Elliptic integrals

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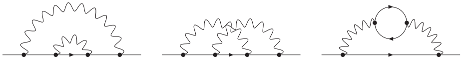
Introduction

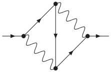
- The simplest Feynman integrals evaluate to multiple polylogarithms
  - all one-loop integrals
  - many of the higher loop integrals we have calculated
- The next-to-simplest Feynman integrals involve an **elliptic curve** (content of this talk).
- There are even more complicated Feynman integrals (several elliptic curves, Calabi-Yau's, not covered here).

We do not have to go very far to encounter elliptic integrals in precision calculations: The simplest example is the two-loop electorn self-energy in QED:

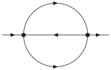
There are three Feynman diagrams contributing to the two-loop electron self-energy in QED with a single fermion:



All master integrals are (sub-) topologies of the **kite graph**:



One sub-topology is the **sunrise graph** with three equal non-zero masses:



(Sabry, ’62)

Fantastic Beasts and Where to Find Them

Where is the elliptic curve?

For the sunrise it's very simple: The second graph polynomial defines an elliptic curve in Feynman parameter space:

$$-p^2 x_1 x_2 x_3 + (x_1 + x_2 + x_3)(x_1 x_2 + x_2 x_3 + x_3 x_1) m^2 = 0.$$

Fantastic Beasts and Where to Find Them

Indications for elliptic Feynman integrals:

- Maximal cut with a square root of a cubic or quartic polynomial
- Irreducible second-order differential operator
- Direct integration: An integration step with a square root of a cubic or quartic polynomial
- A differential equation in dlog form, with algebraic arguments, where not all square roots can be rationalised

Comments for the experts

- Maximal cut:
  - The elliptic curve from the maximal cut for the sunrise integral is not isomorphic to the one from Feynman parameters. They are isogenic, meaning one lattice is a sub-lattice of the other.
- Irreducible second-order differential operator:
  - Not every irreducibly second-order differential operator is the Picard-Fuchs operator of an elliptic curve. The ones which are are tabulated.
    (Movasati and Reiter, '09)
- Direct integration/dlog form:
  - This does not necessarily imply that the result cannot be expressed in polylogs.

Drell-Yan not rationalisable:	Besier, Festi, Harrison, Naskrecki, '19;
Drell-Yan in polylogs:	Heller, von Manteuffel, Schabinger, '19;
dlog not polylogarithmic:	Brown, Duhr, '20

Which approach?

- Physics is about numbers.
- Best strategy depends on the number of kinematic variables the process depends on.
- For a **physical observable** we usually only need a few digits for the highest term in perturbation theory.
- For **amplitudes** we may need quadruple precision in singular limits (soft/collinear).
- For **master integrals** / **special functions** we may want $O(100) - O(1000)$ digits to use PSLQ.

Options

From **many scales / automated** to **fewer scales / fast evaluations**:

- Purely numerical: Sector decomposition, numerical integration in loop momentum space.
- Semi-numerical: Unitarity methods, numerical integration of a differential equation
- Semi-analytical: Expansion in a small parameter
- Analytical: Reduction to standarised special functions

This talk: Reduction to standarised special functions, which we can evaluate fast to $\mathcal{O}(100) - \mathcal{O}(1000)$ digits.

Section 1

Algebraic curves of genus zero and one

Algebraic curves

- Ground field $\mathbb{C}$
- **Algebraic curve** in $\mathbb{CP}^2$ **defined by** a **homogeneous polynomial** $P(x,y,z)$:

$$P(x,y,z) = 0$$

  We usually work in the chart $z = 1$.
- If the curve is smooth and d the degree of P the **genus** of the curve is given by

$$g = \frac{1}{2}(d-1)(d-2).$$

Elliptic curves

Definition (Elliptic curve over $\mathbb{C}$)

An algebraic curve in $\mathbb{CP}^2$ of genus one with one marked point.

Example (Weierstrass normal form)

In the chart $z = 1$:

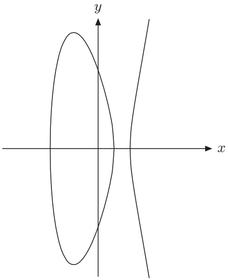
$$y^2 = 4x^3 - g_2 x - g_3$$

Example (Quartic form)

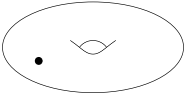
In the chart $z = 1$:

$$y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

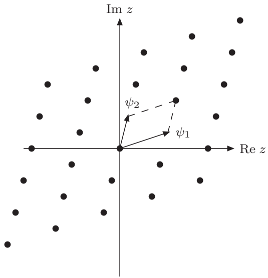
Three shades of an elliptic curve



Complex algebraic curve
$y^2 = 4x^3 - g_2 x - g_3$



Real Riemann surface of genus one with one marked point



Complex plane modulo lattice: $\mathbb{C}/\Lambda$

Periods of an elliptic curve

Example

The Legendre form:

$$y^2 = x(x-1)(x-\lambda)$$

The periods are

$$\psi_1 = 2\int\limits_0^\lambda \frac{dx}{y} = 4K\left(\sqrt{\lambda}\right) \qquad \psi_2 = 2\int\limits_1^\lambda \frac{dx}{y} = 4iK\left(\sqrt{1-\lambda}\right)$$

$K(x)$: complete elliptic integral of the first kind

Notation

Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$, where

$$\tau = \frac{\psi_2}{\psi_1}$$

and require $\mathrm{Im}(\tau) > 0$.

Definition (The complex upper half-plane)

$$\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$$

Definition (The nome squared)

$$\bar{q} = e^{2\pi i \tau}$$

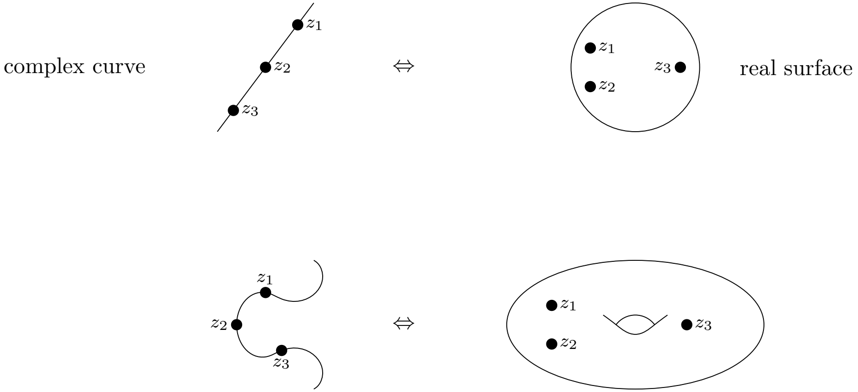
For $\tau \in \mathbb{H}$ we have $|\bar{q}| < 1$.

Section 2

Moduli spaces

Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.



Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3$.

Sphere has a unique shape

Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$

Coordinates are $(\mathbf{z_1}, ..., \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.

One coordinate describes the shape of the torus

Use translation to fix $z_n = 0$

Coordinates are $(\tau, \mathbf{z_1}, ..., \mathbf{z_{n-1}})$

Section 3

Iterated integrals

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda = \gamma^* \omega_j.$$

The iterated integral is defined by

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) = \int\limits_0^\lambda d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) ... \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Chen '77

Translation to physics

- Manifold M: kinematic space, coordinates are the kinematic variables.
- $\gamma(0)$: Boundary point
- $\gamma(1)$: Point,where we would like to evaluate the integral.

Standard special functions

- **Defined by giving the ω's**.
- Alternatively, we may specify primitives of the differential one-forms:

$$\omega = d\Omega.$$

  Ω is related to the symbol.
- May do this for genus zero and genus one.

Iterated integrals on $\mathcal{M}_{0,n}$

We are interested in differential one-forms, which have only simple poles:

$$\omega^{\mathrm{mpl}}(z_j) = \frac{dy}{y - z_j}.$$

Multiple polylogarithms:

$$G(z_1, \dots, z_k; y) = \int_0^y \frac{dy_1}{y_1 - z_1} \int_0^{y_1} \frac{dy_2}{y_2 - z_2} \dots \int_0^{y_{k-1}} \frac{dy_k}{y_k - z_k}, \quad z_k \neq 0$$

Iterated integrals on $\mathcal{M}_{1,n}$

- Coordinates are $(\tau, z_1, ..., z_{n-1})$
- Decompose an arbitrary path along $d\tau$ and dz_j
- Two classes of iterated integrals:
  1. Integration along z
  2. Integration along τ
- What are the differential one-forms we want to integrate?

Differential one-forms on $\mathcal{M}_{1,n}$

1. *From modular forms ($f_k(\tau)$ modular form):*

$$\omega_k^{\mathrm{modular}} = 2\pi i \, f_k(\tau) d\tau$$

2. *From the Kronecker function:*

$$\omega_k^{\mathrm{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}(z,\tau)\, dz + (k-1)\, g^{(k)}(z,\tau) \frac{d\tau}{2\pi i} \right]$$

Modular forms

A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

1. f transforms under modular transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}_2(\mathbb{Z})$$

2. f is holomorphic on $\mathbb{H}$,
3. f is holomorphic at $i\infty$.

Modular forms for congruence subgroups: Require transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

The Kronecker function

Define the **first Jacobi theta function** $\theta_1(z, \bar{q})$ by

$$\theta_1(z, \bar{q}) = -i \sum_{n=-\infty}^{\infty} (-1)^n \bar{q}^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{i\pi(2n+1)z}.$$

The **Kronecker function** $F(z, \alpha, \tau)$:

$$F(z, \alpha, \tau) = \theta_1'(0, \bar{q}) \frac{\theta_1(z+\alpha, \bar{q})}{\theta_1(z, \bar{q})\,\theta_1(\alpha, \bar{q})} = \frac{1}{\alpha} \sum_{k=0}^{\infty} \mathbf{g}^{(\mathbf{k})}(\mathbf{z}, \tau) \alpha^k$$

We are interested in the coefficients $g^{(k)}(z, \tau)$ of the Kronecker function.

The coefficients $g^{(k)}(z,\tau)$ of the Kronecker function

$\bar{q}$-expansion:

$$
\begin{aligned}
g^{(0)}(z,\tau) &= 1, \\
g^{(1)}(z,\tau) &= -2\pi i \left[\frac{1+\bar{w}}{2(1-\bar{w})} + \overline{\mathrm{E}}_{0,0}(\bar{w};1;\bar{q}) \right], \\
g^{(k)}(z,\tau) &= -\frac{(2\pi i)^k}{(k-1)!} \left[-\frac{B_k}{k} + \overline{\mathrm{E}}_{0,1-k}(\bar{w};1;\bar{q}) \right], \qquad k>1,
\end{aligned}
$$

where $\bar{w} = \exp(2\pi i z)$, B_k denotes the k-th Bernoulli number and

$$
\begin{aligned}
\overline{\mathrm{E}}_{n;m}(\bar{w};\bar{v};\bar{q}) &= \mathrm{ELi}_{n;m}(\bar{w};\bar{v};\bar{q}) - (-1)^{n+m}\mathrm{ELi}_{n;m}\left(\bar{w}^{-1};\bar{v}^{-1};\bar{q}\right) \\
\mathrm{ELi}_{n;m}(\bar{w};\bar{v};\bar{q}) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\bar{w}^j}{j^n}\frac{\bar{v}^k}{k^m}\bar{q}^{jk}
\end{aligned}
$$

Iterated integrals on $\mathcal{M}_{1,n}$: Integration along z

Differential one-forms:

$$\omega_k^{\text{Kronecker},z}(z_j,\tau) = (2\pi i)^{2-k} g^{(k-1)}(z-z_j,\tau)\,dz$$

Elliptic multiple polylogarithms:

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_r \\ z_1 & \dots & z_r \end{smallmatrix}; z; \tau\right) = (2\pi i)^{n_1+\dots+n_r-r}\, I\left(\omega_{n_1+1}^{\text{Kronecker},z}(z_1,\tau), \dots, \omega_{n_r+1}^{\text{Kronecker},z}(z_r,\tau); z\right)$$

Broedel, Duhr, Dulat, Tancredi, '17

- $\tau = \text{const}$
- meromorphic version, only simple poles
- not double periodic!

Comments on elliptic multiple polylogarithms

- There exist non-equivalent definitions of elliptic multiple polylogarithms.
- Desirable properties are
  - double periodic
  - meromorphic
  - only simple poles
- It is not possible to have all three properties.
  - double periodic + meromorphic
    Levin, Racinet, '07
  - double periodic + only simple poles
    Brown, Levin, '11
  - meromorphic + only simple poles
    Broedel, Duhr, Dulat, Tancredi, '17

More variants of elliptic polylogarithms

Some authors consider iterated integrals of the form

$$\int\limits_0^{y_0} dy \; f(y) G(z_1, \dots, z_{k-1}; y),$$

where **only the outermost integration is non-polylogarithmic**, for example

$$f(y) = \frac{1}{\sqrt{(y-z_1)(y-z_2)(y-z_3)(y-z_4)}}$$

Can be reduced to $\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_r \\ z_1 & \dots & z_r \end{smallmatrix}; z; \tau \right)$.

Remiddi, Tancredi, '17;
Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm, '17;
Hidding, Moriello, '17;

Iterated integrals on $\mathcal{M}_{1,n}$: Integration along τ

Differential one-forms:

$$\omega_k^{\mathrm{Kronecker},\tau}(z_j) = \frac{(k-1)}{(2\pi i)^{k-2}} g^{(k)}(z_j,\tau) \frac{d\tau}{2\pi i} = \frac{(k-1)}{(2\pi i)^{k}} g^{(k)}(z_j,\tau) \frac{d\bar{q}}{\bar{q}}$$

- Integrate in $\bar{q}$
- No poles in $0 < |\bar{q}| < 1$.
- Possibly a simple pole at $\bar{q} = 0$ (“trailing zero”)

Comments on integration on $\mathcal{M}_{1,n}$

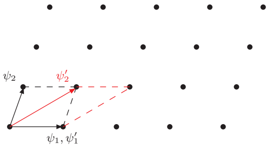
- It is advantageous to integrate in τ:
  - Analytic expressions shorter
  - Easier to evaluate numerically
- Boundary condition at $\tau = i\infty$:
  - Elliptic curve degenerates, geometric genus equals zero
  - Feynman integrals expressible in terms of multiple polylogarithms

Section 4

Modular transformations

Modular transformations

Let's assume we choose as periods (ψ_2, ψ_1), while somebody else made the choice (ψ_2', ψ_1').



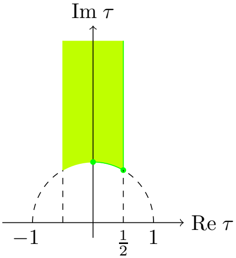
The two choices are related by a (2×2)-matrix γ:

$$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \gamma \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$

This is called a **modular transformation**.

Transformation should be invertible: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

Modular transformations



By a modular transformation we may map τ to the fundamental domain, resulting in

$$|\bar{q}| \leq e^{-\pi\sqrt{3}} \approx 0.0043,$$

resulting in a fast converging series.

Modular transformations

Let us now consider the transformation on $\mathcal{M}_{1,n}$

$$\tau' = \gamma(\tau), \quad z_j' = \frac{z_j}{c\tau + d}, \quad \textit{with} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Modular transformations

Let f be a modular form of modular weight k for a congruence subgroup of level N and consider the coordinate transformation

$$\tau \;=\; \gamma^{-1}\left(\tau'\right) \;=\; \frac{a\tau'+b}{c\tau'+d}, \qquad \gamma^{-1} \in \Gamma.$$

We have

$$\begin{aligned} I\left(f;\tau\right) &= 2\pi i \int\limits_{i\infty}^{\tau} f\left(\tilde{\tau}\right) d\tilde{\tau} \\ &= 2\pi i \int\limits_{\gamma(i\infty)}^{\gamma(\tau)} \left(c\tilde{\tau}'+d\right)^{k-2} \underbrace{\left(f|_k\gamma^{-1}\right)\left(\tilde{\tau}'\right)}_{\in \mathcal{M}_k(\Gamma_N)} d\tilde{\tau}'. \end{aligned}$$

For $k \neq 2$ we pick up a power of the automorphic factor $(c\tilde{\tau}'+d)$ and **leave the space of iterated integrals of modular forms**!

Transformations of multiple polylogarithms

In the case of multiple polylogarithms

$$G(z_1, \dots, z_k; y)$$

the transformations

$$y' = 1 - y, \quad y' = \frac{1}{y}, \quad y' = \frac{1}{1-y}, \quad y' = \frac{1-y}{1+y},$$

don't leave the space of multiple polylogarithms.

Modular transformations

Let $J = (J_1, J_2, \dots)$ denote a basis of master integrals of uniform weight. The definition of J may depend on the choice of the periods.

Solution:

In order not to leave the space of iterated integrals on $\mathcal{M}_{1,n}$ a base transformation

$$\tau' = \gamma(\tau), \qquad z_j' = \frac{z_j}{c\tau + d},$$

needs to be accompanied by a fibre transformation

$$J' = U J.$$

S.W., '20

Modular transformations

Drawback:

As a modular transformation is always accompanied by a fibre transformation, there is no black-box numerical evaluation algorithm just for iterated integrals of $\omega^{\mathrm{modular}}$ and $\omega^{\mathrm{Kronecker}}$.

Feynman integrals are linear combinations of iterated integrals of $\omega^{\mathrm{modular}}$ and $\omega^{\mathrm{Kronecker}}$.

Feynman integrals transform nicely, individual iterated integrals not.

Section 5

Numerics

GiNaC

GiNaC was initiated in 1999 by Ch. Bauer, A. Frink and R. Kreckel at the University of Mainz.

Despite it's name, it is a **computer algebra system**. Allows **symbolic calculations in C++**.

Shipped with major linux distributions (Ubuntu, Debian, Fedora, ...).

Available at `http://www.ginac.de`.

GiNaC contains a sub-package to evaluate multiple polylogarithms and elliptic multiple polylogarithms with arbitrary precision.

J. Vollinga, S.W., (2004); M. Walden, S.W., (2020)

Example

```
ginsh - GiNaC Interactive Shell (GiNaC V1.8.0)
  __,  _______  Copyright (C) 1999-2020 Johannes Gutenberg University Mainz,
 (__) *       | Germany.  This is free software with ABSOLUTELY NO WARRANTY.
  ._) i N a C | You are welcome to redistribute it under certain conditions.
<-------------' For details type `warranty;'.

Type ?? for a list of help topics.
> Digits=50;
50
> evalf(G({0,0,1},1));
-1.2020569031595942853997381615114499907649862923404988817946
```

Elliptic integration kernels

- **Kernels related to $g^{(k)}(z, \tau)$**:

```
Kronecker_dtau_kernel(k, z_j); Kronecker_dz_kernel(k, z_j, tau);
```

- **Kernels related to modular forms**:

```
Eisenstein_kernel(k, a, b, K); modular_form_kernel(k, P, qbar);
```

- **Kernels related to ELi-functions**:

```
ELi_kernel(n, m, x, y); Ebar_kernel(n, m, x, y);
```

- **User-defined kernels**:

```
user_defined_kernel(f, y);
```

Example

```
Digits = 50;

ex tau = 10*I;
ex qbar = evalf(exp(2*Pi*I*tau));

ex z = 0.9;

ex g_2 = Kronecker_dtau_kernel(2,z);
ex g_3 = Kronecker_dtau_kernel(3,z);

std::cout << iterated_integral(lst{g_3,g_2},qbar).evalf() << std::endl;

3.2253571394850843286565907071596312651610339124775388346726E-27
-4.3708908563005738541231070983779780523160206237307714561 53E-85*I
```

Section 6

Conclusions

Conclusions and recommendations

High precision numerical values for elliptic Feynman integrals:

- **Standarised special functions**: Iterated integrals of $\omega_k^{\mathrm{modular}}$ and $\omega_k^{\mathrm{Kronecker}}$ on the moduli space $\mathcal{M}_{1,n}$.
- Choose an **integration path**, that first stays on the hypersurface $\bar{q} = 0$ and then integrate for $z_j = \mathrm{const}$ in $\bar{q}$.
  - Integration on the hypersurface $\bar{q} = 0$ gives multiple polylogarithms.
  - Integration along τ (or $\bar{q}$) gives elliptic iterated integrals.
- **Advantages**:
  - There are no poles along the integration path for the $\bar{q}$-integration, except possibly at $\bar{q} = 0$ ("trailing zero").
  - By a modular transformation we can always achieve $|\bar{q}| \leq 0.0043$. This gives a very fast converging series expansion.