

- Analytic integration
- Feynman parameter integration (removing the need for sector decomposition)
- Helping to set D=4 exactly in reductions to master integrals
- Extracting logarithms (small fermion masses)

# Example: two-loop cross-box



**two-loop limits** **single soft** **single collinear**

$$F_{Xbox} = F^{(2)}_{Xbox} + F^{(1s)}_{Xbox} + F^{(1c)}_{Xbox},$$

***double -soft*** ***double -collinear***

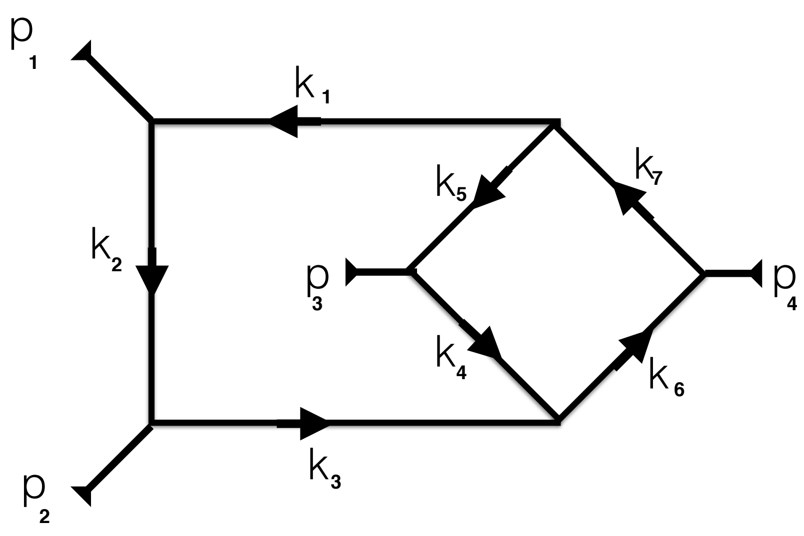
$$F^{(2)}_{Xbox} = \frac{N_5}{A_1A_2A_3A_4A_5A_6A_7},$$

$$\begin{aligned}N_5 = &\left(1-\frac{A_{13}}{s}\right)^2 + \frac{A_2}{tu}\left(A_2+s-A_{13}\right)\\ &-\left(1-\frac{A_1}{s}\right)\left(\frac{A_5}{t}+\frac{A_7}{u}\right)-\left(1-\frac{A_3}{s}\right)\left(\frac{A_4}{u}+\frac{A_6}{t}\right)+\frac{A_2A_{4567}}{tu}\\ &-\frac{A_3}{s}\left(\frac{A_7}{t}+\frac{A_5}{u}\right)-\frac{A_1}{s}\left(\frac{A_6}{u}+\frac{A_4}{t}\right)+\frac{(t-u)^2}{s^2}\frac{A_1A_3}{tu}\end{aligned}$$

$$\begin{aligned}F^{(1c)}_{Xbox} = &-\left[\frac{1}{A_1A_2}-\frac{1}{B_1B_2}\right]\frac{1}{s(1-x_1)}\left\{\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_1=-x_1p_1}-\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_2=0}\right\}\\ &-\left[\frac{1}{A_2A_3}-\frac{1}{B_2B_3}\right]\frac{1}{s(1-x_3)}\left\{\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_3=-x_2p_2}-\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_2=0}\right\}\\ &-\left[\frac{1}{A_4A_5}-\frac{1}{B_4B_5}\right]\left[\frac{N_5}{A_1A_2A_3A_6A_7}\right]_{k_5=-x_3p_3}\\ &-\left[\frac{1}{A_6A_7}-\frac{1}{B_6B_7}\right]\left[\frac{N_5}{A_1A_2A_3A_4A_5}\right]_{k_5=-x_4p_4}.\end{aligned}$$

$$F^{(1s)}_{Xbox} = -\frac{1}{A_1A_2A_3}\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_2=0}$$

# Example: two-loop cross-box



$$\mathrm{X_{box}}^{\mathrm{fin}} \equiv \int \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{d^d k_5}{i\pi^{\frac{d}{2}}} F_{Xbox} = \mathcal{O}(\epsilon^0).$$

$$s^3 \mathrm{X_{box}}^{\mathrm{fin}} = \frac{f_{\mathrm{X_{box}}}(y)}{y} + \frac{f_{\mathrm{X_{box}}}(1-y)}{1-y},$$

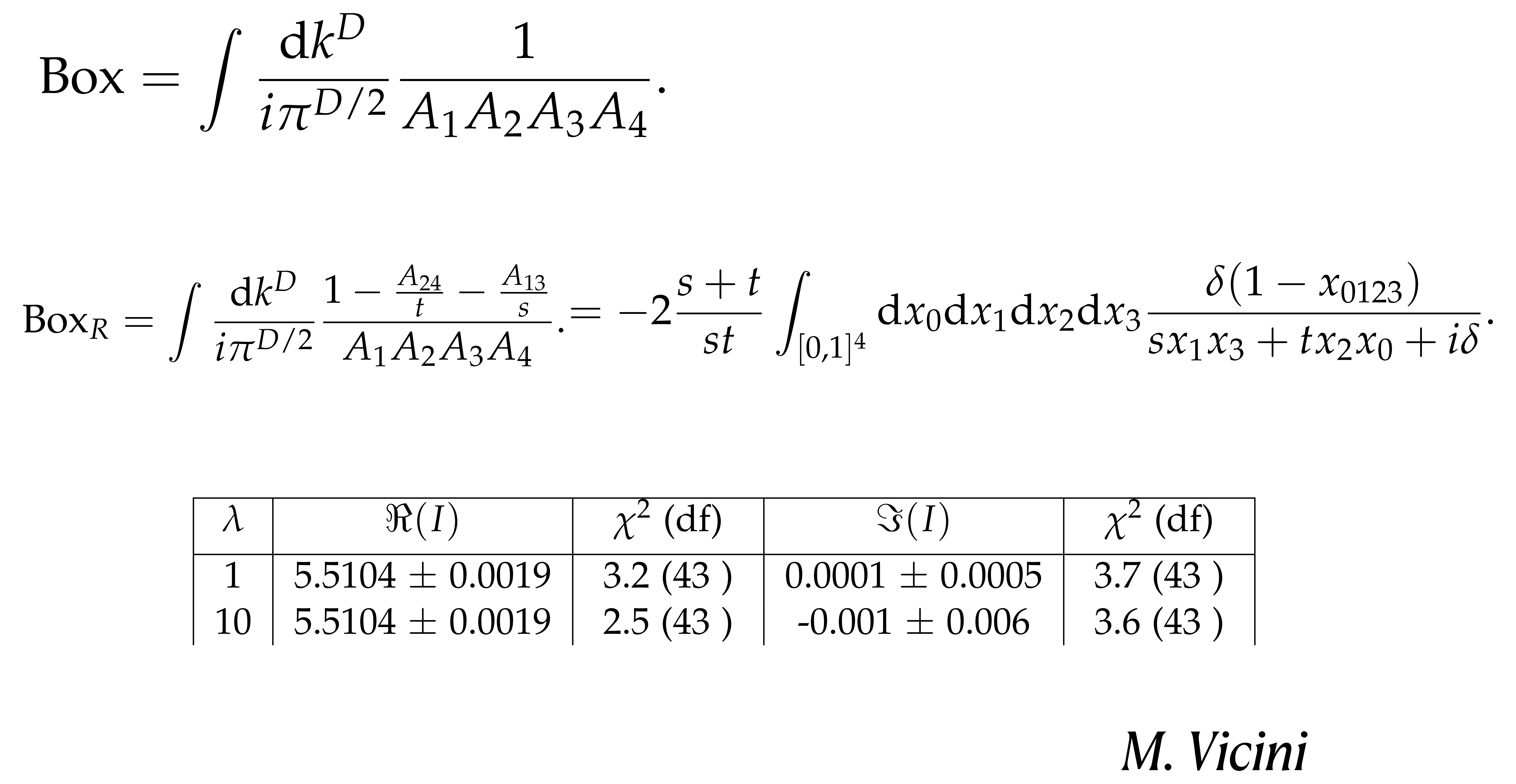
$$f_{\mathrm{X_{box}}}(y) = \left[G_R(y) + i\pi G_I(y)\right] \log\left(\frac{\mu^2}{s}\right) + E_R(y) + i\pi E_I(y)$$

$$\begin{aligned} E_R(y) = \; & -8\,\pi^2\,\mathrm{Li}_2(y) + 8\,\mathrm{Li}_2(y)\,\log(1-y)^2 - 28\,\log(y)\,\mathrm{Li}_2(y)\,\log(1-y) - 18\,\mathrm{Li}_2(y)\,\log(y)^2 \\ & +44\,\mathrm{Li}_3(y)\,\log(1-y) + 96\,\mathrm{Li}_3(y)\,\log(y) - 188\,\mathrm{Li}_4(y) + \frac{17}{36}\,\pi^4 + \frac{1}{12}\,\log(1-y)^4 \\ & +7\,\log(y)\,\log(1-y)\,\pi^2 - \frac{25}{6}\,\pi^2\,\log(1-y)^2 - \frac{3}{2}\,\log(y)^2\,\pi^2 + \log(y)\,\log(1-y)^3 \\ & +44\,S_{12}(y)\,\log(1-y) - 52\,S_{12}(y)\,\log(y) + 84\,S_{13}(y) + 88\,S_{22}(y) - 44\,\zeta_3\,\log(1-y) \\ & -4\,\log(y)\,\zeta_3 - \frac{1}{4}\,\log(y)^4 + \log(y)^3\,\log(1-y) - \frac{9}{2}\,\log(y)^2\,\log(1-y)^2, \end{aligned}$$

...

Maximum Transendentality

# Feynman parameterisation



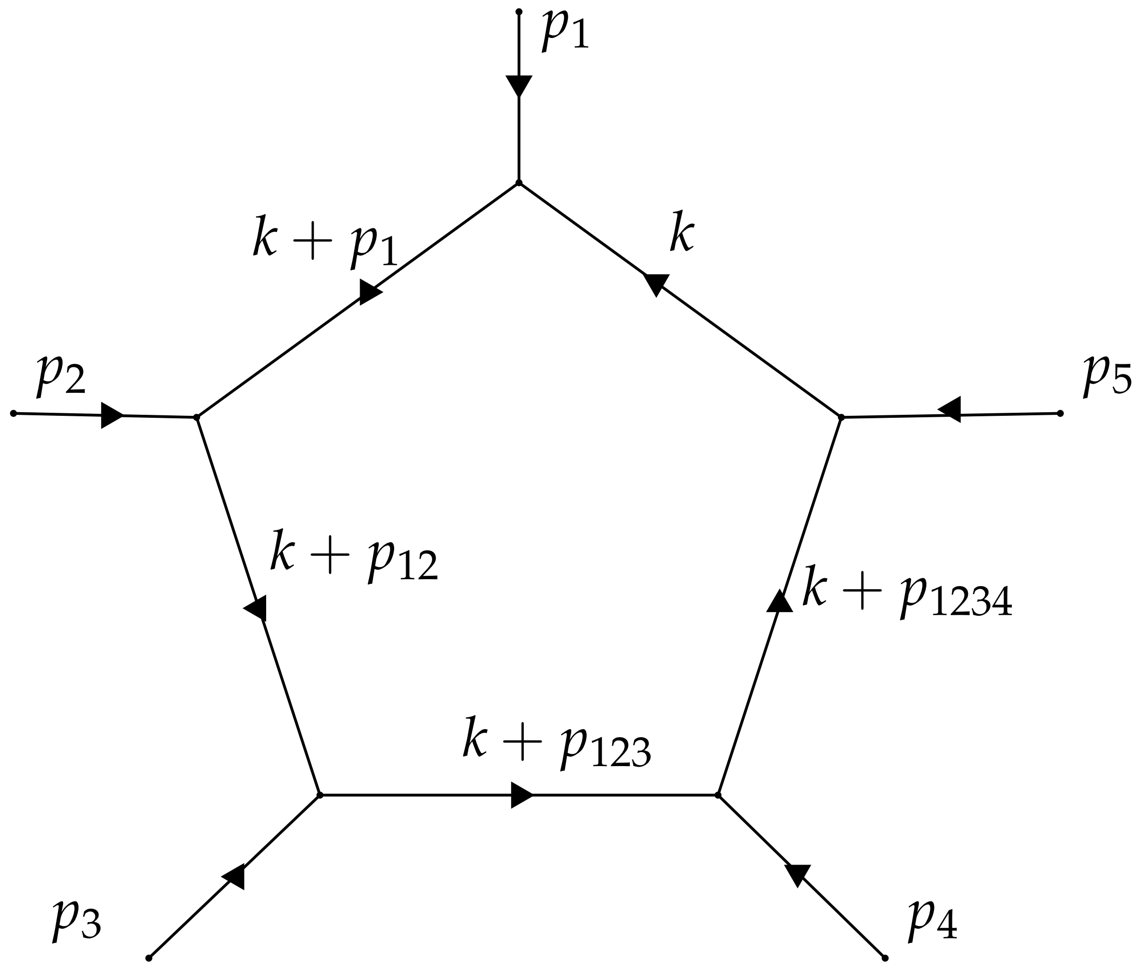
$$\text{Box} = \int \frac{\mathrm{d}k^D}{i\pi^{D/2}} \frac{1}{A_1 A_2 A_3 A_4}.$$

$$\text{Box}_R = \int \frac{\mathrm{d}k^D}{i\pi^{D/2}} \frac{1 - \frac{A_{24}}{t} - \frac{A_{13}}{s}}{A_1 A_2 A_3 A_4}. = -2\frac{s+t}{st} \int_{[0,1]^4} \mathrm{d}x_0 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \frac{\delta(1 - x_{0123})}{s x_1 x_3 + t x_2 x_0 + i\delta}.$$

| $\lambda$ | $\Re(I)$ | $\chi^2$ (df) | $\Im(I)$ | $\chi^2$ (df) |
|---|---|---|---|---|
| 1 | 5.5104 ± 0.0019 | 3.2 (43 ) | 0.0001 ± 0.0005 | 3.7 (43 ) |
| 10 | 5.5104 ± 0.0019 | 2.5 (43 ) | -0.001 ± 0.006 | 3.6 (43 ) |

*M. Vicini*

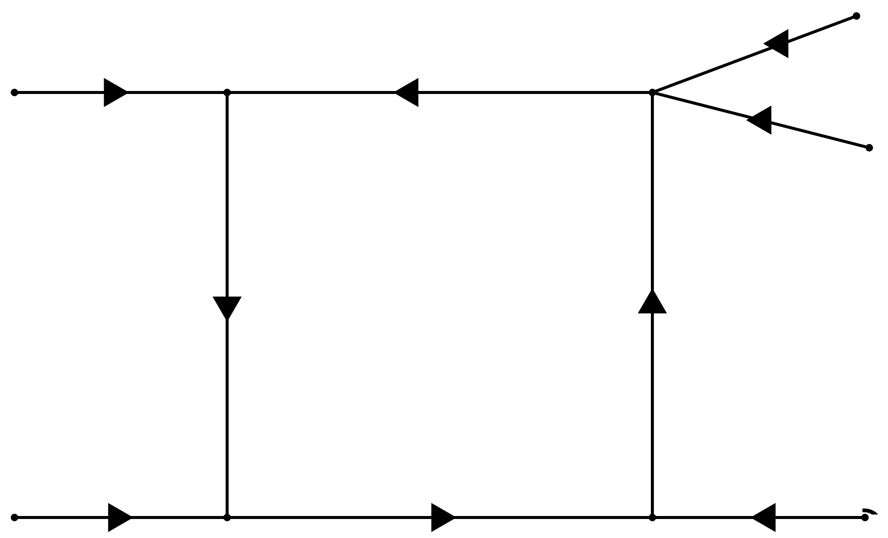
# Feynman parameterisation
# N=4 5-point one-loop amplitude



$$\mathcal{M}_5^{(\text{1-loop})}\big|_{\text{IR-finite}} = \int \frac{\mathrm{d}k^D}{i\pi^{D/2}} \frac{\mathcal{N}(k)}{A_1A_2A_3A_4A_5}$$

$$\begin{aligned}\mathcal{N}(k) = -[&(k+p_{1234})^2b_5 + (k+p_{123})^2b_4 + (k+p_{123})^2(k+p_{1234})^2a_5\\ &+(k)^2(k+p_{1234})^2a_1 + (k+p_{12})^2(k+p_{123})^2a_4 + (k+p_{12})^2b_3+\\ &(k+p_1)^2b_2 + (k)^2(k+p_1)^2a_2 + (k+p_1)^2(k+p_{12})^2a_3 + (k)^2b_1] + o\end{aligned}$$

$$b_1 := s_{23}s_{34}\left(\frac{s_{23}s_{34} + s_{12}(-s_{23}+s_{51}) + s_{45}(-s_{34}+s_{51})}{R} - \frac{1}{2}\right)$$



| $\lambda$ | $\Re(I)$ | $\chi^2$ (df) | $\Im(I)$ | $\chi^2$ (df) |
|---|---|---|---|---|
| 0.5 | 2.693 ± 0.002 | 0.4 (43) | -3.141 ± 0.002 | 0.9 (43) |
| 1.0 | 2.6931 ± 0.0011 | 0.4 (43) | -3.1414 ± 0.0009 | 1.1 (43) |
| 14 | 2.69± 0.03 | 0.01(43) | -3.14 ± 0.04 | 0.09 (43) |
| 20 | 2.69 ± 0.08 | 0.02 (43) | -3.14 ± 0.11 | 0.06 (43) |

*M. Vicini*

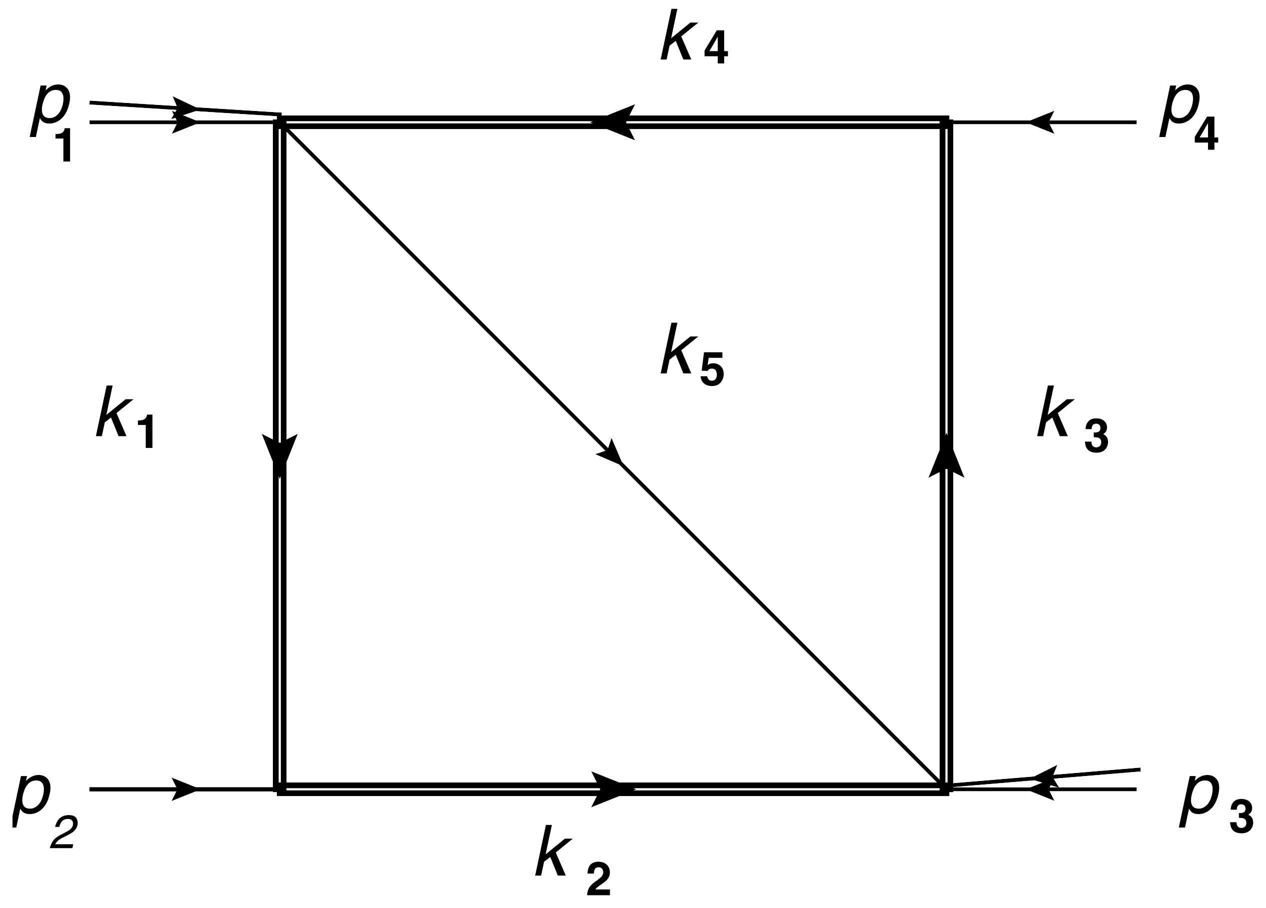
# Small mass expansions

$$I[f_m] = \int dk_i \, f(k_i, m) \, ,$$

$$I[f_m] = \int f_{\text{approx}}(k_i, m) + \int \left[ f(k_i, m) - f_{\text{approx}}(k_i, m) \right]$$

$$I[f_m] = \int f_{\text{approx}}(k_i, m) + \int \left[ f(k_i, m) - f_{\text{approx}}(k_i, m) \right]_{m \to 0} + \mathcal{O}(m) \, .$$

# Small mass expansions



$$\int \frac{d^dk_1}{i\pi^{\frac{d}{2}}}\frac{d^dk_4}{i\pi^{\frac{d}{2}}}\frac{1}{B_1B_2B_3B_4A_5}$$

$$= \int \frac{d^dk_1}{i\pi^{\frac{d}{2}}}\frac{d^dk_4}{i\pi^{\frac{d}{2}}}\Bigg\{\frac{1}{A_1A_2A_3A_4A_5}+\frac{1}{B_1B_2}\left[\frac{1}{A_3A_4A_5}\right]_{k_1=-x_2p_2} + \frac{1}{B_3B_4}\left[\frac{1}{A_1A_2A_5}\right]_{k_4=x_4p_4} - \frac{1}{B_1B_2B_3B_4}\left[\frac{1}{A_5}\right]_{\substack{k_4=x_4p_4,\\ k_1=-x_2p_2}}\Bigg\}.$$

$$\begin{aligned}
u\,\mathrm{D_{box}}\big|_{\mathrm{fin}}(\mu) &= 2\mathrm{Li}_4(v_1) + 2\mathrm{Li}_4(v_3) - 2\mathrm{Li}_4(v_s) - 2\mathrm{Li}_4(v_t) \\
&-2\mathrm{Li}_3(v_1)L_\mu(m_1^2) - 2\mathrm{Li}_3(v_3)L_\mu(m_3^2) + 2\mathrm{Li}_3(v_s)L_\mu(s) + 2\mathrm{Li}_3(v_t)L_\mu(t) \\
&+\mathrm{Li}_2(v_1)L_\mu^2(m_1^2) + \mathrm{Li}_2(v_3)L_\mu^2(m_3^2) - \mathrm{Li}_2(v_s)L_\mu^2(s) - \mathrm{Li}_2(v_t)L_\mu^2(t) \\
&+\frac{1}{3}\ln(1-v_1)L_\mu^3(m_1^2) + \frac{1}{3}\ln(1-v_3)L_\mu^3(m_3^2) - \frac{1}{3}\ln(1-v_s)L_\mu^3(s) \\
&-\frac{1}{3}\ln(1-v_t)L_\mu^3(t)\,.
\end{aligned}$$

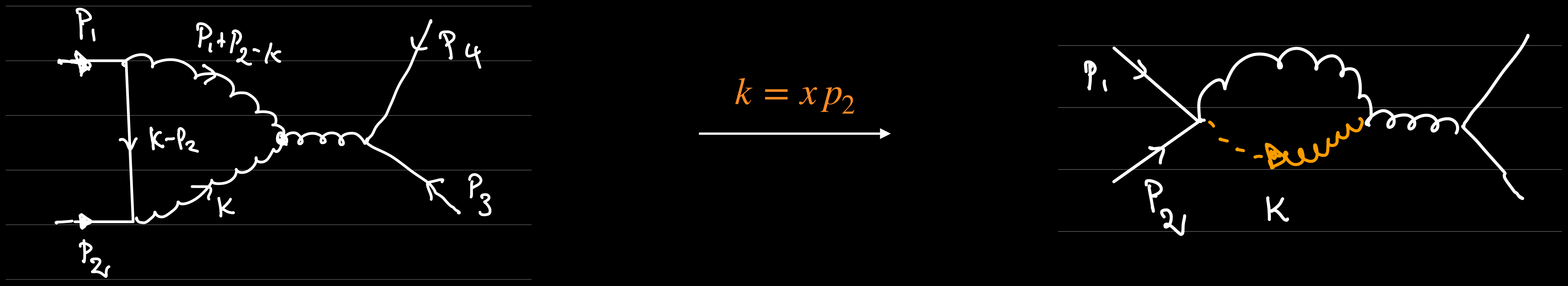
CA, G. Sterman

# Conclusions

- We have witnessed rapid progress in perturbative QCD, matching the precision of the LHC experiments. So far!
- Can we keep up? A need to keep reinventing our field and understanding perturbation theory at deeper levels.
- Infrared factorization  has been crucial historically. This property can be exploited further.
- Achieved a form for a class of two-loop amplitude integrands in which factorisation furnishes **local subtractions** for the removal of infrared singularities, also consistently with a BPHZ subtraction of UV singularities.
- Next challenge: Extend to colourful final states.
- And numerical integration in D=4 exactly.  A lot of progress achieved already in the framework of Loop-Tree-Duality.

# An one-loop example

“Ghost-terms” in triple gluon vertex get annihilated for physical (on-shell) external states.



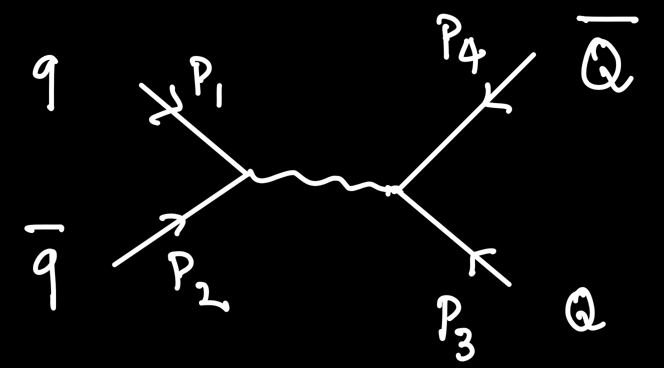
$k = x p_2$

**ghost terms** $\sim \ldots\bar{v}(p_2)(\cancel{p}_1 + \cancel{p}_2)u(p_1) + \ldots\bar{v}(p_2)\left[\cancel{p}_1 + (1-x)\cancel{p}_2\right] u(p_1)$

# An one-loop example

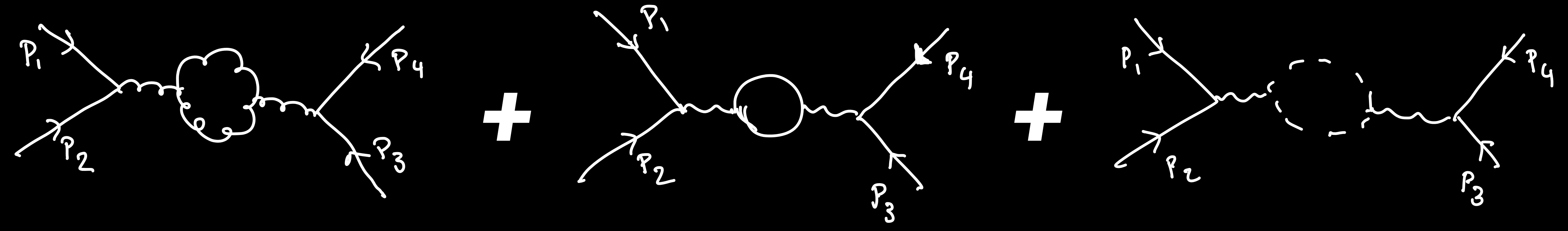
$q(p_1) + \bar{q}(p_2) \to Q(p_3) + \bar{Q}(p_4)$

$\mathscr{A}_0 =$

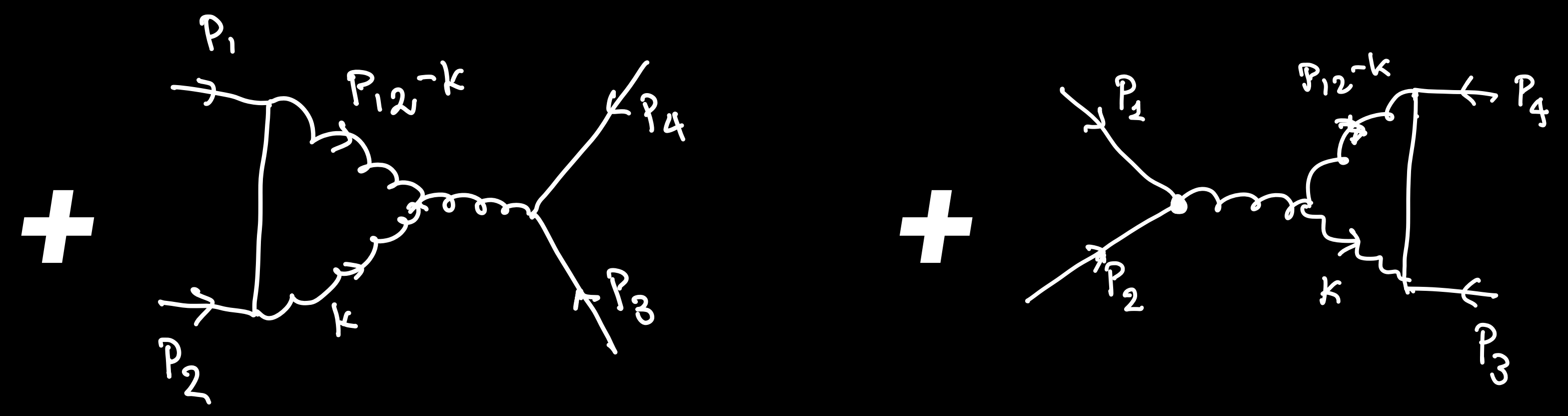


Finite

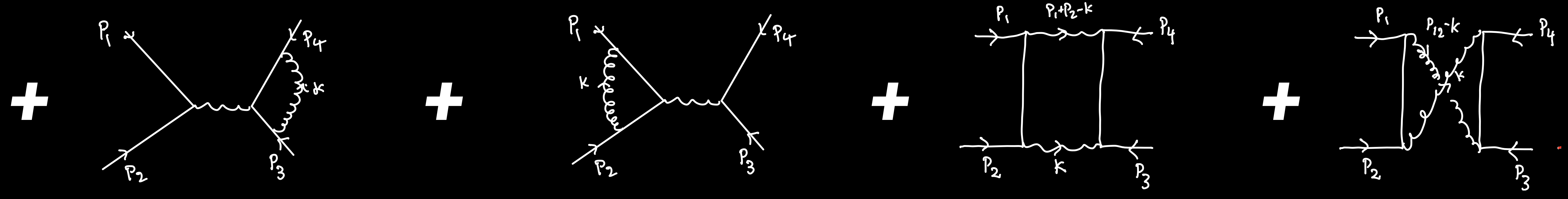
$\mathscr{A}_1(k) =$



Ultraviolet



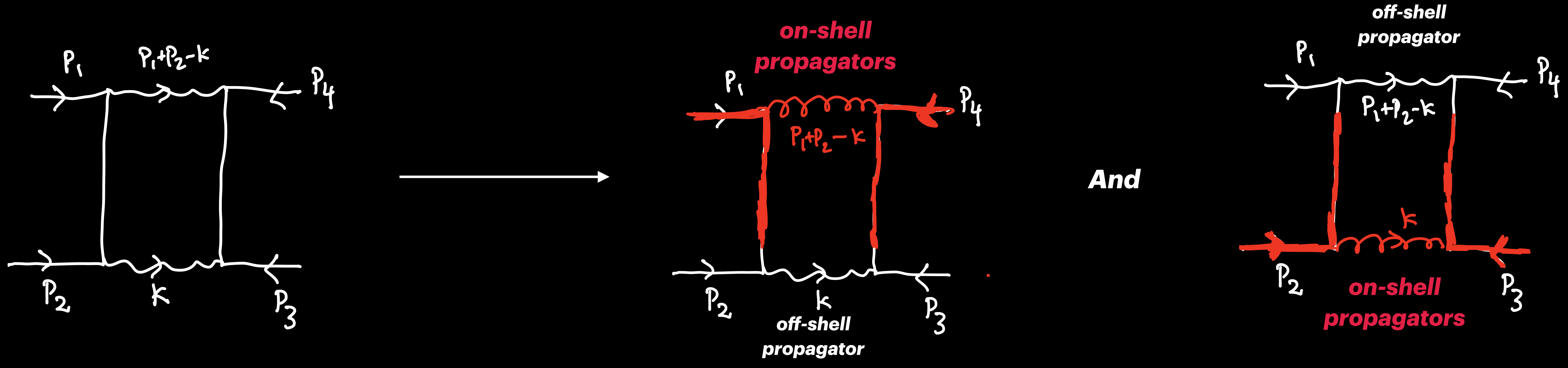
Collinear



Soft and Collinear

# An one-loop example

## Soft singularities



on-shell propagators

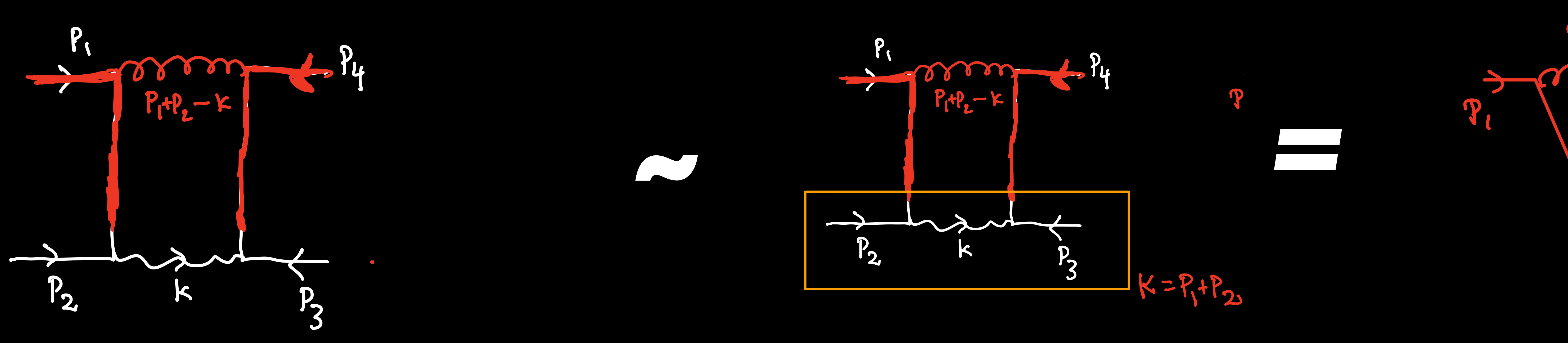
off-shell propagator

And

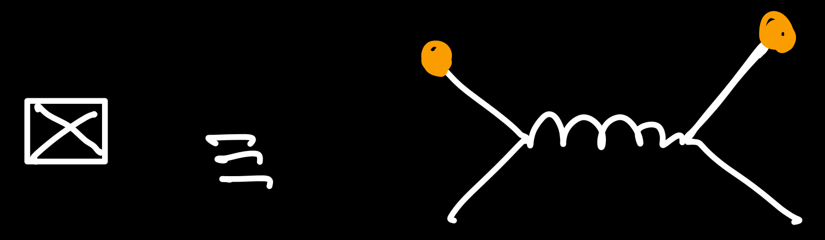
off-shell propagator

on-shell propagators

## Soft approximation



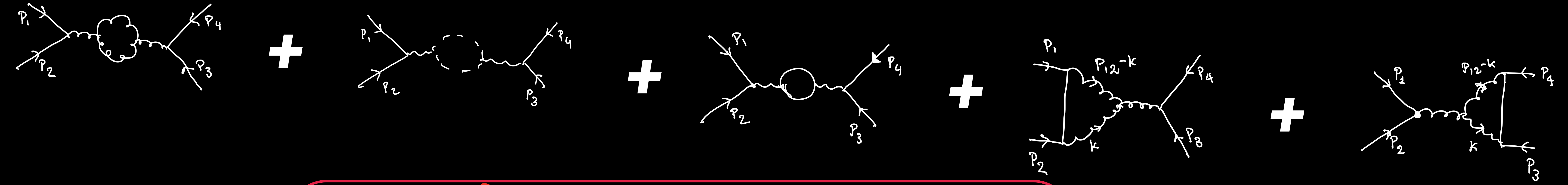
Soft approximation is a “form-factor” one-loop amplitude for the scattering of two partons off a composite external current which corresponds to the tree amplitude

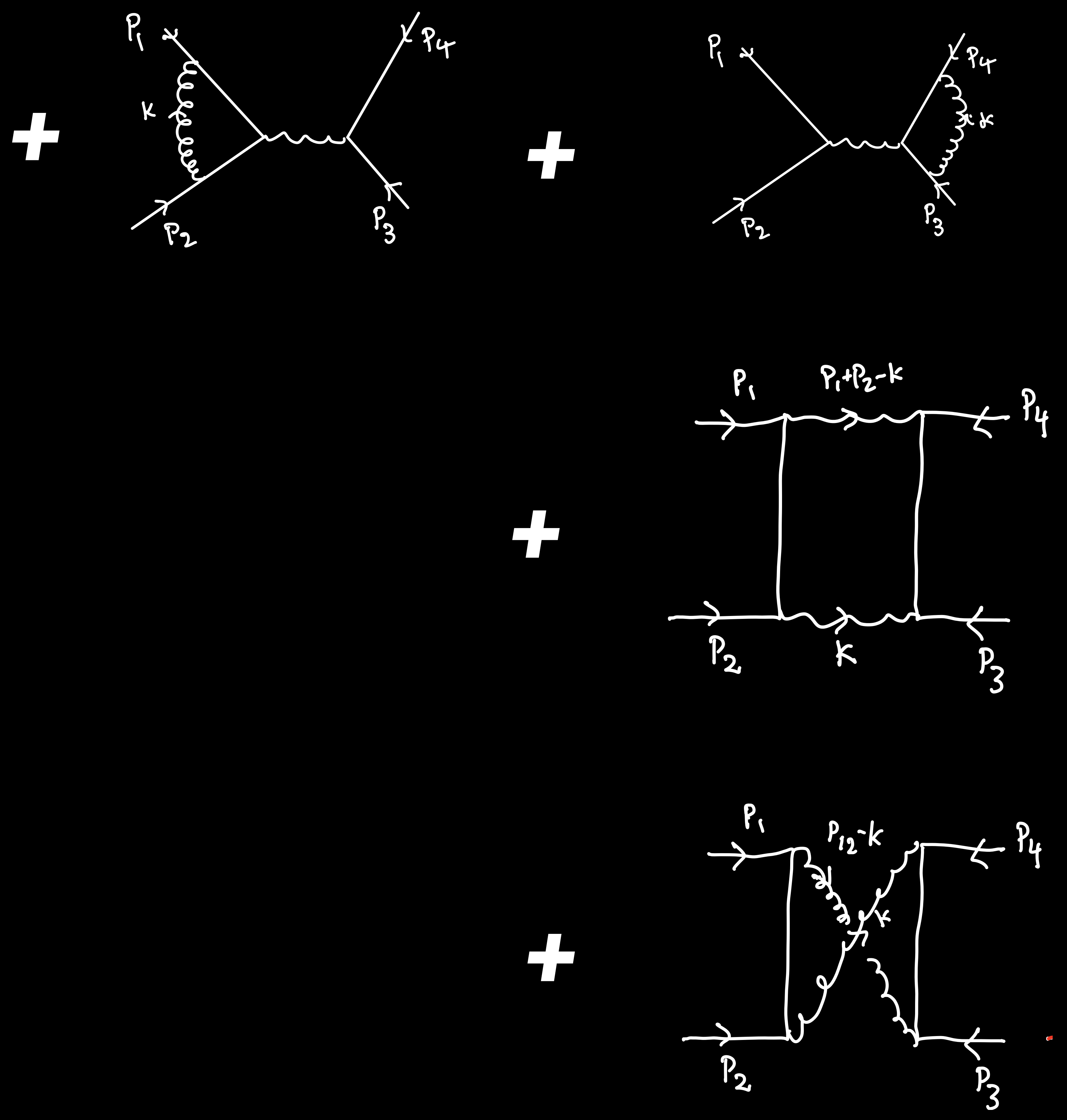


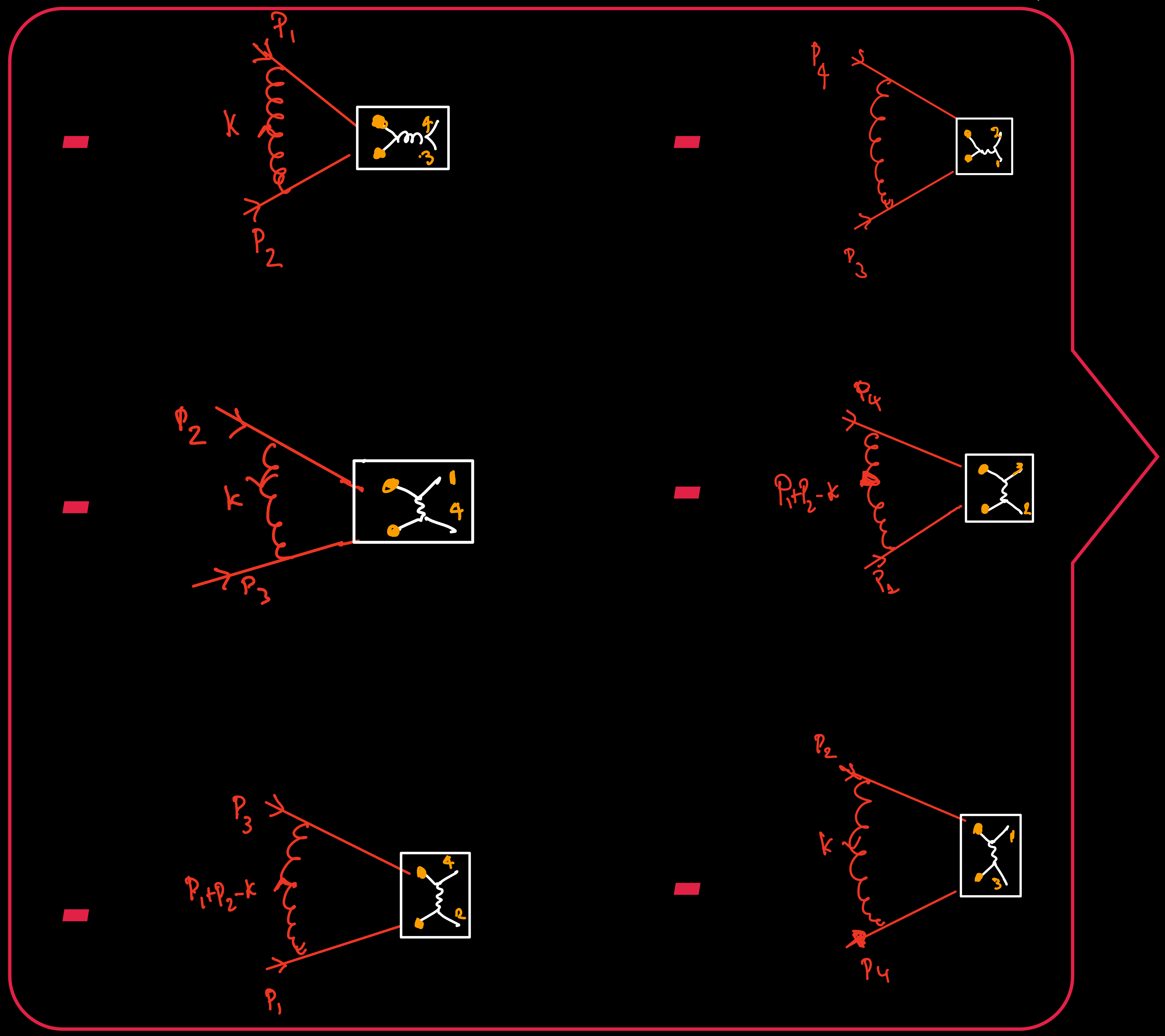
# An one-loop example:

## *removing the soft singularities*

$$\mathscr{A}_1(k) - \mathscr{S}_1(k) =$$



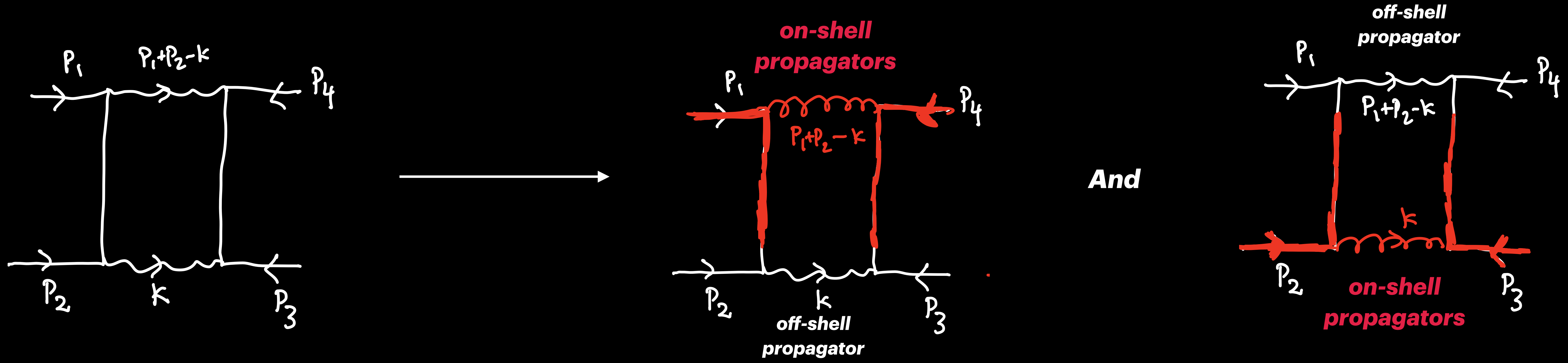




$$\mathcal{S}_1(k) = \sum_{(i,j)} \mathscr{A}_1\left(i+j \to \text{Tree}\right)$$

# An one-loop example

## Soft singularities



**on-shell propagators**

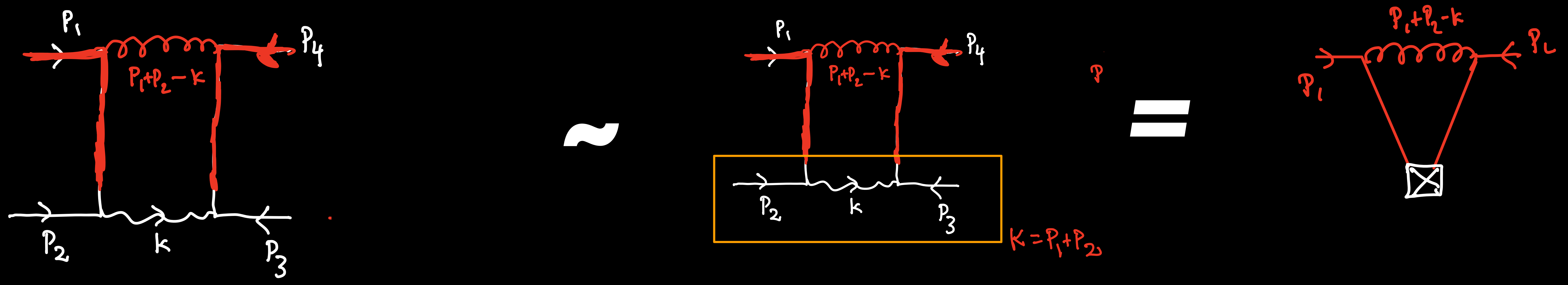
**off-shell propagator**

**And**

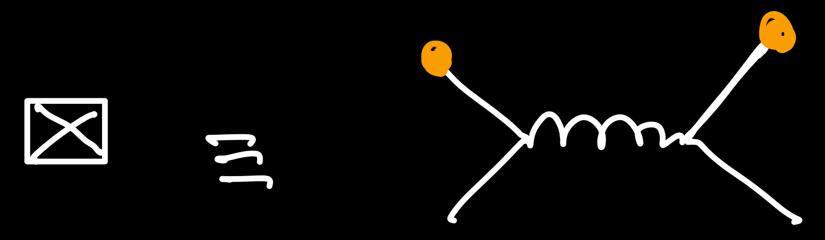
**off-shell propagator**

**on-shell propagators**

## Soft approximation



Soft approximation is a “form-factor” one-loop amplitude for the scattering of two partons off a composite external current which corresponds to the tree amplitude

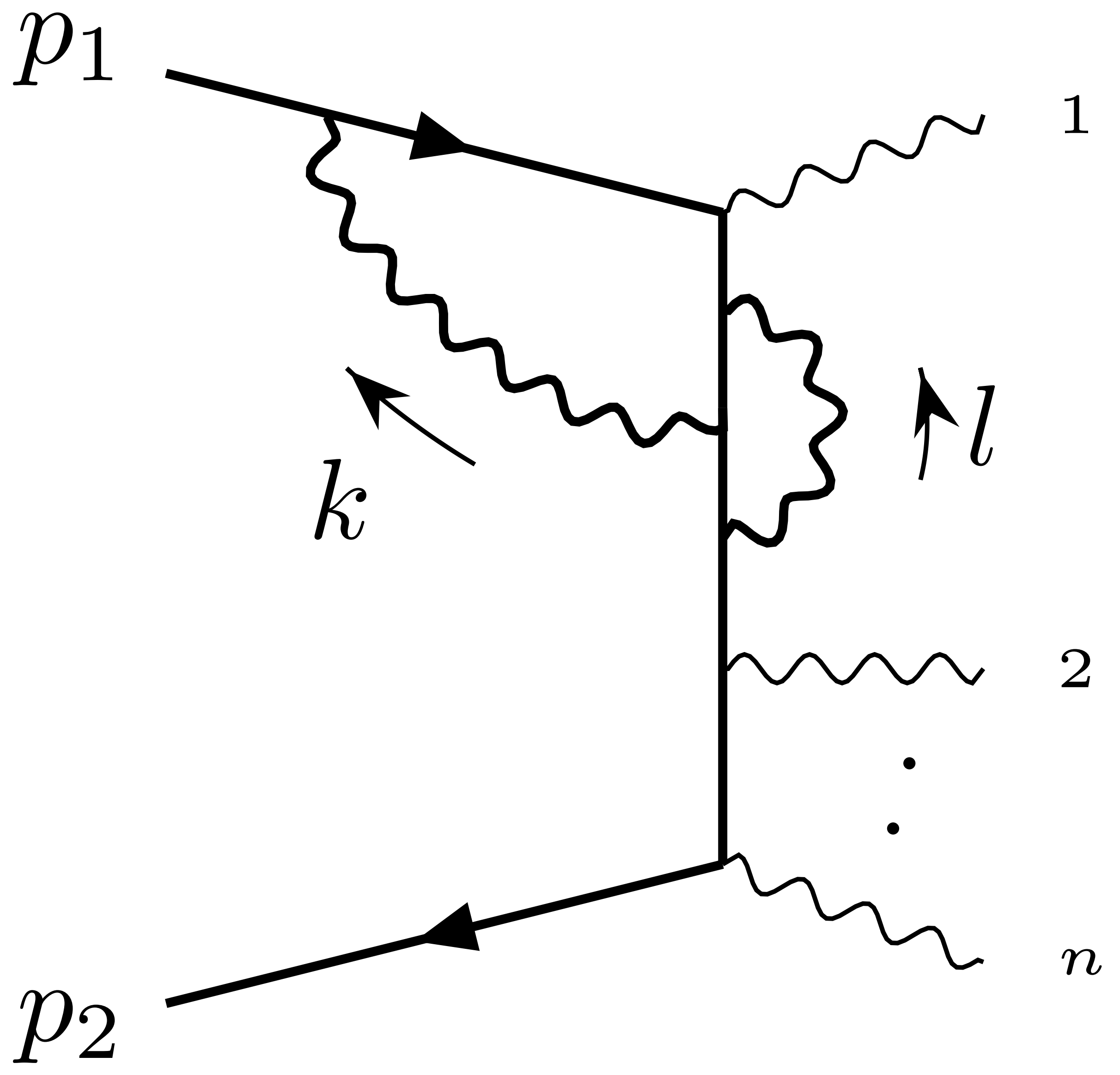


# Ultraviolet Counterterms

- Ultraviolet divergences can be subtracted locally with a usual BPHZ-type of procedure.
- Constructing approximations of the integrand has a simple step, which is power counting
- And a conceptual complication beyond one-loop… mixed UV and IR singularities… we want to subtract UV singularities without spoiling IR factorization.

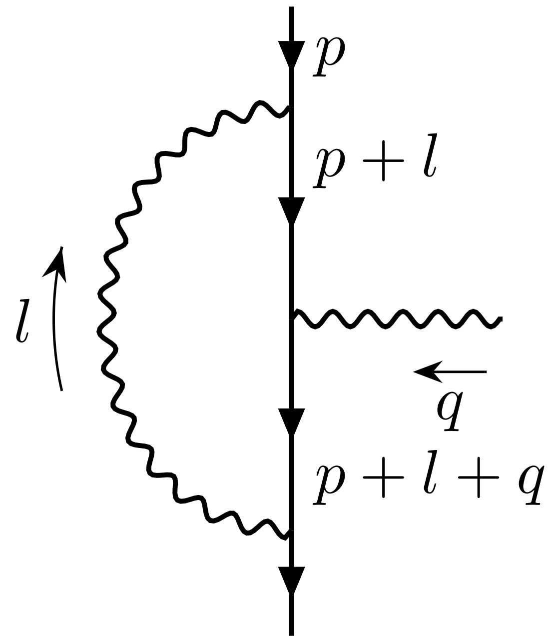
$$\mathscr{M}_1 = \mathscr{M}_{1-finite} + T_{l\to\infty}\mathscr{M}_1$$

$$\mathscr{M}_2 = \mathscr{M}_{2-finite} + T_{k\to\infty}\mathscr{M}_2 + T_{l\to\infty}\mathscr{M}_2$$
$$+T_{k,l\to\infty}\mathscr{M}_2 - T_{k,l\to\infty}\left(T_{k\to\infty}\mathscr{M}_2 + T_{l\to\infty}\mathscr{M}_2\right)$$



# One-loop self-energy/vertex graphs with ultraviolet divergences

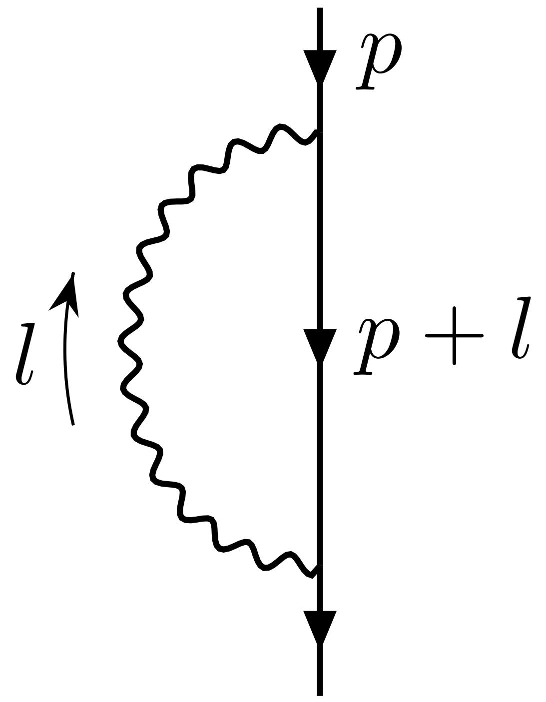
$e\Gamma^{(1),\nu}_{ee\gamma}(p,q,l) \equiv$



$$= (-e^3)\,\frac{\gamma^\mu(\not p+\not l+\not q)\gamma^\nu(\not p+\not l)\gamma_\mu}{(p+q+l)^2(p+l)^2l^2}$$

logarithmic
UV-singularity

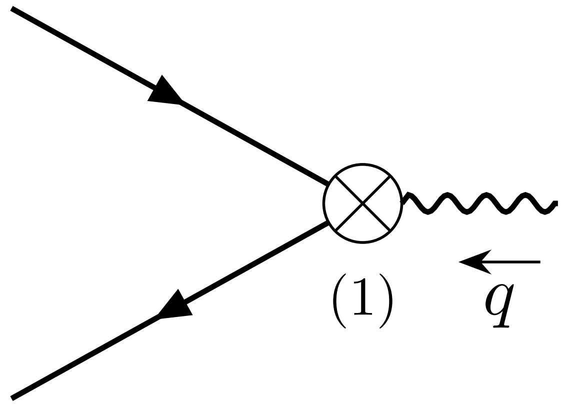
$\Pi_e^{(1)}(p,l) \equiv$



$$= -e^2\,\frac{\gamma^\mu(\not l+\not p)\gamma_\mu}{l^2(l+p)^2}$$

LOCALLY power-like
UV-singularity

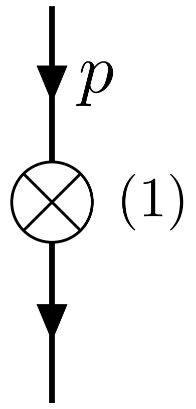
$e\Gamma^{(1,UV),\nu}_{ee\gamma}(l) \equiv$



$$= (-e^3)\,\frac{\gamma^\mu\not l\gamma^\nu\not l\gamma_\mu}{(l^2-M^2)^3}$$

*UV vertex counteterm*

$\Pi_e^{(1,UV)}(p,l) \equiv$



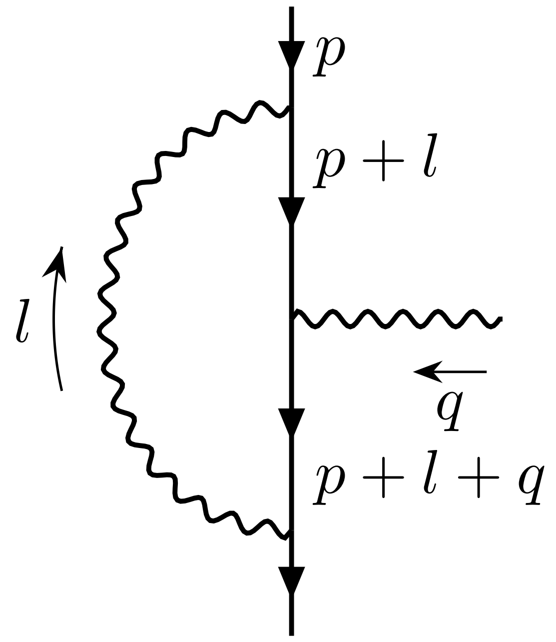
$$= -\,e^2\,\frac{\gamma^\mu(\not l+\not p)\gamma_\mu}{\left[(l+p/2)^2\right]^2}.$$

*Nagy, Soper*

*A possible UV self-energy counteterm*

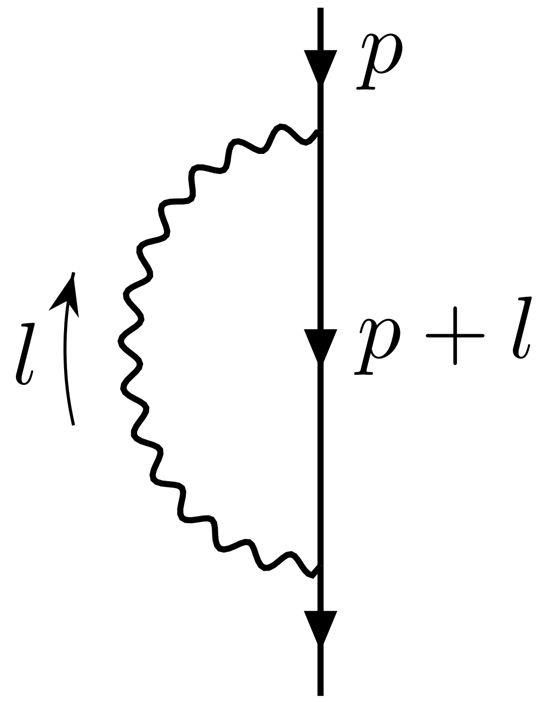
# One-loop self-energy/vertex graphs with ultraviolet divergences

$$e\Gamma^{(1),\nu}_{ee\gamma}(p,q,l) \equiv$$



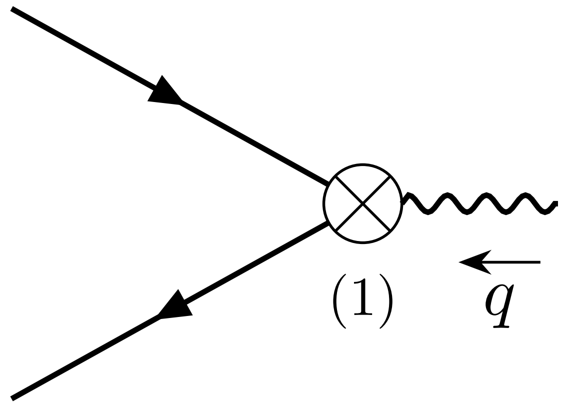
$$= (-e^3)\,\frac{\gamma^\mu(\not p+\not l+\not q)\gamma^\nu(\not p+\not l)\gamma_\mu}{(p+q+l)^2(p+l)^2 l^2}$$

$$\Pi^{(1)}_e(p,l) \equiv$$



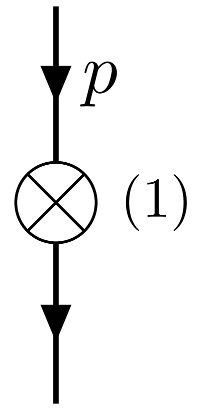
$$= -e^2\,\frac{\gamma^\mu(\not l+\not p)\gamma_\mu}{l^2(l+p)^2}$$

$$e\Gamma^{(1,UV),\nu}_{ee\gamma}(l) \equiv$$



$$= (-e^3)\,\frac{\gamma^\mu \not l \gamma^\nu \not l \gamma_\mu}{(l^2-M^2)^3}$$

$$\Pi^{(1,UV)}_e(p,l) \equiv$$



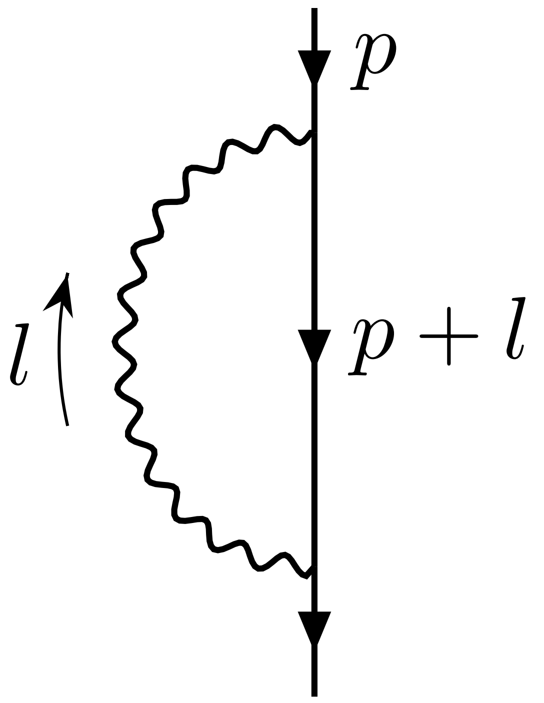
$$= -\,e^2\,\frac{\gamma^\mu(\not l+\not p)\gamma_\mu}{\left[(l+p/2)^2\right]^2}.$$

Recall the Ward-Identity: $$q_\nu\Gamma^{(1),\nu}_{ee\gamma}(p,q,l) = \Pi^{(1)}_e(p,l) - \Pi^{(1)}_e(p+q,l)\,.$$

It is violated by the UV counterterms! $$q_\nu\Gamma^{(1,UV),\nu}_{ee\gamma}(l) \neq \Pi^{(1,UV)}_e(p,l) - \Pi^{(1,UV)}_e(p+q,l),$$

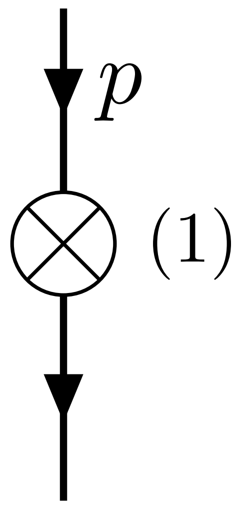
# One-loop self-energy/vertex graphs with ultraviolet divergences

$$\Pi_e^{(1)}(p,l) \equiv$$



$$= -e^2 \frac{\gamma^\mu(\not{l}+\not{p})\gamma_\mu}{l^2(l+p)^2}$$

$$\Pi_e^{(1,UV)}(p,l) \equiv$$

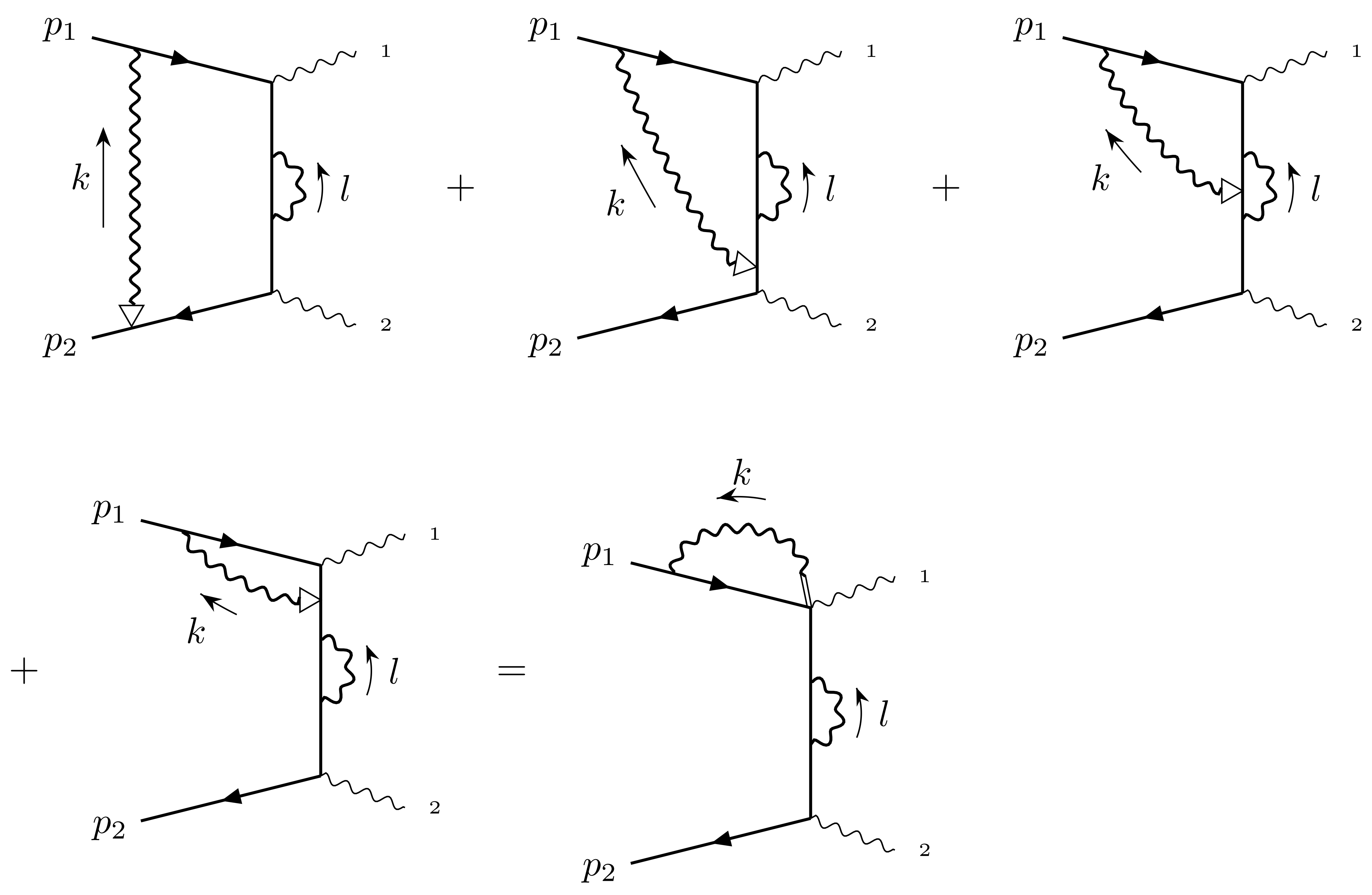


$$= \left(-e^2\right)\left[\frac{\gamma^\mu \not{l}\gamma_\mu}{(l^2-M^2)^2} - \frac{\gamma^\mu \not{l}\not{p}\not{l}\gamma_\mu}{(l^2-M^2)^3}\right]$$

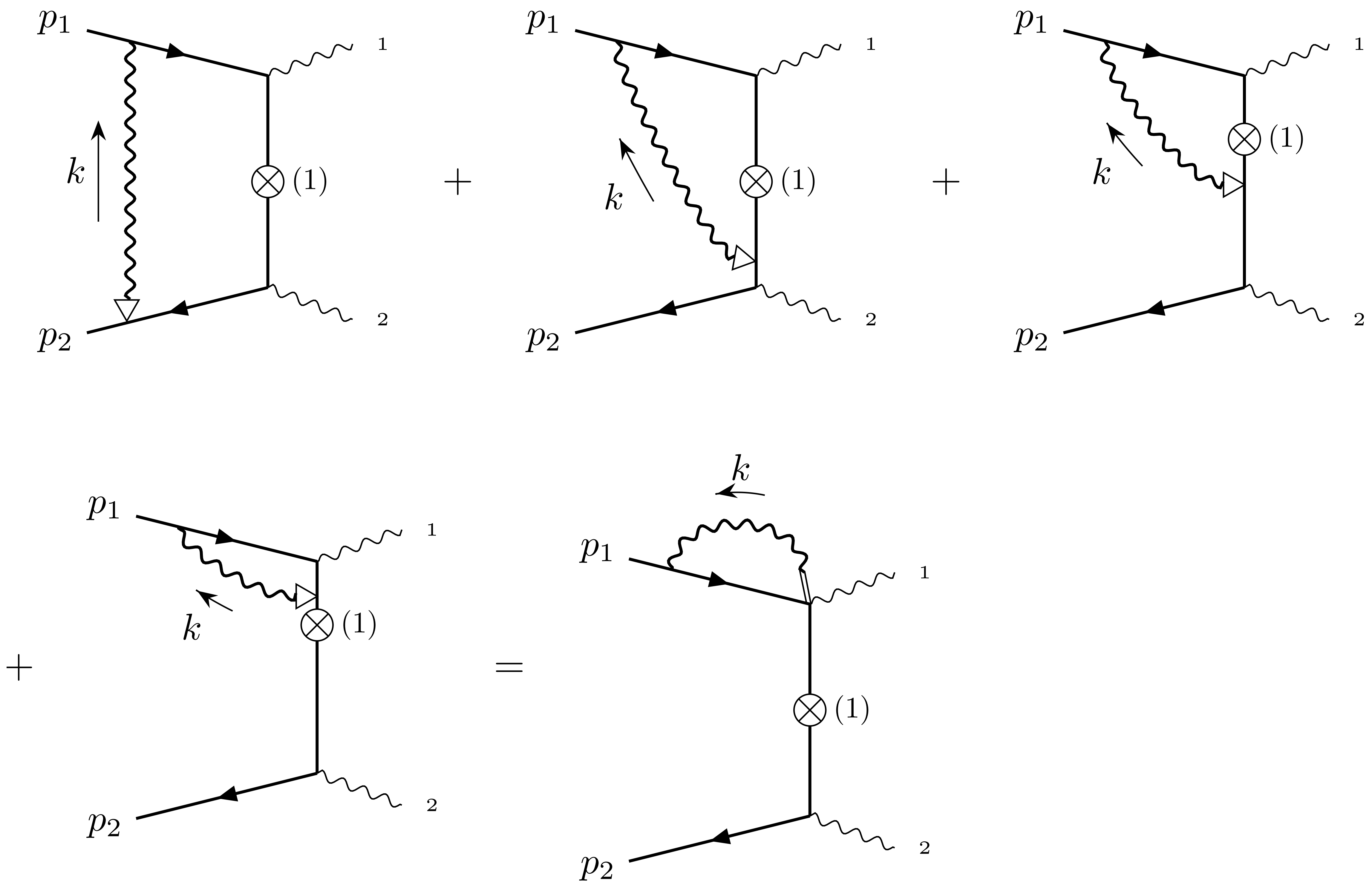
$$= -e^2 \frac{\gamma^\mu \not{l}\gamma_\mu}{(l^2-M^2)^2} - p_\nu \Gamma_{ee\gamma}^{(1,UV),\nu}(l)\,.$$

Respects the Ward-Identity: $q_\nu \Gamma_{ee\gamma}^{(1,UV),\nu}(l) = \Pi_e^{(1,UV)}(p,l) - \Pi_e^{(1,UV)}(p+q,l);$

# Diagrams with mixed UV and IR singularities



Collinear Factorization for the Amplitude



Collinear Factorization for the UV-subtracted Amplitude as well!