ANALYTIC METHODS FOR MULTI-LOOP CALCULATIONS

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Precision calculations for future e+e– colliders: targets and tools
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COMPUTATIONAL CHALLENGES FOR e^+e^- PRECISION PREDICTIONS

- Precise predictions for observables ($\sigma, \mathrm{d}\sigma$) and derived quantities ($m, \ldots$):
  - Scattering amplitudes at higher order perturbation theory:
    $$\mathscr{A} = \sum_{m,n} \alpha_s^m \alpha^n \mathscr{A}^{(m,n)}$$
  - UV renormalization
  - IR subtraction
  - Phase space integration
  - Beyond fixed order: non-relativistic effects, resummation

AMPLITUDE INTEGRAND

talks: Tiziano Peraro, Vasily Sotnikov, Max Zoller

- $\mathscr{A}^{(m,n)}$ integrand from projectors + Feynman diagrams, on-shell techniques, …
- Scheme to regularize UV and IR divergences + treatment of γ_5
- Helicity amplitude $\mathscr{A}^{(m,n)} = s_\lambda \mathscr{A}_\lambda^{(m,n)}$ where s_λ spinor factor and $\mathscr{A}_\lambda^{(m,n)}$ scalar
- $\mathscr{A}_\lambda^{(m,n)} = \sum_i c_i I_i$, different options:

I_i	c_i
Tensor integrals	Tensors
Scalar integrals	Rational scalar functions of kinematics, d
Master integrals (d-fact., phys. denom., canonical, finite, …)	Rational (algebraic,..) scalar functions
Special functions for Laurent exp. in eps (HPL, MPL, eMPL, problem specific, …)	Rational (algebraic,..) scalar functions

Requires IBP reduction ↓

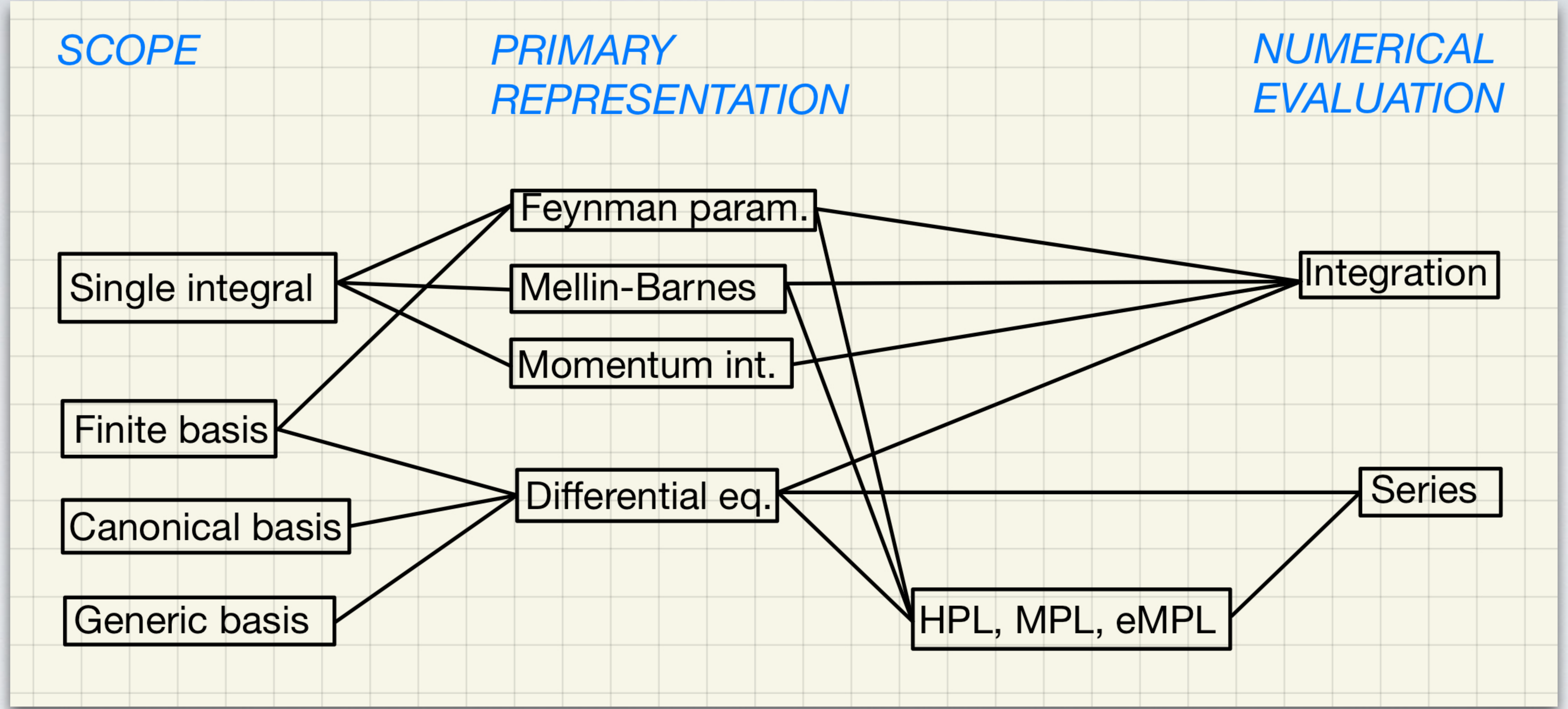
Resolving analytic features ↓

IBP REDUCTIONS

talk: Tiziano Peraro

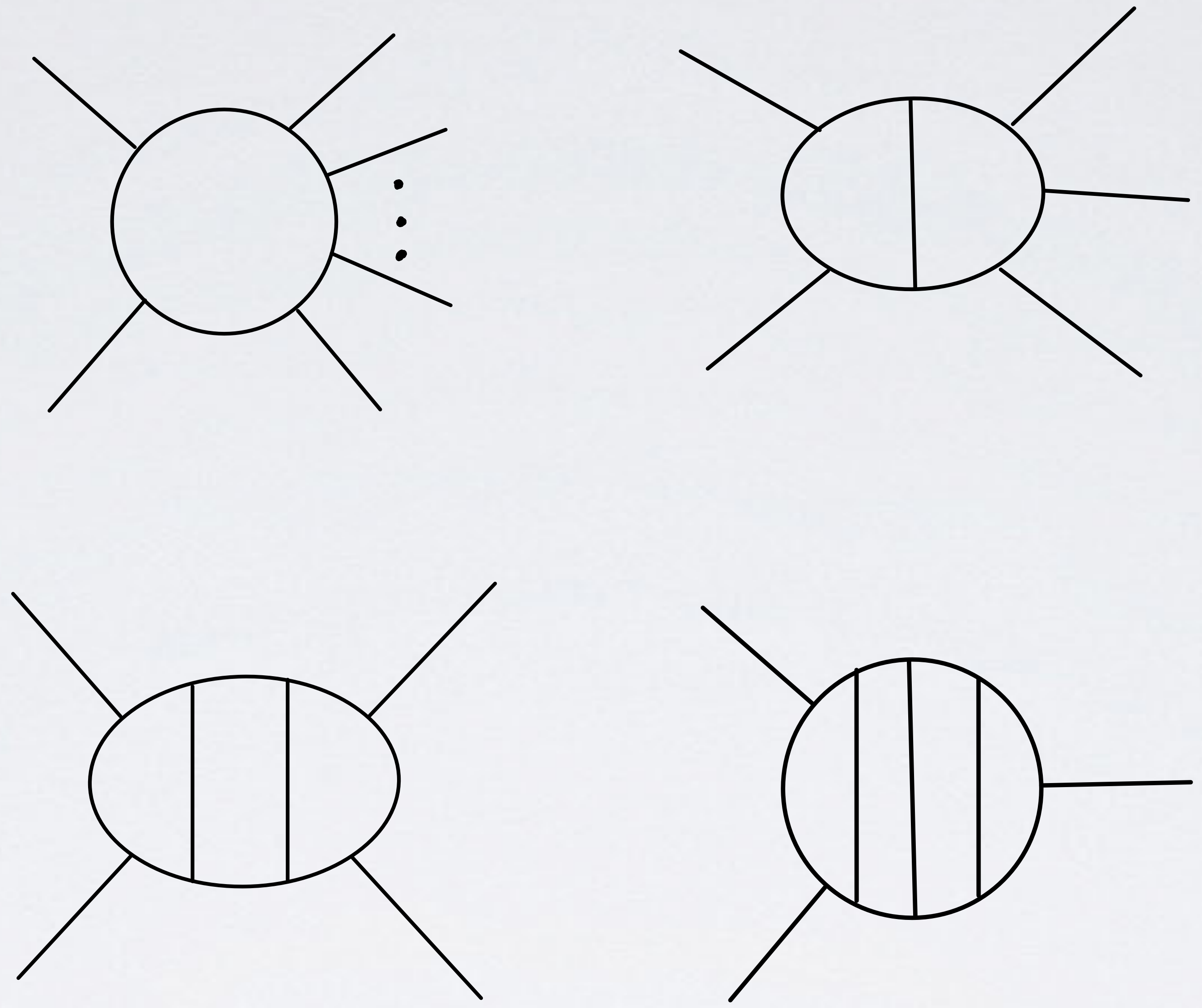
- Linear relations between Feynman integrals
- Algorithmic approach: Laporta
- Alternatives and improvements, symbolic or integer propagator exponents
- Important progress for generic amplitudes:
  - Finite field sampling and rational reconstruction
  - Denominator guessing
  - Syzygies (surface terms)
- Focus:
  - More scales: smarter sampling
  - More loops: smarter memory management
- Future improvements from intersection theory, p-adic numbers, non-commutative Gröbner basis, … ?

EVALUATION OF INTEGRALS



talks (analytical focus): Li Lin Yang, Vasily Sotnikov, Stefan Weinzierl
talks (numerical focus): Valentin Hirschi, Long Chen, Vitalii Materia, Janusz Gluza, Narayan Rana, Martijn Hidding, Xiao Liu

STATE OF THE ART FOR AMPLITUDES



- Depicted amplitudes: frontier in full-color, massless QCD (more in this talk)
- Frontier @ 2-loop leading color, massless QCD: 5 legs, 1 off-shell

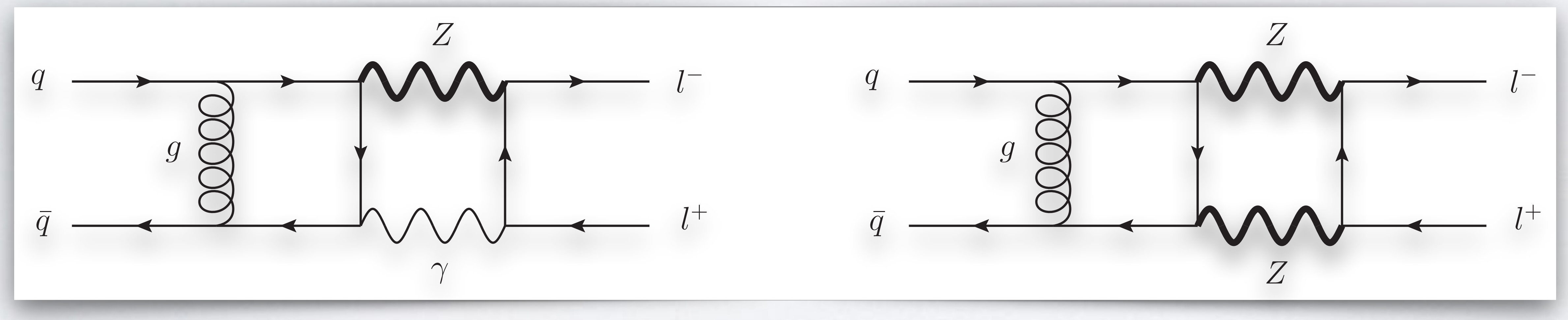
  [Abreu, Febres-Cordero, Its, Klinkert, Page, Sotnikov '21; Badger, Hartanto, Kryś, Zoia '22; Badger, Hartanto, Zoia '22, Hartanto, Poncelet, Popescu, Zoia '22]
- Frontier @ 2-loop with external + internal masses (t,W,Z): 4 legs

  [Bârnreuther, Czakon, Fiedler '13; …; Agarwal, Jones, AvM '20; Heller, AvM, Schabinger, Spiesberger '20; Bonetti, Smirnov, Panzer, Tancredi '21; Brønnum-Hansen, Wang '20, '21; Brønnum-Hansen, Melnikov, Quarroz, Wang '21; Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Ronca, Schubert, Torres Bobadilla Tramontano '21; Chen, Heinrich, Kerner, Klappert, Schlenk '20; Becchetti, Moriello, Schweitzer '21]

talk: Vasily Sotnikov

MIXED EW-QCD CORRECTIONS TO DRELL-YAN e^+e^- PRODUCTION

- High invariant mass l^+l^- production: search for new physics
- EW corrections important due to large Sudakov logs
- Two independent calculations at large invariant mass:



  - *[Bonciani, Buonocore, Grazzini, Kallweit, Rana, Tramontana, Vicini 2021]* talk: Narayan Rana
    based on semi-numerical two-loop amplitudes *[Armadillo, Bonciani, Devoto, Rana, Vicini 2022]*
  - *[Buccioni, Caola, Chawdhry, Devoto, Heller, AvM, Melnikov, Röntsch, Signorile-Signorile 2022]*
    based on analytical two-loop amplitudes *[Heller, AvM, Schabinger, Spiesberger 2020]* (this talk)

γ_5 SCHEMES

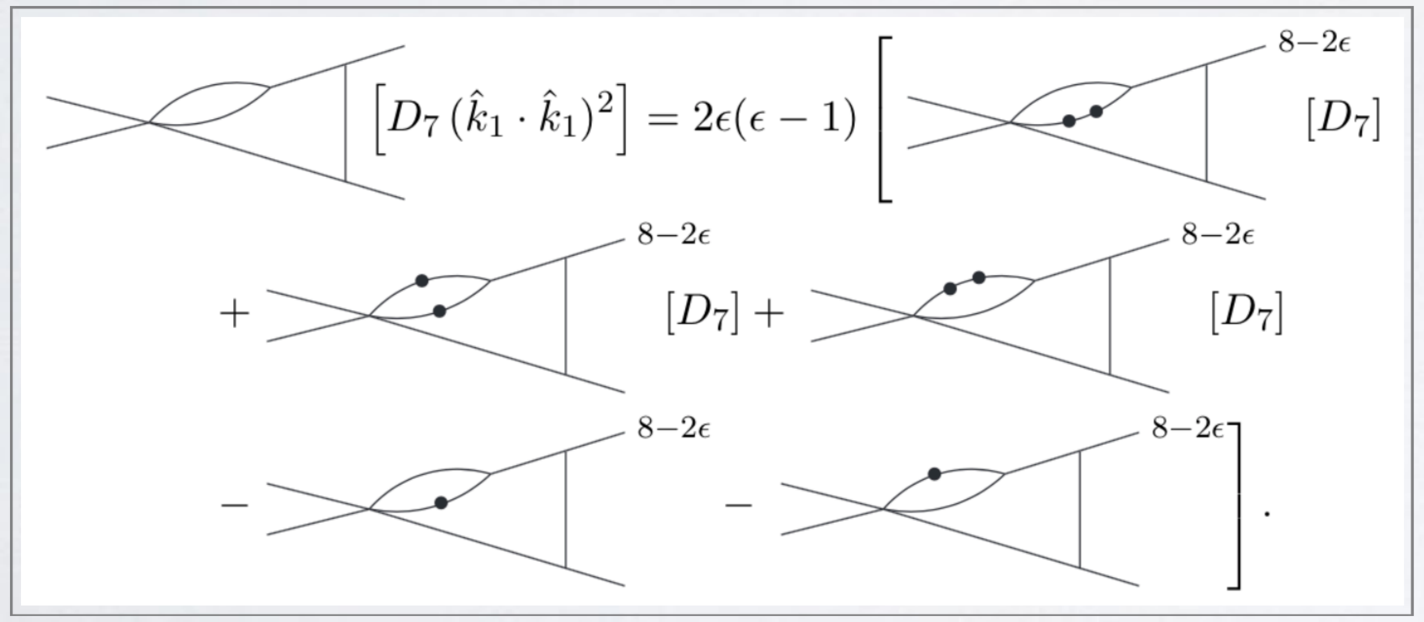
- Problem: γ_5 really a ***4-dimensional*** object: $\mathrm{tr}\,\{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\} = -4i\,\varepsilon_{\mu\nu\rho\sigma}$
- What to do in ***dimensional regularization*** ?
- ***Split*** d dimensional space → 4 dim (bar) + ϵ dim (hat) $k^\mu = \bar{k}^\mu + \hat{k}^\mu$, $\gamma^\mu = \bar{\gamma}^\mu + \hat{\gamma}^\mu$
  keep $\epsilon_{\mu\nu\rho\sigma}$ 4-dimensional: $\varepsilon^{\mu\nu\rho}{}_\alpha\,\varepsilon_{\mu\nu\rho\beta} = -6\,\bar{g}_{\alpha\beta}$
- Option 1: *'t Hooft, Veltman ['72], Breitenlohner, Maison ['77]* (HVBM) $\{\bar{\gamma}_\mu, \gamma_5\} = 0$, $[\hat{\gamma}_\mu, \gamma_5] = 0$
  - give up ***anti-commutativity***
  - violates Ward identities
- Option 2: *Kreimer ['90]* $\{\gamma_\mu, \gamma_5\} = 0$
  - give up ***cyclicity*** of Dirac trace
  - requires reading point
  - possibly symmetrization

SETUP AND μ TERMS

- 3 different setups for our calculation:
    1. HVBM scheme + projectors + mu-terms
    2. Kreimer's scheme + projectors + mu-terms
    3. Kreimer's scheme + tensor reduction of integrals / spin-summed interference
- HVBM + Kreimer: short Dirac chains
- HVBM: split indices before Dirac trace, Kreimer: no split for Dirac trace

$$\varepsilon^{\mu}{}_{\alpha\delta\kappa}\,\varepsilon_{\mu\beta\eta\lambda} = \bar{g}_{\alpha\lambda}\bar{g}_{\beta\kappa}\bar{g}_{\delta\eta} - \bar{g}_{\alpha\lambda}\bar{g}_{\beta\delta}\bar{g}_{\eta\kappa} - \bar{g}_{\alpha\eta}\bar{g}_{\beta\kappa}\bar{g}_{\delta\lambda} + \bar{g}_{\alpha\beta}\bar{g}_{\delta\lambda}\bar{g}_{\eta\kappa} + \bar{g}_{\alpha\eta}\bar{g}_{\beta\delta}\bar{g}_{\kappa\lambda} - \bar{g}_{\alpha\beta}\bar{g}_{\delta\eta}\bar{g}_{\kappa\lambda}\,,$$

- Tensor integrals with ϵ ***dim loop momenta*** (μ ***terms***) from parametric representation,



+ standard reduction techniques

SYMMETRY RESTORATION

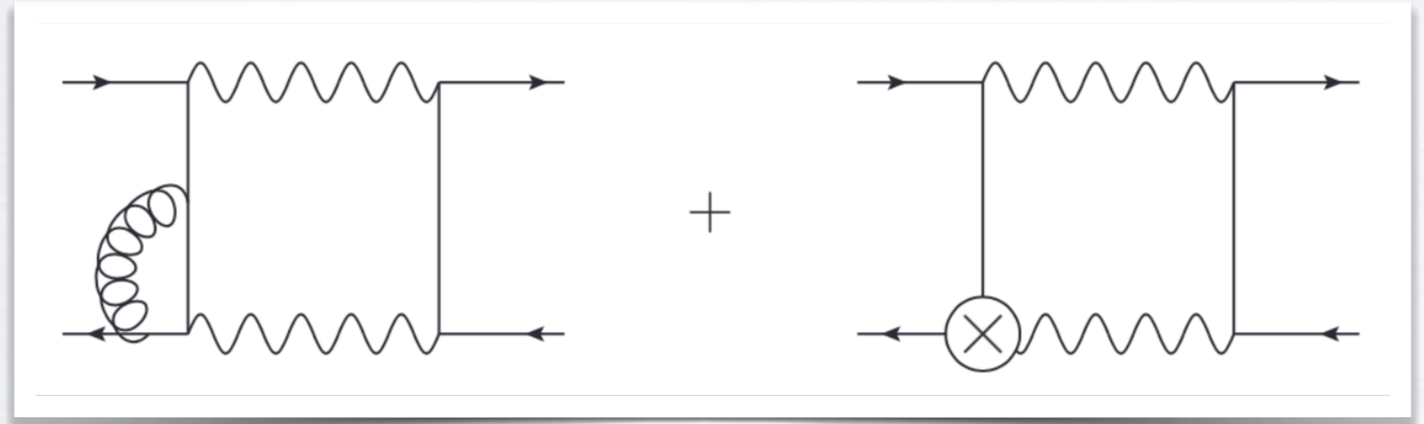
- In HVBM, corrections to ***vector and axial-vector currents*** differ
- Restore ***Ward identities*** by adding counter terms
- For vertex, ***require***

$$\bar{\mathcal{A}}^{(0,1)}_{Z\bar{q}q}(s) = -\frac{a_q}{v_q}\,\bar{\mathcal{V}}^{(0,1)}_{Z\bar{q}q}(s)$$

- Implement by adding ***counter terms***

$$\begin{aligned}\delta Z^{(0,1)}_{Z\bar{q}q} &= \bar{\mathcal{A}}^{(0,1)}_{Z\bar{q}q}(s) - \mathcal{A}^{(0,1)}_{Z\bar{q}q}(s)\\ &= 2\,a_q \frac{(2-\epsilon)\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)e^{\gamma_E\epsilon}}{(1-\epsilon)\Gamma(2-2\epsilon)} C_F e^{i\pi\epsilon}\left(\frac{\mu^2}{s}\right)^{\epsilon}\end{aligned}$$

- Remark 1: need also symmetry restoring counter terms in ***boxes***



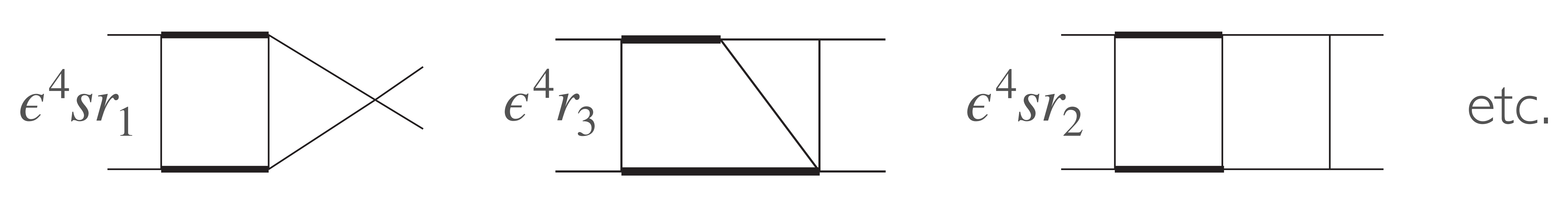
- Remark 2: we keep also the ***higher order ϵ terms*** for the counter term

γ_5 SCHEMES DEPENDENCES

- γ^5 ***scheme dependence*** (after adding symmetry restoring counter terms):
  - $\mathcal{O}(\epsilon^1)$, $\mathcal{O}(\epsilon^2)$ one-loop remainders and $\mathcal{O}(1/\epsilon)$, $\mathcal{O}(\epsilon^0)$ bare two-loop:
    scheme and reading point dependent
  - Finite remainders:
    coincide between schemes

- Note: towards general counter-terms for HVBM in
  [Belusca-Maito, Ilakovac, Kühler, Mador-Bozinovic, Stöckinger 2021]

NON RATIONALIZABLE ROOTS



- Use ***differential equations***, integrals allow for ϵ basis *[Henn '13]:* $d\vec{m} = \epsilon \, \mathrm{dlog}(l_a) \, \hat{A}^{(a)} \, \vec{m}$
- Leading singularities introduce: $r_1 = \sqrt{s(s-4m^2)}$, $r_2 = \sqrt{-st\left(4m^2(t+m^2) - st\right)}$, $r_3 = \sqrt{s\left(t^2(s-4m^2) + sm^2(m^2-2t)\right)}$
- Reparametrization $s = -m^2(1-w)^2/w$, $t = -m^2 w(1+z)^2/(z(1+w)^2)$ rationalizes 2 out of 3 roots, $r = \sqrt{4(1-w)^2 w z^2 + (w+z)^2(1+wz)^2}$ is ***not rationalizable*** *[van Straten '14, Besier, Festi, Harrison, Naskrecki '19]*
- Our approach:
  - Systematically simplify letters
  - Construct multiple polylogarithms a la *[Duhr, Gangl, Rhodes '11]*, but for algebraic arguments
- See also:
  - Other work on polylogarithmic iterated integrals with algebraic letters: *[Caron-Huot, Henn '14, Ablinger, Blümlein, Raab, Schneider '14, Papadopoulos, Tommasini, Wever '15, Bonciani et al '16, AvM, Tancredi '17, Abreu et al '20, Chicherin, Sotnikov '20, Syrrakos '20, Kreel, Weinzierl '21, Chicherin, Sotnikov, Zoia '21, Henn, Peraro, Xu, Zhang '22, …]* and many more
  - Cases with elliptical curves: *[…, Abreu, Becchetti, Duhr, Ozcelik '22]* *talk: Stefan Weinzierl*

ANALYTICAL RESULT

- Match diff. eqs. against ***analytical ansatz***

talks: Narayan Rana, Martijn Hidding, Xiao Liu

- Transport boundary constants: high precision numerics with ***power-log series solutions***
  [Lee, Smirnov, Smirnov '18, Liu, Ma, Wang '18, Heller, AvM, Schabinger '19, Moriello '19, Hidding '20, Armadillo, Bonciani, Devoto, Rana, Vicini '22, …]

- ***Analytical result*** in terms of standard multiple polylogs possible despite non-rationalizable roots:

$$m_{32} = \epsilon^3 \left[4\,\mathrm{Li}_3\left(\frac{l_1 l_2 l_6 l_7 l_{10} l_{13}}{l_{14} l_{15} l_{16}}\right) - 2\,\mathrm{Li}_3\left(\frac{l_2^3 l_6 l_7^2}{l_{15} l_{16}}\right) + \ldots + 4\,\mathrm{Li}_2\left(\frac{l_6 l_{14} l_{16}}{l_7 l_9 l_{15}}\right) \ln(l_3) + \ldots \right]$$
$$+\epsilon^4 \left[-\mathrm{Li}_{2,2}\left(-\frac{l_1^2 l_3 l_{15}}{l_2^2 l_7 l_{14}}, \frac{l_2^2 l_7 l_{15}}{l_1 l_3 l_6 l_1 4}\right) + \ldots + \frac{701}{4}\,\mathrm{Li}_4\left(\frac{l_1 l_3^2 l_6^2 l_9 l_{14}}{l_2 l_7 l_{13} l_{15} l_{16}}\right) + \ldots \right] + O(\epsilon^5)$$

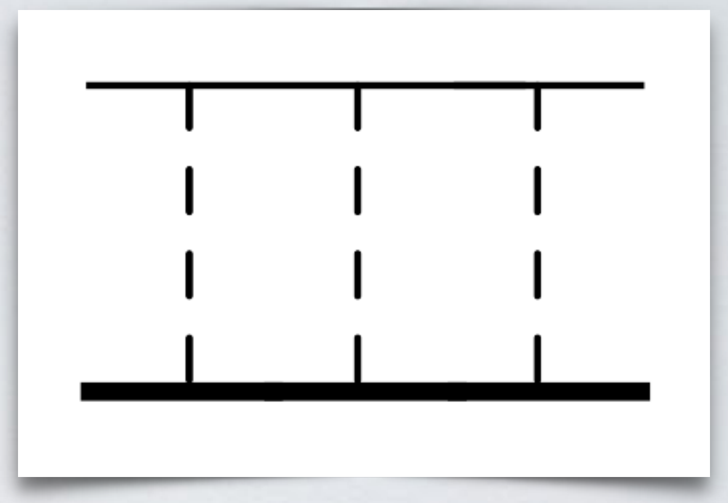
[Heller, AvM, Schabinger 2019]

- No spurious letters, simple constants

- $\approx$ **0.7 s** numerical evaluation of ***full amplitude*** (incl. ZZ, WW contrib., generic point) with HandyG *[Naterop, Signer, Ulrich '19]*

- "Good MPLs" crucial for performance, requires dividing phase space

MORE MASSES, MORE ROOTS



- $e\mu$ **scattering** integrals *[Heller '21]:*
  - Leading singularities introduce four roots $[x, y = (m_1^2 \pm m_2^2)/(2s),\ z = t/s]$

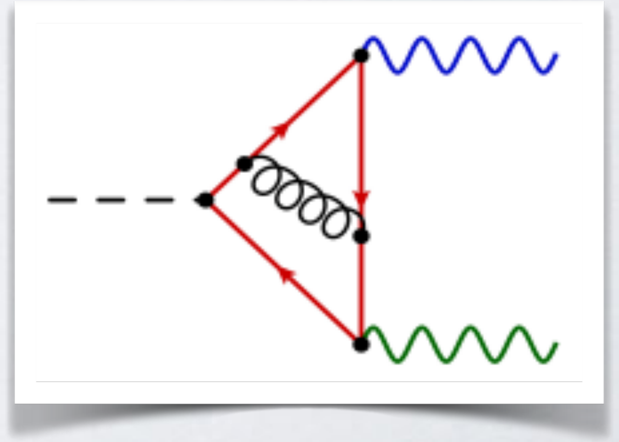
$$r_1 = \sqrt{1-4x+4y^2}, \quad r_2 = \sqrt{-4xz+4yz+z^2}, \quad r_3 = \sqrt{-4xz-4yz+z^2},$$
$$r_4 = \sqrt{1-4x+4y^2+2z-4xz+z^2}.$$

  - ϵ dlog basis, initially 30 complicated letters, deg 4 in x,y,z
  - Algorithm gives 20 simplified letters (no rationalization at all)

$$\mathcal{L}_R = \left\{x-y, x+y, -1+4x-4y^2-z, z\right\}$$

$$\mathcal{L}_A = \Big\{r_1, r_2, r_3, r_4, \frac{1}{2}(-1+2y+r_1), \frac{1}{2}(1+2y+r_1), \frac{1}{2}(r_2+z), \frac{1}{2}(r_3+z),$$
$$\frac{1}{2}(-1+2x-z+r_4), \frac{1}{2}(r_2+r_1+r_4), \frac{1}{2}(r_3+r_1+r_4), \frac{1}{2}(r_2-r_1-r_4),$$
$$\frac{1}{2}(r_3-r_1-r_4), \frac{1}{2}(r_1-r_3-r_4), \frac{1}{2}(r_1-r_2-r_4), \frac{1}{2}(z(-2+4x-z)+r_2r_3)\Big\}$$

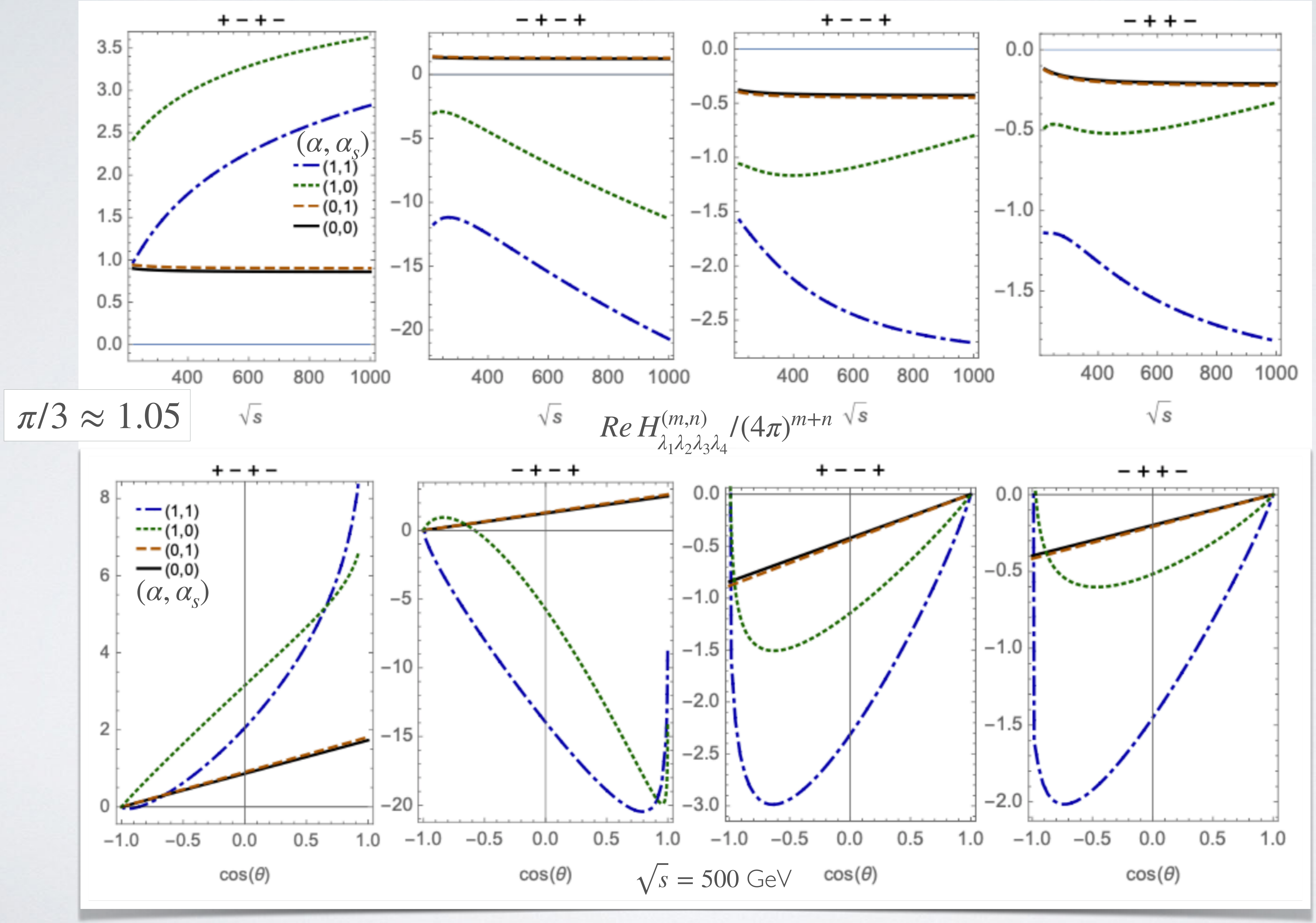
  - Analytic solution in terms of G functions



- $H \to ZZ$ **decay**, mixed EW-QCD 2-loop corrections *[Chaubey, Kaur, Shivaji '22]*:
  - Letter simplification, direct integration

BACK TO DY: HELICITY AMPLITUDES

[Heller, AvM, Schabinger, Spiesberger 2020]



CROSS SECTION AT HIGH ENERGIES

[Buccioni, Caola, Chawdhry, Devoto, Heller, AvM, Melnikov, Röntsch, Signorile-Signorile 2022]

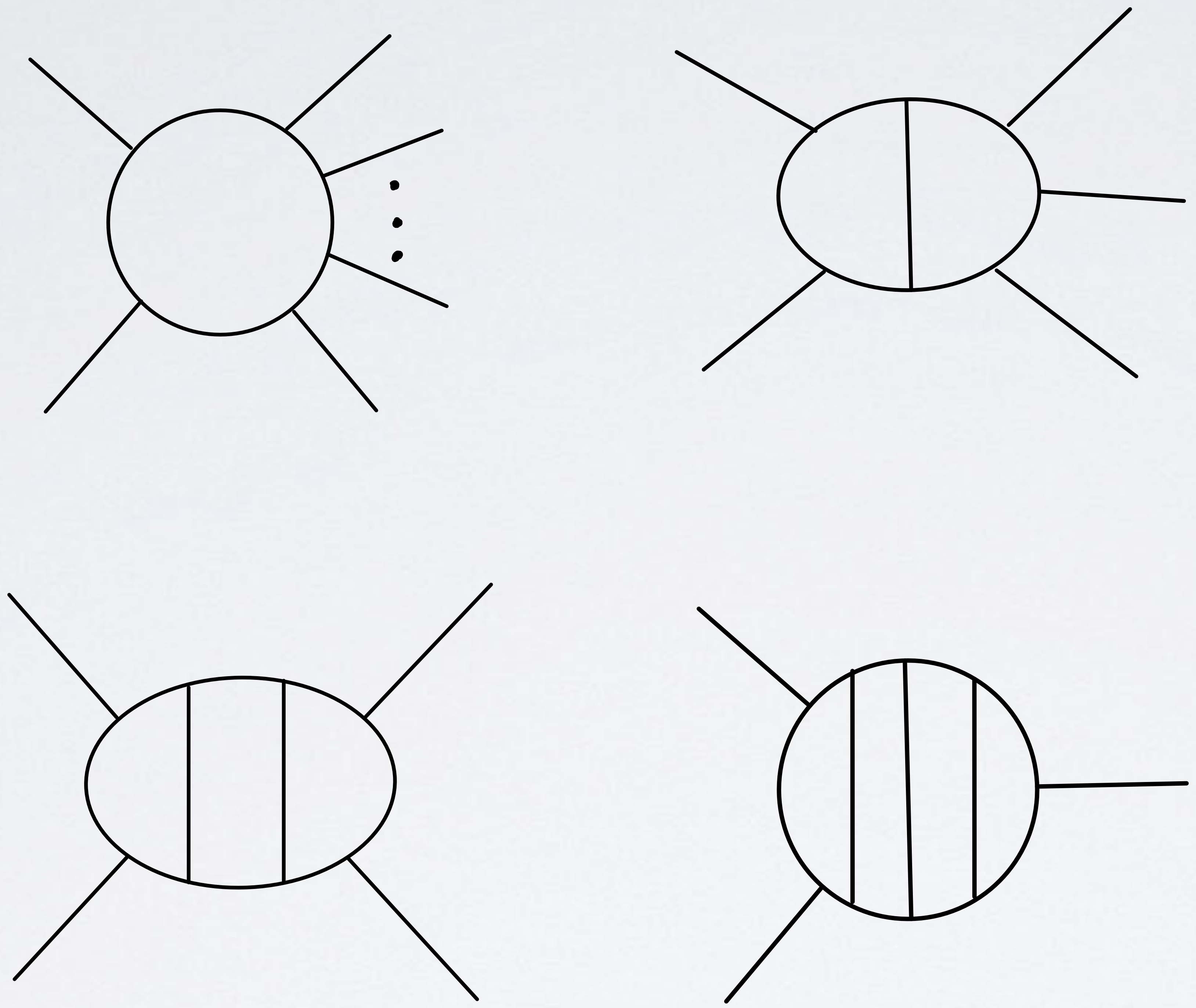
σ[fb]	$\sigma^{(0,0)}$	$\delta\sigma^{(1,0)}$	$\delta\sigma^{(0,1)}$	$\delta\sigma^{(2,0)}$	$\delta\sigma^{(1,1)}$
$q\bar{q}$	1561.42	340.31	−49.907	44.60	−16.80
$\gamma\gamma$	59.645		3.166		
qg		0.060		−32.66	1.03
$q\gamma$			−0.305		−0.207
$g\gamma$					0.2668
gg				1.934	
sum	1621.06	340.37	−47.046	13.88	−15.71

σ [fb]	$\sigma^{(0,0)}$	$\delta\sigma^{(1,0)}$	$\delta\sigma^{(0,1)}$	$\delta\sigma^{(2,0)}$	$\delta\sigma^{(1,1)}$	$\delta\sigma^{(1,1)}_{\text{fact.}}$	$\sigma_{\text{QCD}\times\text{EW}}$
$\Phi^{(1)}$	1169.8	254.3	−30.98	10.18	−10.74	−6.734	$1392.6^{+0.75\%}_{-0\%}$
$\Phi^{(2)}$	368.29	71.91	−11.891	2.85	−4.05	−2.321	$427.1^{+0.41\%}_{-0.02\%}$
$\Phi^{(3)}$	82.08	14.31	−4.094	0.691	−1.01	−0.7137	$91.98^{+0.22\%}_{-0.14\%}$
$\Phi^{(4)} \times 10$	9.107	1.577	−1.124	0.146	−0.206	−0.1946	$9.500^{+0\%}_{-0.97\%}$

- Nested local IR subtractions
- $p_{T,l^\pm} > 30$ GeV, $|y_{l^\pm}| < 2.5$, $\sqrt{p_{T,l^-} - p_{T,l^+}} > 35$ GeV, $\mu_R = \mu_F = m_{ll}/2$
- $\phi^{(i)}$: $(200-300,\ 300-500,\ 500-1500,\ > 1500)$ GeV intervals
- G_μ scheme: G_μ, m_w, m_z input

- Mixed EW-QCD corrections order -1% already for energies > 200 GeV, larger magnitude than NNLO QCD
- Reduction in uncertainty even larger, notably: scale dependence, choice of EW input parameter scheme
- Factorizable corrections dominate at large energies (Sudakov logs)

FULL-COLOR MASSLESS QCD AMPLITUDES



$q\bar{q} \to \gamma\gamma j$

[Agarwal, Buccioni, AvM, Tancredi '21]

$gg \to \gamma\gamma j$

[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

$q\bar{q} \to \gamma\gamma$

[Caola, AvM, Tancredi '20]

$gg \to \gamma\gamma$

[Bargiela, Caola, AvM, Tancredi '21]

$q\bar{q} \to q'\bar{q}'$, $gg \to gg$, $q\bar{q} \to gg$:

[Chakraborty, Caola, Gambuti, AvM, Tancredi '21,'21,ta]

$q\bar{q} \to \gamma^*$, $gg \to H$

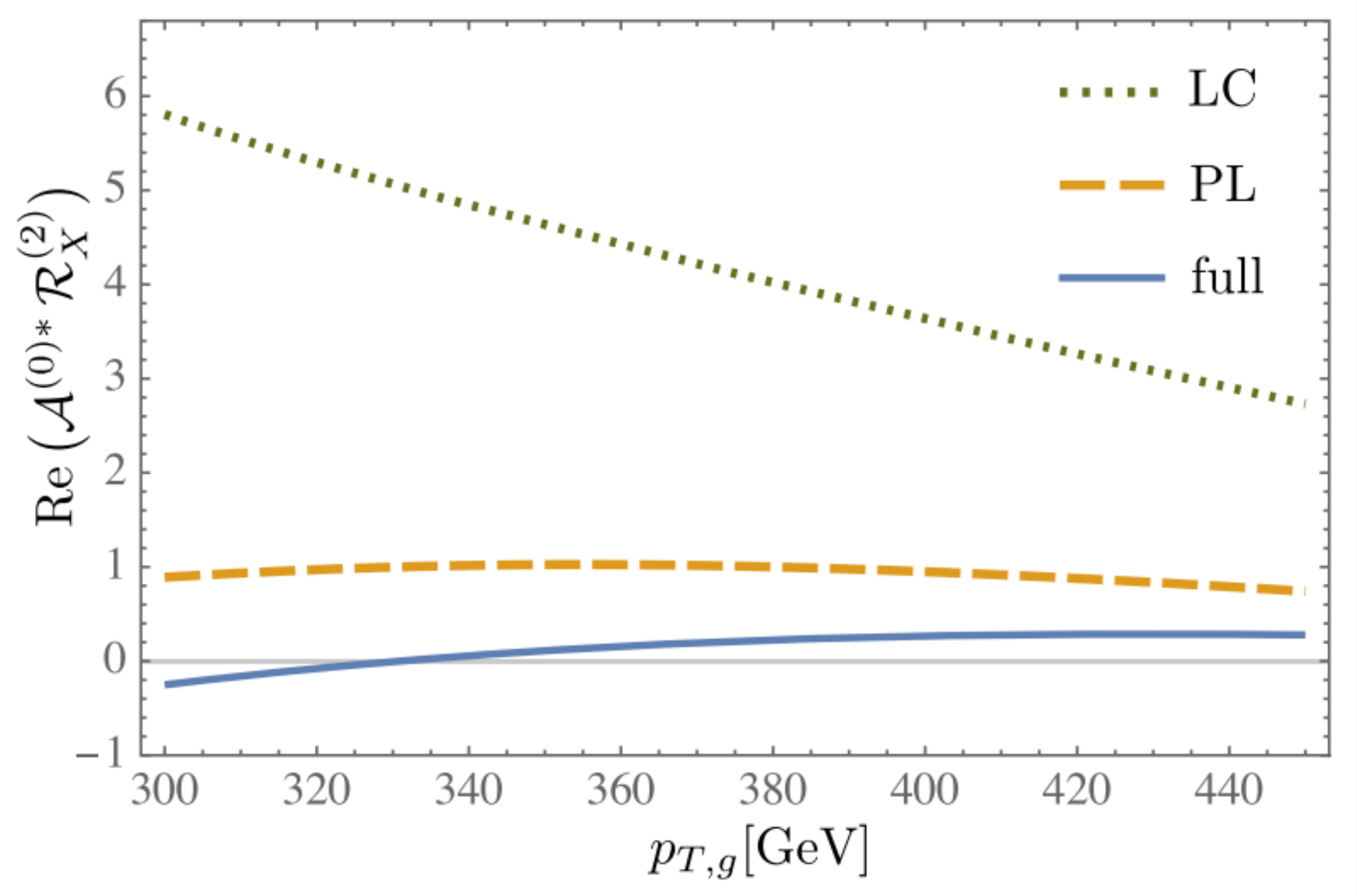
[Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

$b\bar{b} \to H$

[Chakraborty, Huber, Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

$q\bar{q} \to \gamma\gamma g$ @ 2 LOOPS

- Master integrals in public package PentagonFunctions *[Chicherin, Sotnikov '20]*
- $q\bar{q} \to \gamma\gamma g$ helicity amplitudes: *[Agarwal, Buccioni, AvM, Tancredi '21]*
    - Compact analytical results for helicity amplitudes
    - Leading color not obviously a good approximation:



PARTIAL FRACTIONS FOR AMPLITUDES

multivariateapart.nb

Univariate partial fractions separate terms with different poles:

In[1]:= $\text{Apart}\left[\frac{1}{x\,(1+x)},\, x\right]$

Out[1]= $\frac{1}{x} - \frac{1}{1+x}$

Let's consider a multivariate example:

In[2]:= $\text{multi} = \frac{2\,y - x}{y\,(x+y)\,(y-x)};$

Naive iteration introduces spurious poles (here 1/x) for multivariate case:

In[3]:= $\text{Apart}[\text{multi},\, y]$

Out[3]= $\frac{1}{x\,y} + \frac{1}{2\,x\,(-x+y)} - \frac{3}{2\,x\,(x+y)}$

Solution: multivariate partial fractions using methods from polynomial ideal theory:

In[4]:= ``<< MultivariateApart` ``

```
MultivariateApart -- Multivariate partial fractions. By Matthias Heller and Andreas von Manteuffel.
```

In[5]:= `MultivariateApart[multi]`

Out[5]= $-\frac{1}{2\,(x-y)\,y} + \frac{3}{2\,y\,(x+y)}$

Note: minimize denominator degrees ($\neq$ Leinartas)

- PFD: significant ***reduction in size***
- Easy to identify ***linear relations*** between coefficients
- Easy to ***generate fast code*** even for complicated amplitudes
- Representation can be tuned for ***numerical stability*** !

  see $q\bar{q} \to \gamma\gamma j$ @ 2-loops

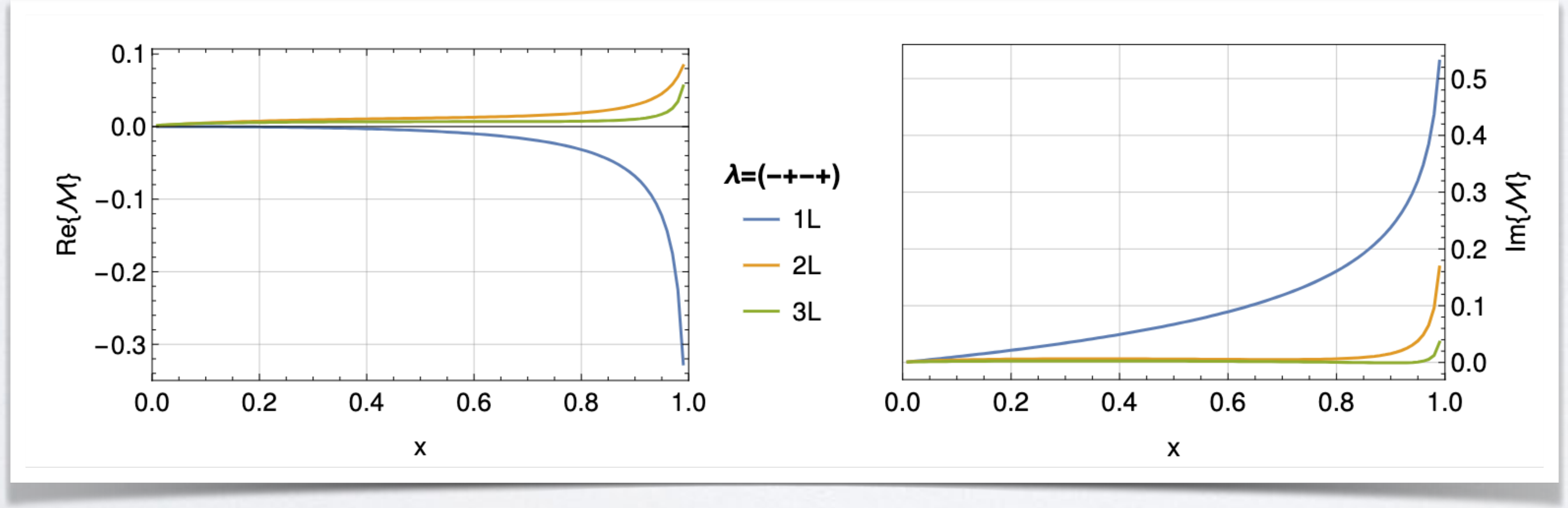
  [Agarwal, Buccioni, AvM, Tancredi '21]

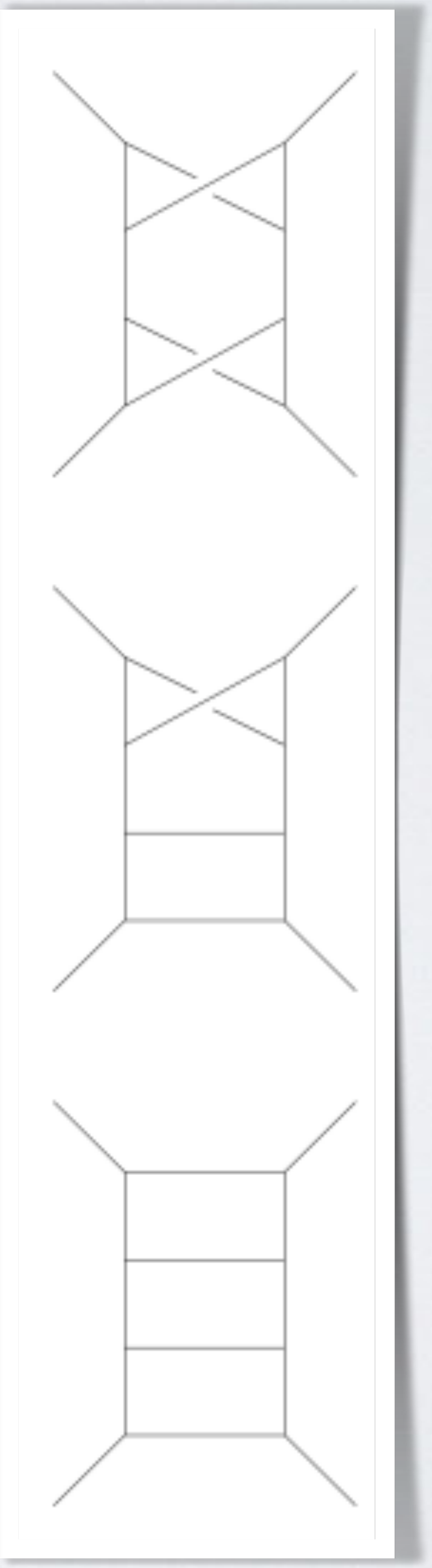
[Pak '11, Abreu et al '19, Boehm et al '20, Heller et al '21, Bendle et al '21]

$gg \to \gamma\gamma$ @ 3 LOOPS

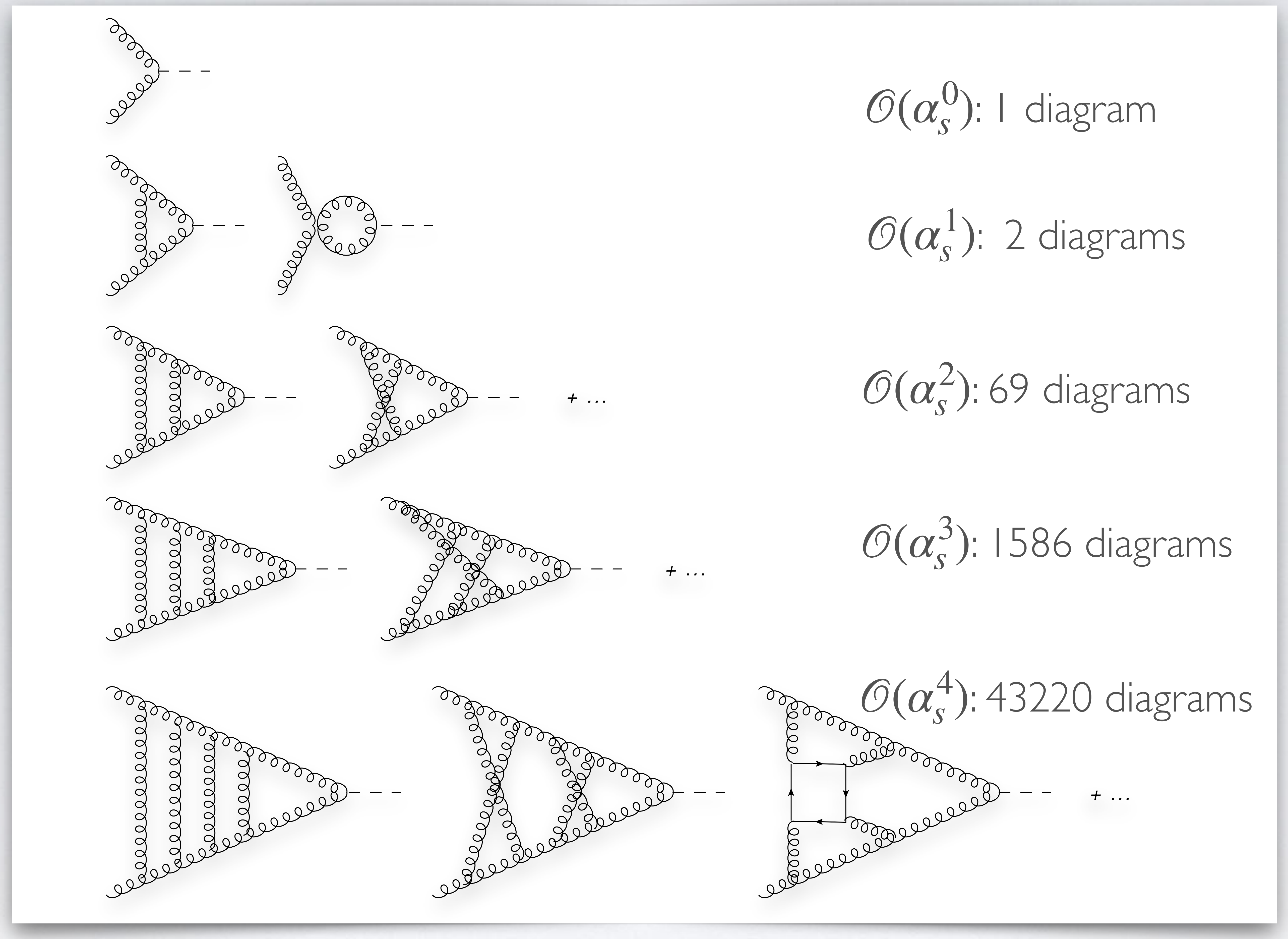
- Master integrals in terms of HPLs: *[Henn, Mistlberger, Smirnov, Wasser ’20]*
- $gg \to \gamma\gamma$ helicity amplitudes: *[Bargiela, Caola, AvM, Tancredi ’21]*
  - Symbolic intermediate expressions sizable but allow for easy crossings, simple workflow
  - Compact analytical results for amplitudes

	1L	2L	3L
Number of diagrams	6	138	3299
Number of inequivalent integral families	1	2	3
Number of integrals before IBPs and symmetries	209	20935	4370070
Number of master integrals	6	39	486
Size of the `Qgraf` result [kB]	4	90	2820
Size of the `Form` result before IBPs and symmetries [kB]	276	54364	19734644
Size of helicity amplitudes written in terms of MIs [kB]	12	562	304409
Size of helicity amplitudes written in terms of HPLs [kB]	136	380	1195





FORM FACTORS @ 4 LOOPS

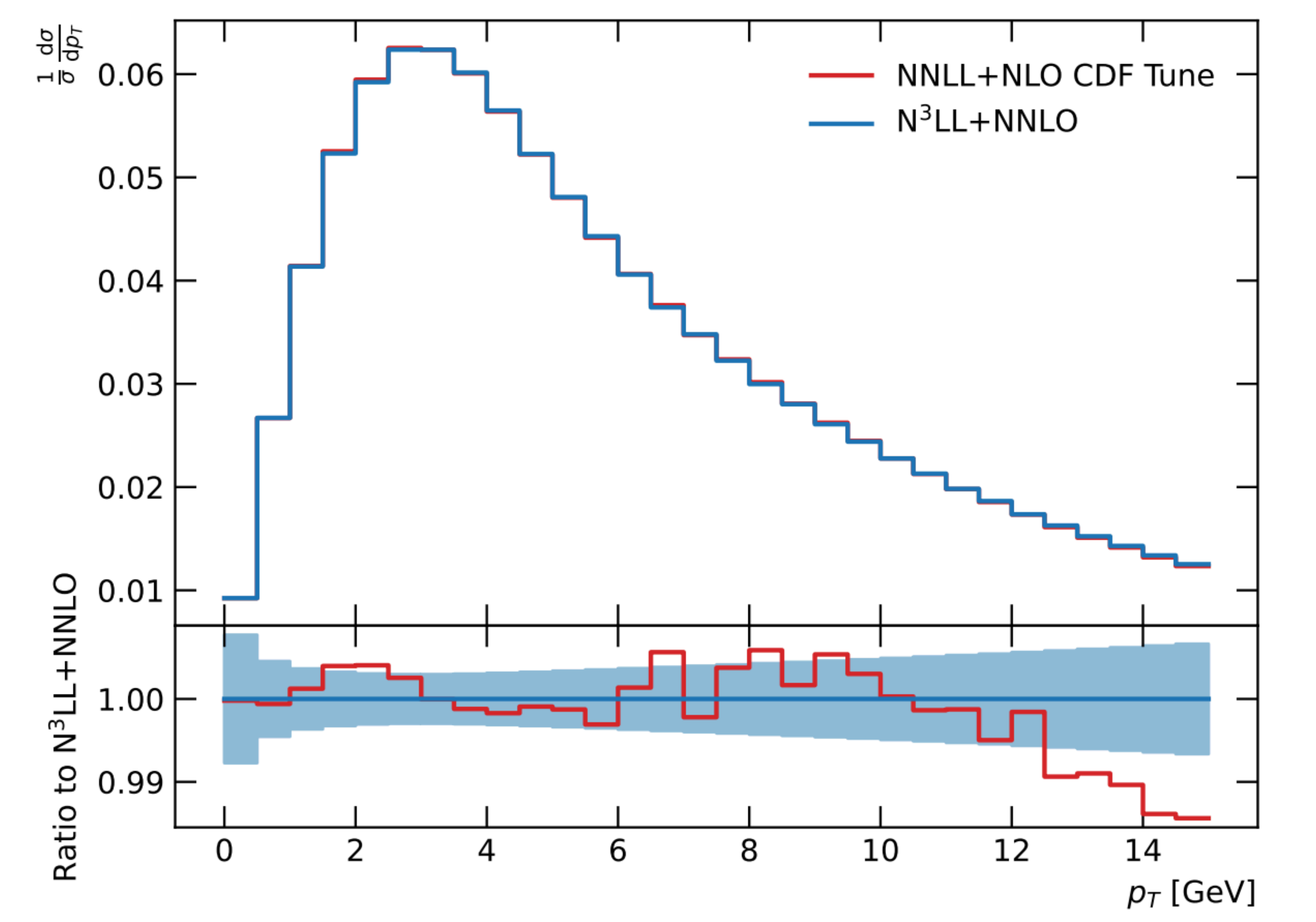


- Consider $q\bar{q}\gamma^*$, ggH, $b\bar{b}H$ form factors at 4-loops:
  - Virtual N4LO for Drell-Yan, Higgs prod./decay
  - Universal IR features: cusp ($1/\epsilon^2$) and collinear ($1/\epsilon$) anomalous dimensions

ANOMALOUS DIMS & RESUMMATION

Order	Anomalous Dimension γ_i (B)	Γ_{cusp} (A)	Fixed Order Matching (Y)
LL	-	1-loop	-
NLL	1-loop	2-loop	-
NLL' (+ NLO)	1-loop	2-loop	α_s
NNLL (+ NLO)	2-loop	3-loop	α_s
NNLL' (+ NNLO)	2-loop	3-loop	α_s^2
N^3LL (+ NNLO)	3-loop	4-loop	α_s^2
N^3LL' (+ N^3LO)	3-loop	4-loop	α_s^3
N^4LL (+ N^3LO)	4-loop	5-loop	α_s^3

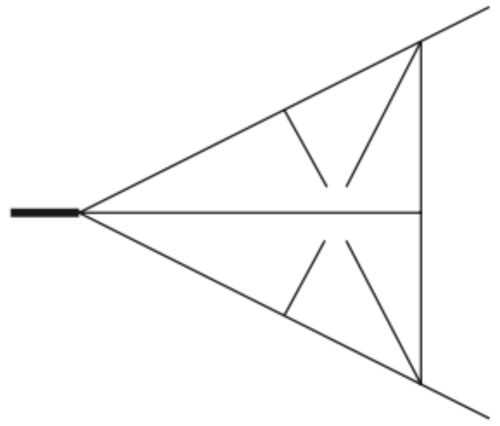
[Isaacson, Fu, Yuan '21]



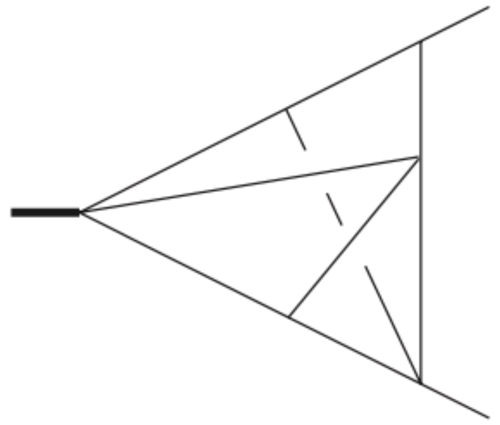
- W at small pT: important for mass measurement at hadron colliders
- Fixed order breaks in this regime, requires resummation
- N3LL or higher needs four-loop cusp anomalous dim., some further works:
  - Hbb @ N3LL *[Ajjath, Chakraborty, Das, Mukherjee, Ravindran '19]* + many more
  - Energy-energy correlation @ N4LL *[Duhr, Mistlberger, Vita '21; Moult, Zhu, Zhu '21]*

IBP DETAILS

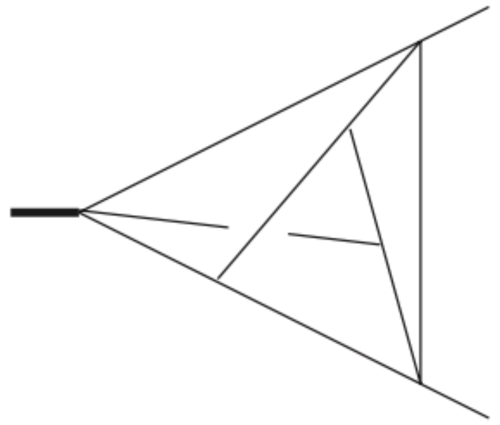
- Reduction of dots: “no-numerator syzygies” in Lee-Pomeransky rep. *[Lee ‘14; Bitoun, Bogner, Klausen, Panzer ’17]*
    - Need ***higher-order annihilators***.
- Reduction of numerators: mostly plain Laporta, some with “no-dot syzygies” in Baikov rep. *[Gluza, Kajda, Kosower ‘11; Schabinger ’11; Its ’15; Larsen, Zhang ’15; Böhm, Georgoudis, Larsen Schulze, Zhang ’18; …]*
    - Used ***linear algebra approach*** *[Agarwal, Jones, AvM ’20]*.
- O(25k) sectors, up to ***O(10^8) eqs. per sector***
- Reduction tables: several TB compressed (checksums!)
- Inter-sector relation:



$$= \frac{4(2d-7)}{3d-11}$$



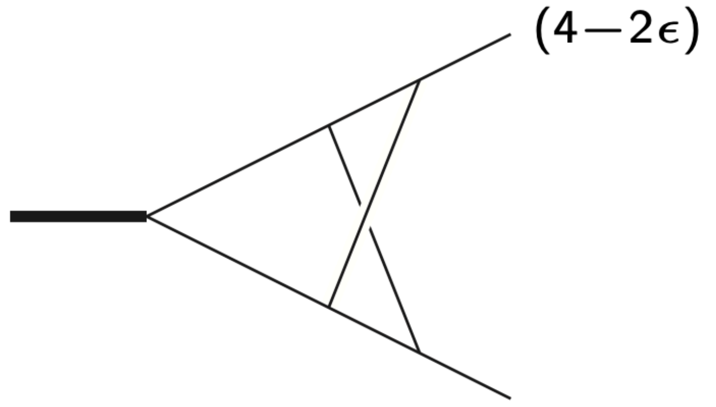
$$+ \frac{5(5-d)}{3d-11}$$



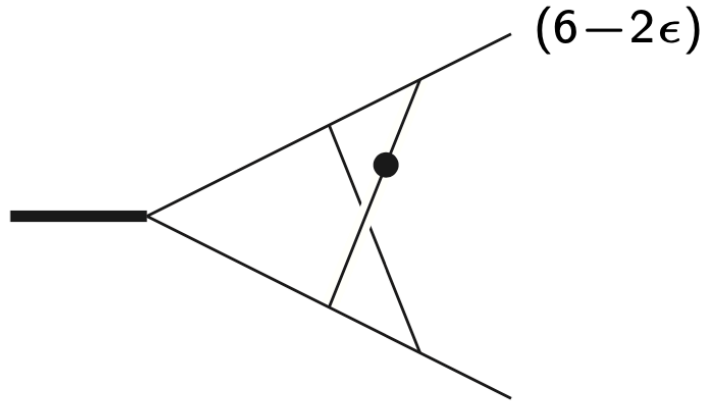
$$+ \text{subsectors},$$

METHOD OF FINITE INTEGRALS

- Observation *[Panzer 2014; AvM, Panzer, Schabinger 2014]*:
  - any ***divergent*** loop integral expressible in terms of **finite** basis integrals



$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$

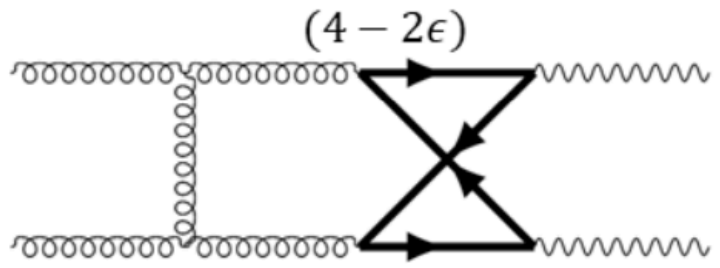
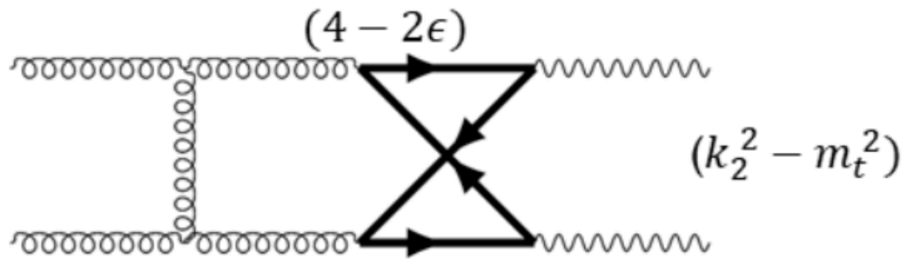
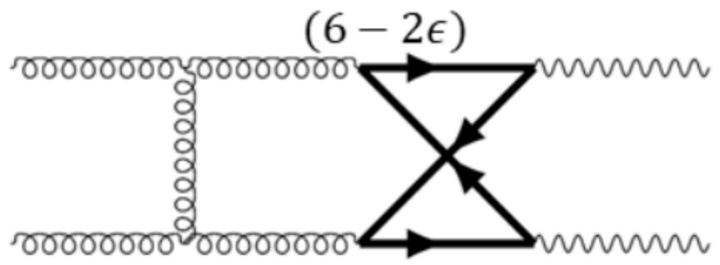
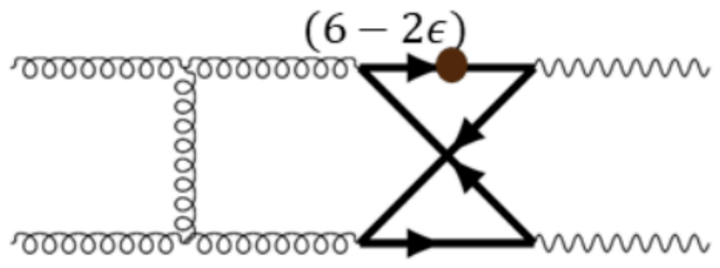


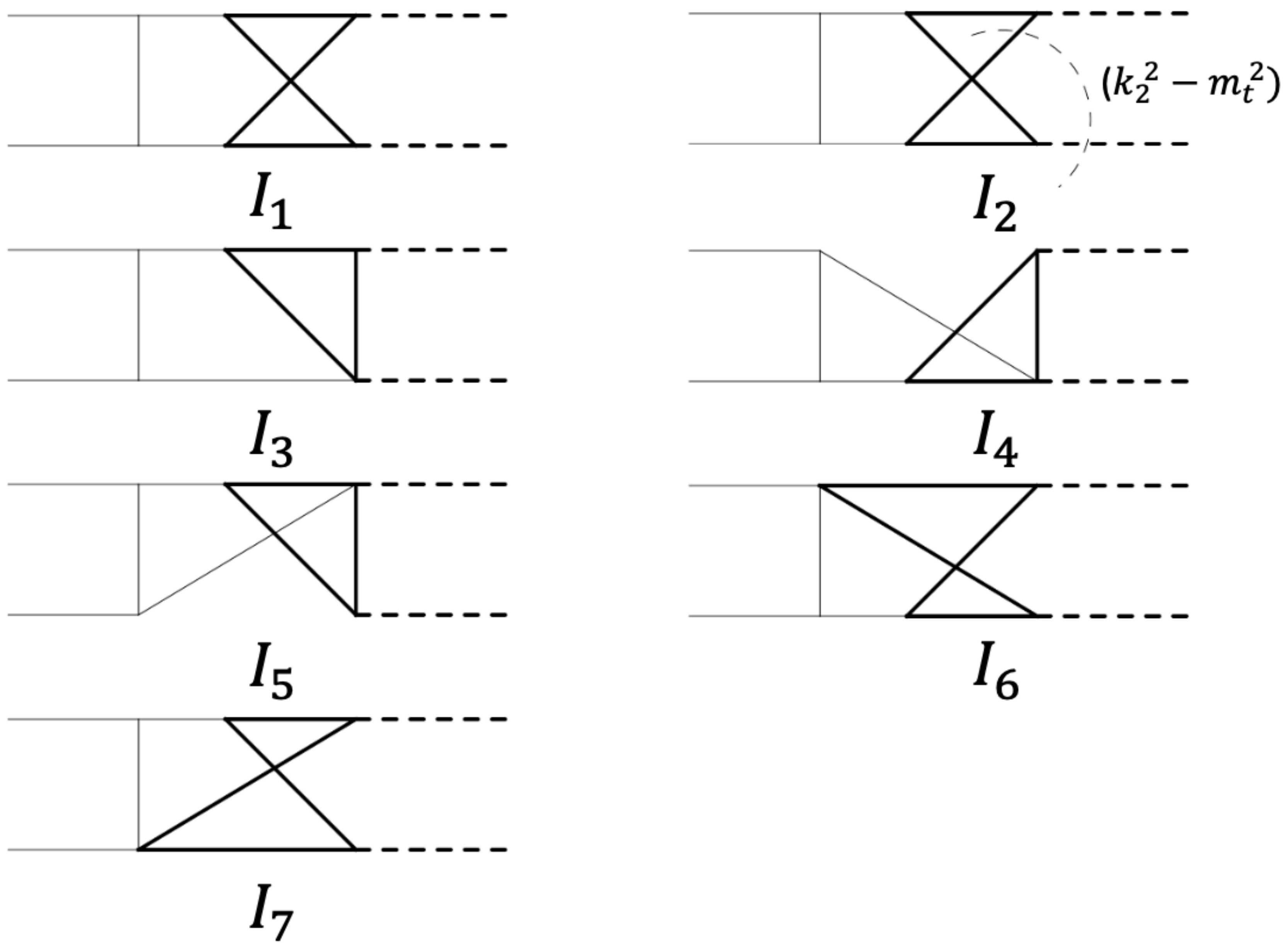
$$+\ldots$$

- Expand integrands of ***finite*** integrals around $\epsilon = (4-d)/2 \approx 0$
  - If linearly reducible: integrate ***analytically*** with HyperInt *[Panzer 2014]*
  - Improved ***numerical*** evaluations, used for HH *[Borowka, Greiner, Heinrich, Jones, Kerner '16]*, Hj *[Jones, Kerner, Lusioni '18]*, ZH *[Chen, Davies, Heinrich, Jones, Kerner, Mishima, Schlenk, Steinhauser '22]* …

talks: Long Chen, Vitalii Materia

GENERALIZED FINITE INTEGRALS

Integral	Rel.Err.	Timing(s)
	~2*10^-3	45
	~4*10^-2	63
	~8*10^-6	55
	~8*10^-4	60
Linear combination	~1*10^-4	18



$$I = (m_z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_z^2 - t)I_7$$

$$I(\nu_1, \ldots, \nu_N) = (-1)^{r+\Delta t}\,\Gamma(\nu - L\,d/2) \int \left(\prod_{j\in\mathcal{N}_T} \mathrm{d}x_j\right) \left(\prod_{j\in\mathcal{N}_t} \frac{x^{\nu_j - 1}}{\Gamma(\nu_j)}\right) \delta\left(1 - \sum_{j\in\mathcal{N}_T} x_j\right)$$
$$\left[\left(\prod_{j\in\mathcal{N}_{\backslash T}} \frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}}\right) \left(\prod_{j\in\mathcal{N}_{\Delta t}} \frac{\partial^{|\nu_j|+1}}{\partial x_j^{|\nu_j|+1}}\right) \frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu - L\,d/2}}\right]_{x_j=0\ \forall\ j\in\mathcal{N}_{\backslash T}} \qquad (\nu_j \in \mathbb{Z}).$$

[Agarwal, AvM, Jones 2020]

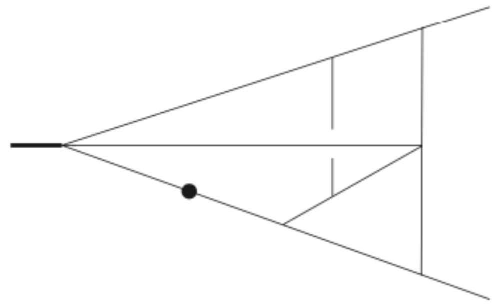
Supported by pySecDec

[Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017]

“NICE” FINITE INTEGRALS

- Example: 10 terms in ϵ for weight 6 in conventional basis:

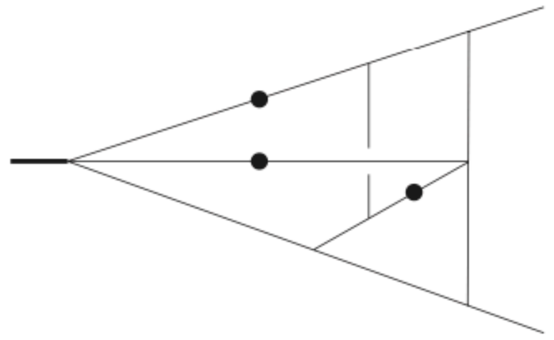
(4−2ϵ)



$$
\begin{aligned}
= &\frac{1}{\epsilon^8}\left(-\frac{1}{144}\right) + \frac{1}{\epsilon^7}\left(-\frac{1}{12}\right) + \frac{1}{\epsilon^6}\left(\frac{1}{24}\zeta_2 - \frac{7}{36}\right) + \frac{1}{\epsilon^5}\left(\frac{29}{24}\zeta_3 + \frac{1}{2}\zeta_2 - \frac{1}{72}\right) \\
&+ \frac{1}{\epsilon^4}\left(\frac{71}{16}\zeta_2^2 + \frac{29}{2}\zeta_3 + \frac{39}{16}\zeta_2 + \frac{335}{144}\right) + \frac{1}{\epsilon^3}\left(\frac{1819}{24}\zeta_5 - \frac{23}{6}\zeta_2\zeta_3 + \frac{213}{4}\zeta_2^2 + \frac{1211}{24}\zeta_3 + \frac{431}{48}\zeta_2\right. \\
&\left.+ \frac{47}{18}\right) + \frac{1}{\epsilon^2}\left(-\frac{1285}{24}\zeta_3^2 + \frac{80579}{1008}\zeta_2^3 + \frac{1819}{2}\zeta_5 - 46\zeta_2\zeta_3 + \frac{25787}{160}\zeta_2^2 + \frac{417}{8}\zeta_3 - \frac{1175}{48}\zeta_2 - \frac{7277}{72}\right) \\
&+ \frac{1}{\epsilon}\left(\frac{434203}{192}\zeta_7 - \frac{7139}{24}\zeta_2\zeta_5 - \frac{54139}{120}\zeta_2^2\zeta_3 - \frac{1285}{2}\zeta_3^2 + \frac{80579}{84}\zeta_2^3 + \frac{5571}{2}\zeta_5 - \frac{9005}{24}\zeta_2\zeta_3 + \frac{967}{480}\zeta_2^2\right. \\
&\left.- \frac{4045}{8}\zeta_3 - \frac{733}{24}\zeta_2 + \frac{57635}{72}\right) - \frac{2023}{12}\zeta_{5,3} - \frac{30581}{4}\zeta_3\zeta_5 - \frac{6829}{24}\zeta_2\zeta_3^2 + \frac{45893321}{100800}\zeta_2^4 + \frac{434203}{16}\zeta_7 \\
&- \frac{7139}{2}\zeta_2\zeta_5 - \frac{54139}{10}\zeta_2^2\zeta_3 - \frac{10706}{3}\zeta_3^2 + \frac{7987951}{3360}\zeta_2^3 + \frac{1309}{12}\zeta_5 - \frac{30317}{24}\zeta_2\zeta_3 - \frac{43847}{96}\zeta_2^2 + \frac{32335}{24}\zeta_3 \\
&+ \frac{2553}{4}\zeta_2 - \frac{334727}{72} + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

- Only 1 term for weight 6 for a nice finite integral:

(6−2ϵ)



$$
= -\frac{3}{2}\zeta_3^2 - \frac{4}{3}\zeta_2^3 + 10\zeta_5 + 2\zeta_2\zeta_3 - \frac{1}{5}\zeta_2^2 - 6\zeta_3 + \mathcal{O}(\epsilon)
$$

ANALYTICAL CUSP ANOMALOUS DIMENSION

$$\Gamma_4^r = N_f^3 C_R \left(\frac{64}{27}\zeta_3 - \frac{32}{81}\right)$$
$$+ N_f^2 C_A C_R \left(-\frac{224}{15}\zeta_2^2 + \frac{2240}{27}\zeta_3 - \frac{608}{81}\zeta_2 + \frac{923}{81}\right)$$
$$+ N_f^2 C_F C_R \left(\frac{64}{5}\zeta_2^2 - \frac{640}{9}\zeta_3 + \frac{2392}{81}\right)$$
$$+ N_f C_A^2 C_R \left(\frac{2096}{9}\zeta_5 + \frac{448}{3}\zeta_3\zeta_2 - \frac{352}{15}\zeta_2^2 - \frac{23104}{27}\zeta_3 + \frac{20320}{81}\zeta_2 - \frac{24137}{81}\right)$$
$$+ N_f C_A C_F C_R \left(160\zeta_5 - 128\zeta_3\zeta_2 - \frac{352}{5}\zeta_2^2 + \frac{3712}{9}\zeta_3 + \frac{440}{3}\zeta_2 - \frac{34066}{81}\right)$$
$$+ N_f C_F^2 C_R \left(-320\zeta_5 + \frac{592}{3}\zeta_3 + \frac{572}{9}\right)$$
$$+ N_f \frac{d_F^{abcd} d_R^{abcd}}{N_R} \left(-\frac{1280}{3}\zeta_5 - \frac{256}{3}\zeta_3 + 256\zeta_2\right)$$
$$+ \frac{d_A^{abcd} d_R^{abcd}}{N_R} \left(-384\zeta_3^2 - \frac{7936}{35}\zeta_2^3 + \frac{3520}{3}\zeta_5 + \frac{128}{3}\zeta_3 - 128\zeta_2\right)$$
$$+ C_A^3 C_R \left(-16\zeta_3^2 - \frac{20032}{105}\zeta_2^3 - \frac{3608}{9}\zeta_5 - \frac{352}{3}\zeta_3\zeta_2 + \frac{3608}{5}\zeta_2^2 + \frac{20944}{27}\zeta_3 - \frac{88400}{81}\zeta_2 + \frac{84278}{81}\right)$$

where $R = F, A$ for $r = q, g$. Note the quartic Casimir (dd) contributions.

$$\Gamma_4^{\mathcal{N}=4} = \frac{d_A^{abcd} d_A^{abcd}}{N_A}\left(-384\zeta_3^2 - \frac{7936}{35}\zeta_2^3\right) + C_A^4\left(-16\zeta_3^2 - \frac{20032}{105}\zeta_2^3\right),$$

$$\ln(F) = \sum_{L=1}^{\infty} a^L z^{L\epsilon}\left(-\frac{\Gamma_L}{2(L\epsilon)^2} - \frac{\mathcal{G}_L}{2L\epsilon} + L_L^{\text{fin}}\right)$$

N=4 SYM: *[Henn, Mistlberger, Korchemsky '19; Huber, AvM, Panzer, Schabinger, Yang '19]*

- Wilson line method (with a conjecture): *[Henn, Peraro, Stahlhofen, Wasser '19; Brüser, Grozin, Henn, Stahlhofen '19; Henn, Mistlberger, Korchemsky '19]*
- Quartics from form factors: *[Lee, Smirnov, Smirnov, Steinhauser '19]*
- Full calculation from QCD form factors: *[AvM, Panzer, Schabinger '20]*

ANALYTICAL COLLINEAR ANOMALOUS DIMENSIONS

$$
\begin{aligned}
\gamma_4^q = {} & C_F^4\Big(11760\,\zeta_7 - 768\,\zeta_5\zeta_2 + \frac{256}{5}\zeta_3\zeta_2^2 - 2304\,\zeta_3^2 - \frac{33776}{35}\zeta_2^3 - 5040\,\zeta_5 - 240\,\zeta_3\zeta_2 - \frac{1368}{5}\zeta_2^2 + 4008\,\zeta_3 - 900\,\zeta_2 + \frac{4873}{12}\Big) \\
& + C_F^3 C_A\Big(-21840\,\zeta_7 + 4128\,\zeta_5\zeta_2 + \frac{512}{5}\zeta_3\zeta_2^2 + 6440\,\zeta_3^2 + \frac{634376}{315}\zeta_2^3 - 1952\,\zeta_5 - \frac{3976}{3}\zeta_3\zeta_2 + \frac{8668}{5}\zeta_2^2 - 6520\,\zeta_3 + 2334\,\zeta_2 - \frac{2085}{2}\Big) \\
& + C_F^2 C_A^2\Big(17220\,\zeta_7 - 4208\,\zeta_5\zeta_2 - \frac{128}{5}\zeta_3\zeta_2^2 - \frac{14204}{3}\zeta_3^2 - \frac{43976}{35}\zeta_2^3 + \frac{10708}{9}\zeta_5 + \frac{4192}{9}\zeta_3\zeta_2 - \frac{48680}{27}\zeta_2^2 + \frac{259324}{27}\zeta_3 - \frac{93542}{27}\zeta_2 + \frac{29639}{18}\Big) \\
& + C_F C_A^3\Big(-\frac{45511}{6}\zeta_7 + \frac{1648}{3}\zeta_5\zeta_2 - \frac{4132}{15}\zeta_3\zeta_2^2 + \frac{5126}{9}\zeta_3^2 - \frac{77152}{315}\zeta_2^3 + \frac{175166}{27}\zeta_5 + \frac{15400}{9}\zeta_3\zeta_2 + \frac{186742}{135}\zeta_2^2 - \frac{1751224}{243}\zeta_3 + \frac{1062149}{729}\zeta_2 + \frac{7179083}{26244}\Big) \\
& + \frac{d_F^{abcd}d_A^{abcd}}{N_F}\Big(3484\,\zeta_7 + 1024\,\zeta_5\zeta_2 - \frac{736}{5}\zeta_3\zeta_2^2 - \frac{3344}{3}\zeta_3^2 + \frac{27808}{315}\zeta_2^3 - \frac{1840}{9}\zeta_5 - 1792\,\zeta_3\zeta_2 + \frac{224}{15}\zeta_2^2 - \frac{7808}{9}\zeta_3 - \frac{2176}{3}\zeta_2 + 192\Big) \\
& + n_f C_F^3\Big(368\,\zeta_3^2 - \frac{117344}{315}\zeta_2^3 + \frac{3872}{3}\zeta_5 - \frac{512}{3}\zeta_3\zeta_2 - \frac{668}{5}\zeta_2^2 - \frac{1120}{9}\zeta_3 + 322\,\zeta_2 + \frac{27949}{108}\Big) \\
& + n_f C_F^2 C_A\Big(-\frac{3400}{3}\zeta_3^2 + \frac{5744}{35}\zeta_2^3 - \frac{4472}{3}\zeta_5 + \frac{3904}{9}\zeta_3\zeta_2 + \frac{105488}{135}\zeta_2^2 - \frac{23518}{81}\zeta_3 + \frac{673}{27}\zeta_2 - \frac{1092511}{972}\Big) \\
& + n_f C_F C_A^2\Big(\frac{6916}{9}\zeta_3^2 + \frac{24184}{315}\zeta_2^3 + \frac{6088}{27}\zeta_5 - \frac{3584}{9}\zeta_3\zeta_2 - \frac{17164}{45}\zeta_2^2 + \frac{140632}{243}\zeta_3 - \frac{445117}{729}\zeta_2 + \frac{326863}{1944}\Big) \\
& + n_f\frac{d_F^{abcd}d_F^{abcd}}{N_F}\Big(\frac{1216}{3}\zeta_3^2 + \frac{9472}{315}\zeta_2^3 - \frac{21760}{9}\zeta_5 + 128\,\zeta_3\zeta_2 - \frac{320}{3}\zeta_2^2 - \frac{5312}{9}\zeta_3 + \frac{4544}{3}\zeta_2 - 384\Big) \\
& + n_f^2 C_F^2\Big(\frac{1040}{9}\zeta_5 - \frac{224}{9}\zeta_3\zeta_2 - \frac{8032}{135}\zeta_2^2 - \frac{4232}{81}\zeta_3 + \frac{1972}{27}\zeta_2 + \frac{9965}{486}\Big) \\
& + n_f^2 C_F C_A\Big(-\frac{1184}{9}\zeta_5 + \frac{256}{9}\zeta_3\zeta_2 + \frac{152}{15}\zeta_2^2 + \frac{14872}{243}\zeta_3 + \frac{41579}{729}\zeta_2 - \frac{97189}{17496}\Big) \\
& + n_f^3 C_F\Big(\frac{128}{135}\zeta_2^2 + \frac{1424}{243}\zeta_3 + \frac{16}{27}\zeta_2 - \frac{37382}{6561}\Big)
\end{aligned}
$$

$$
\gamma_4^{\mathcal{N}=4} = -300\,\zeta_7 - 256\,\zeta_5\zeta_2 - 384\,\zeta_4\zeta_3 + \frac{1}{N_c^2}\Big[5226\,\zeta_7 + 1536\,\zeta_5\zeta_2 - 552\,\zeta_4\zeta_3\Big]
$$

(N=4 planar color: *[Dixon '17]*)

$$
\begin{aligned}
\gamma_4^g = {} & C_A^4\Big(-\frac{2671}{6}\zeta_7 - \frac{896}{3}\zeta_5\zeta_2 - \frac{2212}{15}\zeta_3\zeta_2^2 - \frac{286}{9}\zeta_3^2 - \frac{674696}{945}\zeta_2^3 + \frac{19232}{27}\zeta_5 + \frac{1588}{3}\zeta_3\zeta_2 + \frac{249448}{135}\zeta_2^2 + \frac{36380}{243}\zeta_3 - \frac{1051411}{729}\zeta_2 + \frac{10672040}{6561}\Big) \\
& + \frac{d^A_{abcd}d^A_{abcd}}{N_A}\Big(3484\,\zeta_7 + 1024\,\zeta_5\zeta_2 - \frac{736}{5}\zeta_3\zeta_2^2 - \frac{3344}{3}\zeta_3^2 + \frac{39776}{315}\zeta_2^3 + \frac{2720}{9}\zeta_5 - 2336\,\zeta_3\zeta_2 - \frac{1808}{15}\zeta_2^2 - \frac{12512}{9}\zeta_3 + 64\,\zeta_2 + \frac{128}{9}\Big) \\
& + n_f C_A^3\Big(-\frac{596}{9}\zeta_3^2 + \frac{148976}{945}\zeta_2^3 + \frac{16066}{27}\zeta_5 + 148\,\zeta_3\zeta_2 - \frac{69502}{135}\zeta_2^2 - \frac{260822}{243}\zeta_3 + \frac{155273}{729}\zeta_2 - \frac{421325}{1944}\Big) \\
& + n_f C_A^2 C_F\Big(152\,\zeta_3^2 + \frac{5632}{315}\zeta_2^3 + \frac{8}{9}\zeta_5 - 176\,\zeta_3\zeta_2 - \frac{1196}{45}\zeta_2^2 + \frac{29606}{81}\zeta_3 + \frac{3023}{9}\zeta_2 - \frac{903983}{972}\Big) \\
& + n_f C_A C_F^2\Big(-80\,\zeta_3^2 - \frac{320}{7}\zeta_2^3 - \frac{1600}{3}\zeta_5 + \frac{148}{5}\zeta_2^2 + \frac{1592}{3}\zeta_3 - 2\,\zeta_2 + \frac{685}{12}\Big) + n_f C_F^3\Big(46\Big) \\
& + n_f\frac{d^A_{abcd}d^F_{abcd}}{N_A}\Big(\frac{1216}{3}\zeta_3^2 - \frac{14464}{315}\zeta_2^3 - \frac{30880}{9}\zeta_5 + 1216\,\zeta_3\zeta_2 + \frac{2464}{15}\zeta_2^2 + \frac{2560}{9}\zeta_3 - 64\,\zeta_2 + \frac{448}{9}\Big) \\
& + n_f^2 C_A^2\Big(-\frac{1024}{9}\zeta_5 - 32\,\zeta_3\zeta_2 + \frac{3128}{135}\zeta_2^2 + \frac{37354}{243}\zeta_3 - \frac{13483}{729}\zeta_2 + \frac{611939}{17496}\Big) \\
& + n_f^2 C_A C_F\Big(\frac{304}{9}\zeta_5 + \frac{32}{3}\zeta_3\zeta_2 + \frac{128}{45}\zeta_2^2 - \frac{1688}{81}\zeta_3 - \frac{172}{9}\zeta_2 + \frac{1199}{18}\Big) + n_f^2 C_F^2\Big(-\frac{352}{9}\zeta_3 + \frac{676}{27}\Big) \\
& + n_f^2\frac{d^F_{abcd}d^F_{abcd}}{N_A}\Big(\frac{1024}{3}\zeta_3 - \frac{1408}{9}\Big) + n_f^3 C_A\Big(\frac{256}{135}\zeta_2^2 - \frac{400}{243}\zeta_3 - \frac{16}{81}\zeta_2 - \frac{15890}{6561}\Big) + n_f^3 C_F\Big(\frac{308}{243}\Big)
\end{aligned}
$$

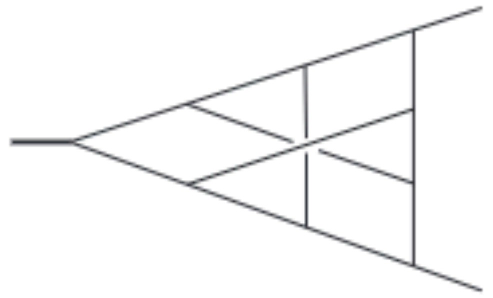
[Agarwal, AvM, Panzer, Schabinger '21]

ANALYTICAL FORM FACTORS @ 4-LOOP QCD

- Partial results for finite parts of form factors @ 4-loop QCD:
  [Henn, Smirnov, Smirnov, Steinhauser ’16; Henn, Smirnov, Smirnov, Steinhauser, Lee ’16; Lee, Smirnov, Smirnov, Steinhauser ’17, ’19]
- Partial results for finite parts of for factors @ 4-loop QCD:
  [AvM, Schabinger ’16, ’19, ’19]
- Complete form factors @ 4-loop QCD:
  [Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser ’21, ’22; Chakraborty, Huber, Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser ’22]
- See also:
  - Recent results for form factors with masses + singlet contrib. @ 3-loop QCD:
    [Fael, Lange, Schönwald, Steinhauser ’22; Czakon, Niggetiedt ’20; Chen, Czakon, Niggetiedt ’21; Gehrmann, Primo ’21]
  - First steps towards inclusive H cross section at 4th order (soft-collinear contributions):
    [Moch, Ruijl, Ueda, Vermaseren, Vogt ’17, ’18; Das, Moch, Vogt ’19, ’20]

METHOD OF DIFFERENTIAL EQUATIONS

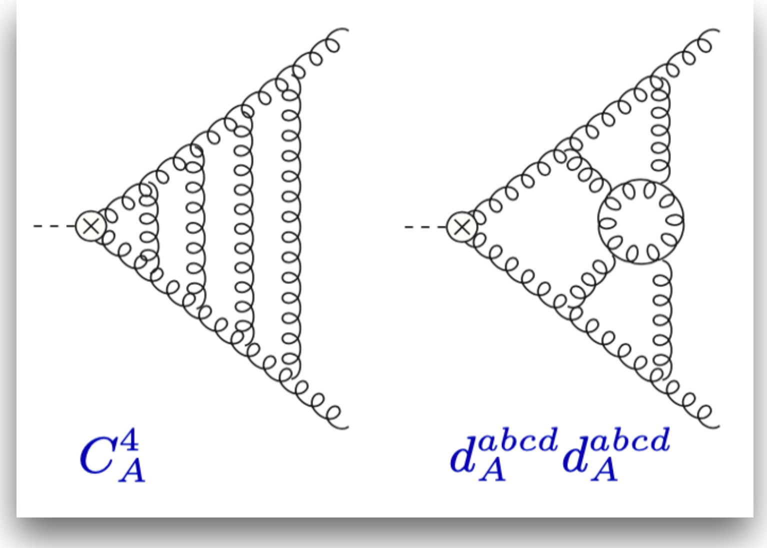
- Take a second leg off-shell, $x = q_1^2/q^2$,
  transport from x=1 (propagator) to x=0 (one-scale FF) *[Henn, Smirnov, Smirnov '13]*
- Reductions with Fire 6 *[A.V. Smirnov, Chukharev '19]*, canonical form *[Henn '13]* with Libra *[Lee '20]*
- Example topology with singularities at x = 0, 1, -1,1/4, 4: *[Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]*
  - 2-scale letters: $\frac{1}{x-1}, \quad \frac{1}{x+1}, \quad \frac{1}{x-4}, \quad \frac{1}{x-1/4}, \quad \frac{1}{(1-x)\sqrt{x}}, \quad \frac{1}{x\sqrt{x-1/4}}, \quad \frac{1}{x\sqrt{1/x-1/4}}$
  - 1-scale G(…, 1) with weights $0, \quad \pm 1, \quad \pm i\sqrt{3}, \quad e^{\pm i\pi/3}, \quad e^{\pm 2i\pi/3}, \quad e^{\pm i\pi/3}/2$ mapped to MZVs

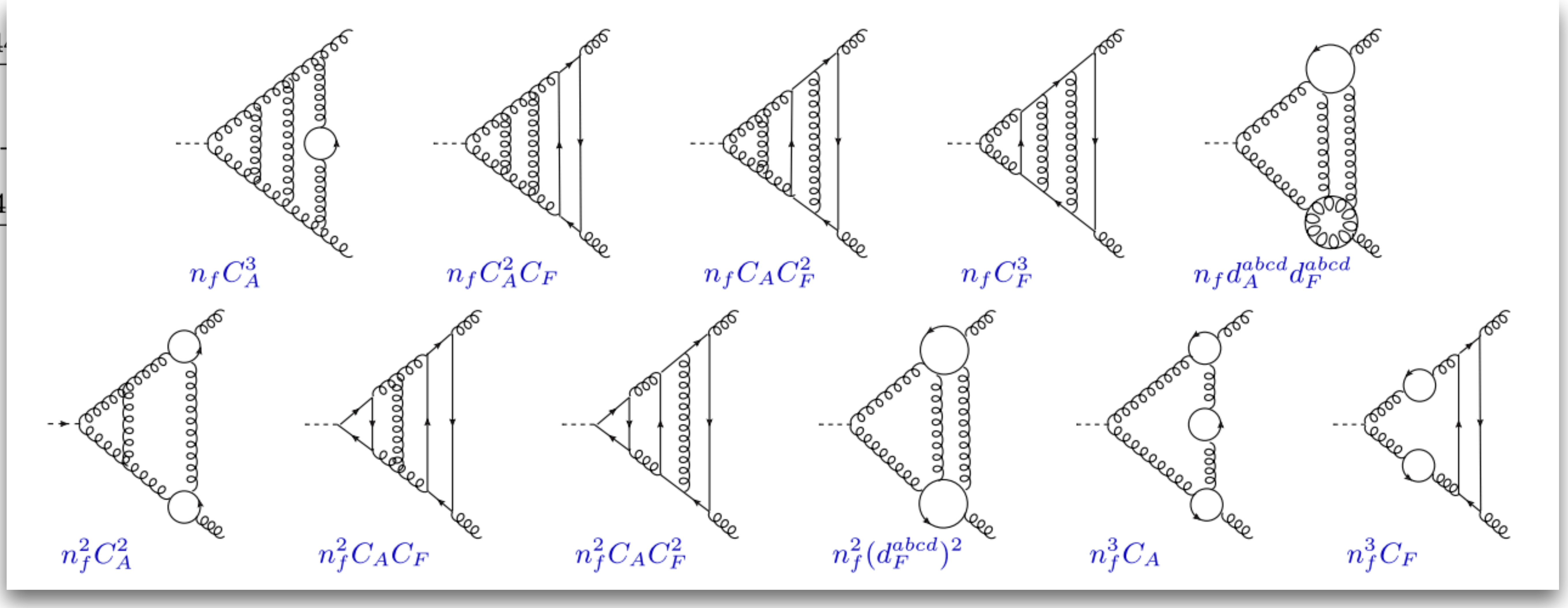


$$= \frac{1}{\epsilon^8}\left(\frac{7}{18}\right) + \frac{1}{\epsilon^7}\left(\frac{55}{24}\right) + \frac{1}{\epsilon^6}\left(-\frac{67}{9}\zeta_2 - \frac{797}{144}\right) + \frac{1}{\epsilon^5}\left(-\frac{442}{9}\zeta_3 - \frac{643}{18}\zeta_2 + \frac{1193}{144}\right) + \frac{1}{\epsilon^4}\left(-\frac{9199}{360}\zeta_2^2 - \frac{3547}{18}\zeta_3 + \frac{7793}{72}\zeta_2 + \frac{1013}{48}\right) + \frac{1}{\epsilon^3}\left(-\frac{2858}{3}\zeta_5 + \frac{27617}{36}\zeta_3\zeta_2 - \frac{3439}{180}\zeta_2^2 + \frac{60893}{72}\zeta_3 - \frac{1897}{8}\zeta_2 - \frac{43895}{144}\right) + \frac{1}{\epsilon^2}\left(\frac{179927}{72}\zeta_3^2 - \frac{40853}{252}\zeta_2^3 - 2780\zeta_5 + \frac{23467}{9}\zeta_3\zeta_2 + \frac{132359}{180}\zeta_2^2 - \frac{66607}{24}\zeta_3 - \frac{5423}{72}\zeta_2 + \frac{311383}{144}\right) + \frac{1}{\epsilon}\left(-\frac{1015395}{32}\zeta_7 + \frac{30493}{2}\zeta_5\zeta_2 + \frac{274199}{90}\zeta_3\zeta_2^2 + \frac{44984}{9}\zeta_3^2 - \frac{540823}{420}\zeta_2^3 + \frac{477281}{24}\zeta_5 - \frac{412181}{36}\zeta_3\zeta_2 - \frac{117101}{30}\zeta_2^2 + \frac{410629}{72}\zeta_3 + \frac{400999}{72}\zeta_2 - \frac{622069}{48}\right) + \frac{122261}{15}\zeta_{5,3} + \frac{1298525}{12}\zeta_5\zeta_3 - \frac{942899}{36}\zeta_3^2\zeta_2 - \frac{121150681}{9000}\zeta_2^4 - \frac{2558101}{16}\zeta_7 + \frac{360793}{6}\zeta_5\zeta_2 - \frac{53821}{18}\zeta_3\zeta_2^2 - \frac{1428953}{72}\zeta_3^2 + \frac{2037031}{168}\zeta_2^3 - \frac{1989461}{24}\zeta_5 + \frac{526387}{12}\zeta_3\zeta_2 + \frac{245017}{18}\zeta_2^2 + \frac{738547}{72}\zeta_3 - \frac{1198061}{24}\zeta_2 + \frac{10519199}{144} + \mathcal{O}(\epsilon), \quad (6)$$

ggH FORM FACTOR @ 4 LOOPS

$$
\begin{aligned}
F^{\text{fin}}_{g,4} = {} & C_A^4\Big(-\frac{181}{30}\zeta_{5,3}+\frac{2377}{6}\zeta_5\zeta_3+\frac{271}{9}\zeta_3^2\zeta_2+\frac{4583689}{27000}\zeta_2^4-\frac{224939}{72}\zeta_7+\frac{5423}{6}\zeta_5\zeta_2+\frac{18931}{90}\zeta_3\zeta_2^2+\frac{418801}{162}\zeta_3^2+\frac{353093}{1620}\zeta_2^3+\frac{1203647}{135}\zeta_5-\frac{1806605}{486}\zeta_3\zeta_2-\frac{778313}{5832}\zeta_2^2-\frac{47586469}{1944}\zeta_3+\frac{32379341}{104976}\zeta_2+\frac{5165679667}{139968}\Big)\\
& +\frac{d_A^{abcd}d_A^{abcd}}{N_A}\Big(260\,\zeta_{5,3}-5092\,\zeta_5\zeta_3-16\,\zeta_3^2\zeta_2-\frac{496766}{525}\zeta_2^4-\frac{6776}{3}\zeta_7-5016\,\zeta_5\zeta_2+\frac{2992}{3}\zeta_3\zeta_2^2+\frac{31588}{3}\zeta_3^2+\frac{1073972}{945}\zeta_2^3-6460\,\zeta_5+\frac{6752}{9}\zeta_3\zeta_2+\frac{24616}{45}\zeta_2^2+\frac{68410}{9}\zeta_3-\frac{4682}{27}\zeta_2-\frac{1310}{\cdots}\Big)\\
& +n_fC_A^3\Big(-\frac{8390}{9}\zeta_7+\frac{991}{9}\zeta_5\zeta_2-\frac{2129}{45}\zeta_3\zeta_2^2-\frac{32425}{324}\zeta_3^2-\frac{702253}{5670}\zeta_2^3+\frac{566977}{540}\zeta_5+\frac{67831}{162}\zeta_3\zeta_2-\frac{2333729}{29160}\zeta_2^2+\frac{9686917}{1944}\zeta_3+\frac{113944685}{104976}\zeta_2-\frac{20463665839}{839808}\Big)\\
& +n_fC_A^2C_F\Big(\frac{16003}{12}\zeta_7+\frac{230}{9}\zeta_5\zeta_2-\frac{44}{15}\zeta_3\zeta_2^2-\frac{1787}{3}\zeta_3^2+\frac{32254}{945}\zeta_2^3+\frac{143197}{36}\zeta_5+\frac{78590}{81}\zeta_3\zeta_2-\frac{44839}{540}\zeta_2^2+\frac{8317937}{1944}\zeta_3-\frac{293267}{3888}\zeta_2-\frac{573672965}{46656}\Big)\\
& +n_fC_AC_F^2\Big(-\frac{9580}{3}\zeta_7-300\,\zeta_5\zeta_2+12\,\zeta_3\zeta_2^2-368\,\zeta_3^2-\frac{39328}{945}\zeta_2^3-\frac{92317}{18}\zeta_5+\frac{193}{3}\zeta_3\zeta_2-5\,\zeta_2^2+\frac{700879}{108}\zeta_3-\frac{217}{36}\zeta_2+\frac{1156175}{1296}\Big)\\
& +n_fC_F^3\Big(3360\,\zeta_7-2940\,\zeta_5-156\,\zeta_3+\frac{169}{2}\Big)\\
& +n_f\frac{d_A^{abcd}d_F^{abcd}}{N_A}\Big(\frac{2464}{3}\zeta_7+1824\,\zeta_5\zeta_2-\frac{1088}{3}\zeta_3\zeta_2^2-\frac{15700}{3}\zeta_3^2-\frac{245536}{945}\zeta_2^3+\frac{108692}{9}\zeta_5+\frac{154\cdots}{9}\cdots\\
& +n_f^2C_A^2\Big(\frac{9452}{81}\zeta_3^2+\frac{15044}{945}\zeta_2^3-\frac{38071}{135}\zeta_5+\frac{3113}{486}\zeta_3\zeta_2+\frac{78953}{3240}\zeta_2^2+\frac{1103621}{1944}\zeta_3-\frac{25105537}{104976}\zeta_2\cdots\\
& +n_f^2C_AC_F\Big(-270\,\zeta_3^2-\frac{10084}{945}\zeta_2^3-\frac{23572}{27}\zeta_5-\frac{944}{9}\zeta_3\zeta_2-\frac{764}{135}\zeta_2^2-\frac{724883}{486}\zeta_3-\frac{4790}{27}\zeta_2+4\cdots\\
& +n_f^2C_F^2\Big(\frac{800}{3}\zeta_3^2+\frac{13696}{945}\zeta_2^3+\frac{3920}{3}\zeta_5+\frac{32}{3}\zeta_3\zeta_2-\frac{212}{15}\zeta_2^2-1592\,\zeta_3+\frac{58}{9}\zeta_2+\frac{32137}{216}\Big)\\
& +n_f^2\frac{d_F^{abcd}d_F^{abcd}}{N_A}\Big(512\,\zeta_3^2-960\,\zeta_5+\frac{384}{5}\zeta_2^2+1520\,\zeta_3-\frac{9008}{9}\Big)\\
& +n_f^3C_A\Big(-\frac{194}{15}\zeta_5+\frac{124}{27}\zeta_3\zeta_2-\frac{944}{405}\zeta_2^2-\frac{17818}{243}\zeta_3+\frac{9430}{729}\zeta_2-\frac{8399887}{52488}\Big)\\
& +n_f^3C_F\Big(\frac{640}{27}\zeta_5-\frac{64}{9}\zeta_3\zeta_2+\frac{112}{45}\zeta_2^2+\frac{4060}{27}\zeta_3+\frac{64}{3}\zeta_2-\frac{233953}{972}\Big)
\end{aligned}
$$



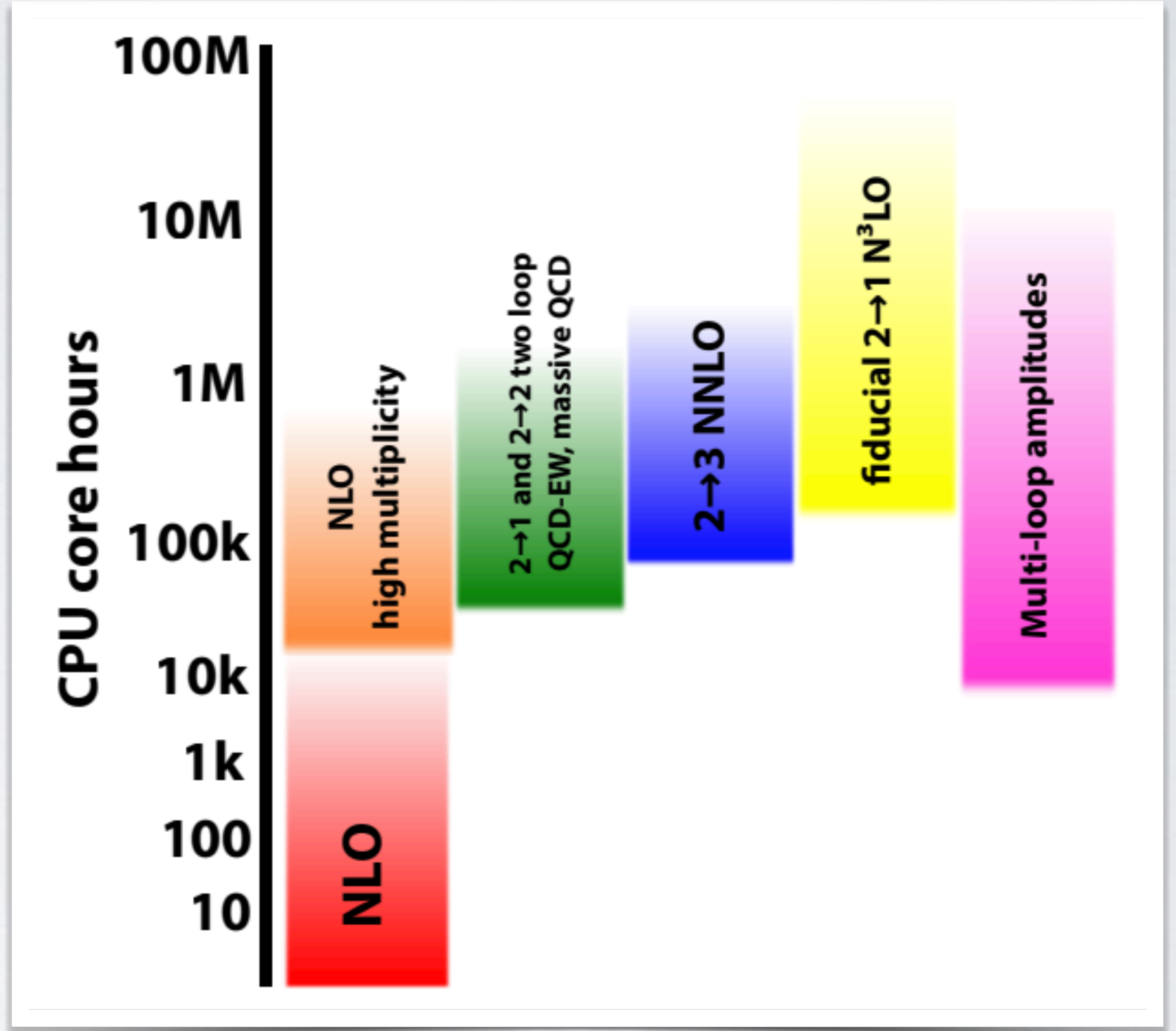


[Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov '21,'22]

Note: all master integrals mapped to finite basis and checked to at least 10^{-4} relative precision with Fiesta 5 *[A.V. Smirnov, Shapurov, Vysotsky '21]*

STATUS AND PROSPECTS FOR e^+e^- PRECISION GOALS

- A lot of overlap with recent technology for LHC processes (mixed EW-QCD, higher loops, …)
- **Numerical methods** easier to automate, avoid expression swell
- **Analytical insights** can enable much better numerical performance
- **Finite-field methods** crucial in many current calculations
- Challenging e^+e^- calculations motivate:
  - Improved **IBP reductions** (fast, low memory, automated)
  - Improved **integral evaluation** (fast, reliable, automated)
  - Improved γ_5 **treatment** (rigorous, automated)



From: *Snowmass survey of 53 recent perturbative calculations* *[Febres-Cordero, AvM, Neumann '22]*