Numerical evaluation of QCD virtual corrections with top quarks in e^+e^- collisions

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Precision calculations for future e^+e^- colliders: targets and tools
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In collaboration with M.Czakon, M.Niggetiedt, R.Poncelet
Based on: [LC, M.Czakon 2201.01797, 2112.03795]
[LC, M.Czakon, M.Niggetiedt 2109.01917]
[M.Czakon, M.Niggetiedt 2001.03008]
[LC, M.Czakon, R.Poncelet 1712.08075]

Outline

Motivation and Background

One of the take-home messages from the week-1

e^+e^- collisions offer a clean environment for studying properties of heavy quarks;
A few ‰ to % precision on cross sections and asymmetries of top-quark pair production above threshold at the on-going future e^+e^- colliders are possible.

[→ Talk by Simon]

Concerning precision **QCD** corrections for massive $Q\bar{Q}$ at lepton colliders:

$e^+e^- \to t\bar{t}$ near-threshold @ $NNNLO$ [Beneke *et al.*, 15-17] [→ Talk by Beneke]

$e^+e^- \to t\bar{t}$ @ $NNLO$ [Gao, Zhu 14; **LC** *et al.* 17] and for $b\bar{b}$ @ $NNLO$ [Bernreuther *et al.* 17]

Recently re-computed purely numerically using the Local-Unitarity method [Capatti *et al.* 22] [→ Talk by Hirschi]

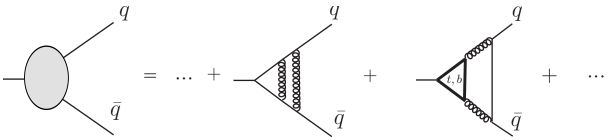
$\sigma^{t\bar{t}}_{\mathrm{NNLO}} = \sigma^{t\bar{t}}_{\mathrm{LO}}\,(1+\Delta_1+\Delta_2):$

$\sqrt{s}$ [GeV]	360	381.3	400	500
Δ_1	0.627	0.352	0.266	0.127
Δ_2	0.281	0.110	0.070	0.020

The QCD correction factors to LO A^b_{FB} at Z-pole ($\mu_R = m_z$) [Bernreuther *et al.* 17]

	$1+A_1$	$1+A_1+A_2$	A_1	A_2
thrust axis:	0.9713	0.9608	−0.0287	−0.0105

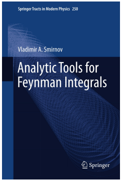
Focus: Virtual QCD corrections to quark FFs



- Quark form-factors (FFs) couple an external color-neutral boson to a pair of quarks
  - $e^+e^- \to Z/\gamma^* \to Q\bar{Q} + X,\ H/Z \to Q\bar{Q} + X$, Drell-Yan process, DIS etc
  - simplest object to extract certain universal QCD quantities
- Massless quark FFs
  - Purely massless QCD corrections analytically to 3-loop [Moch, Vermaseren, Vogt 05; Baikov, Chetyrkin,Lee, Smirnov, Smirnov,Steinhauser 09-10; Gehrmann, Glover, Huber, Ikizlerli, Studerus 10; Gehrmann, Ahmed......]
  - Top-quark loop-induced contributions at 3-loop [**LC**, Czakon, Niggetiedt 21]
  - 4-loop analytic results [Lee, Smirnov, Smirnov, Steinhauser 19; Manteuffel, Panzer, Schabinger..., Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser 22] [→ Talk by Manteuffel]
- Massive quark FFs
  - 2-loop QCD corrections known analytically [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi 04-06,...]
  - Partial 3-loop analytic results [Henn, Smirnov, Smirnov, Steinhauser 16-18; Ablinger, Marquard, Rana, Schneider 18; Lee, Smirnov, Smirnov, Steinhauser 18; Blümlein, Marquard, Rana, Schneider 19]
  - Truncated series-expansion results at 3-loop [Fael, Lange, Schönwald, Steinhauser 22].

Various (numerical) methods for multi-loop integrals

Analytic methods for Feynman integrals [→ Talks by Manteuffel, Weinzierl]



Many generally-applicable (semi) **numerical** approaches for evaluating multi-loop integrals (maybe another book?)

- Numerical evaluation of *integral representations*
  - Sector decomposition [Binoth, Heinrich 00-04] [→ Talk by Maheria]
  - Mellin-Barnes integral representation [Smirnov; Tausk 99] [→ Talk by Gluza]
  - Numerical extrapolation of Feynman parametric integrals (in ϵ and $i\rho$) [Doncker, Yuasa, Kato, Ishikawa, Kapenga, Olagbem 05-18]
  - Loop-Tree-Duality [Catani, Gleisberg, Krauss, Rodrigo, Winter 08] and Local-Unitarity representation [Capatti, Hirschi, Pelloni,Ruijl 20] [→ Talk by Hirschi]
  - . . .
- Numerically solving differential equations (DE) of master integrals (MI)
  - Pure numerical evolution of DE supplemented by deep series expansion [Boughezal, Czakon, Schutzmeier 07; Czakon 08]
  - A sequence of expansions around singular/regular points (DESS, "expansion-and-matching") [Lee, Smirnov, Smirnov 17-18; Fael, Lange, Schönwald, Steinhauser 21]
  - DiffExp [Moriello; Hidding 19] (extensive use of the Frobenius method for a N-th order DE) [→ Talk by Hidding]
  - Auxiliary mass flow [Liu, Ma, Wang 17] (DE w.r.t the auxiliary mass $i\eta$ with boundary at $\eta \to \infty$) [→ Talk by Liu]

Computing MIs by numerically solving DE

- With the emergence of IBP relations [Chetyrkin, Tkachov 81] (and Laporta algorithm [00]), solving DE evolves as a systematic and powerful approach to treat Feynman integrals [Kotikov 91; Remiddi 97]
- The initial applications of the strategy "*pure numerical evolution of DE supplemented by deep series expansion*" to (physical) amplitudes [Boughezal, Czakon, Schutzmeier 07; Czakon 08]
- Further successful applications in the past:
  - 2-loop QCD virtual corrections to $t\bar{t}$ production at LHC [Bärnreuther, Czakon, Fiedler 13; LC, Czakon, Poncelet 17]
  - $B \to X_s \gamma$ at $O(\alpha_s^2)$ [Czakon, Fiedler, Huber, Misiak, Schutzmeier, Steinhauser 15]
  - 3-loop Higgs-gluon form factor with exact top-mass dependence [Czakon, Niggetiedt 20]

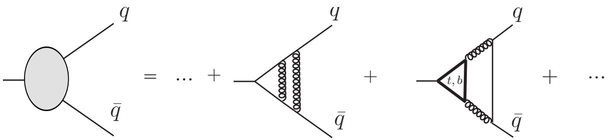
Two major directions of improvements of this approach:

- Better MI basis whose DE is less stiff
- More efficient computer algorithm/tools for solving DE

Outline

Singlet contributions to the massless quark FFs

We work in QCD with $n_f = n_l + 1 = 6$ flavors and only the top quark kept massive.



$$\bar{u}(p_1)\,\Gamma^\mu\, v(p_2)\,\delta_{ij} = \bar{u}(p_1)\,\left(v_q\, F^V \gamma^\mu \,+\, a_q\, F^A \gamma^\mu \gamma_5)\right) v(p_2)\,\delta_{ij}$$

The *non-singlet* and *singlet* part of massless quark FF:

$$F^V \;=\; F^V_{ns} + F^V_s = F^V_{ns} + \sum_f \frac{v_f}{v_q}\, F^V_{s,f}\,,$$

$$F^A \;=\; F^A_{ns} + F^A_s = F^A_{ns} + \sum_f \frac{a_f}{a_q}\, F^A_{s,f}\,,$$

depending on whether the external Z boson couples directly to the external quarks or not.

The computational work-flow

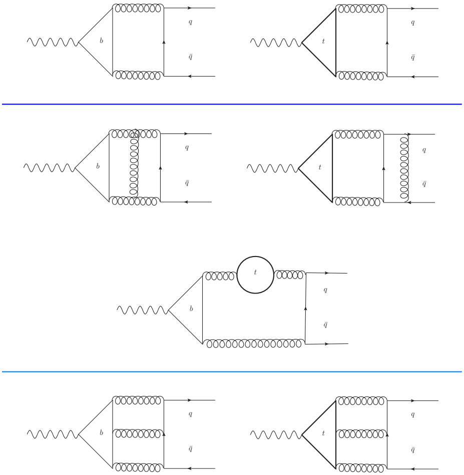
The tool-chain:

- Generating Feynman diagrams
  DiaGen [Czakon] ($\mathcal{O}$(minutes))

- Applying Feynman Rules, Dirac/Lorentz algebra and Color algebra
  FORM [Vermaseren] ($\mathcal{O}$(few minutes to hours))

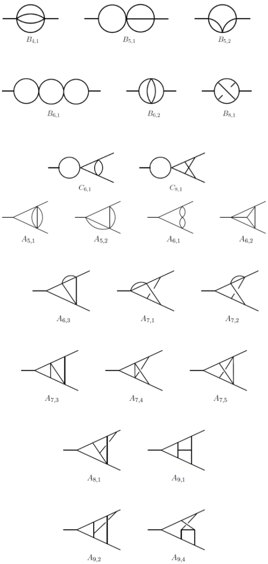
- IBP reduction of loop integrals by Laporta algorithm
  IdSolver [Czakon] ($\mathcal{O}$(hours to days))

- Calculating Master integrals (by DE) (it depends...)



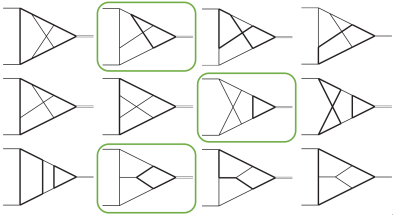
Classification of 3-loop MIs

Purely massless ones [Gehrmann, *et al* 10]:

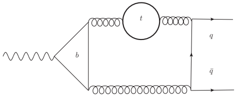


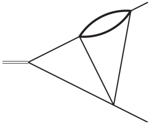
Those dependent on m_t:

- A subset of ggH topologies [Czakon, Niggetiedt 20]



- A new topology arising from:





Solving the ϵ-form DE of MIs analytically

- Derive the DE in $x = \frac{s}{m_t^2}$ by IBP reducing derivatives

$$\frac{\mathrm{d}\, M_i(\epsilon,x)}{\mathrm{d}\, x} = \sum_j A_{ij}(\epsilon,x)\, M_j(\epsilon,x) \quad \Rightarrow \quad \vec{M}(x) = \hat{\mathrm{P}} \exp\left[\int \hat{A}(\epsilon,x)\, \mathrm{d}x\right] \cdot \vec{I}_c(\epsilon)$$

- Transform the DE into an ϵ-form [Henn 13] found by **CANONICA** [Meyer 17]

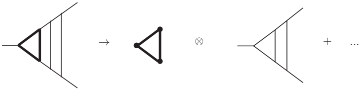
$$\vec{M}_0(x) = T(\epsilon,y) \cdot \vec{M}_n(y)\,, \quad \text{with } x = 2 - y - \frac{1}{y}$$

$$\vec{M}_n(y) = \hat{\mathrm{P}} \exp\left[\epsilon \sum_{a=0,\pm} \int \frac{1}{y-a}\, \mathrm{d}y\right] \cdot \vec{I}_c(\epsilon)$$

$$= \sum_{n=0}^{\infty} \epsilon^n * (\text{a linear combination of HPLs [Remiddi, Vermaseren 99]})$$

- Determine the $\vec{I}_c(\epsilon)$ from boundary at $x = 0$ (the large-mass limit)
  Expansion-by-Subgraph [Chetyrkin 88;Smirnov 90] $\Rightarrow$ heavy-graphs $\otimes$ co-graphs:

single-scale vacuum integrals to 3L;
massless vertex integrals to 2L.



Solving the ϵ-expanded DE numerically

Extract the ODE system

- Set up the DE in $x = \frac{s}{m_t^2}$ by IBP reducing derivatives

$$\frac{\mathrm{d}\, M_i(\epsilon, x)}{\mathrm{d}\, x} = \sum_j A_{ij}(\epsilon, x)\, M_j(\epsilon, x)\,,$$

- Derive the ϵ-free ODE w.r.t x

$$M_i(\epsilon, x) = \sum_{l=\underline{i}}^{\bar{i}} \epsilon^l\, I_{i,l}(x)$$

$$\frac{\mathrm{d}\, I_m(x)}{\mathrm{d}\, x} = \sum_n B_{mn}(x)\, I_n(x)$$

(variables other than x are inserted by numbers)

- $B(x)$: matrix of *rational* functions with a finite set of poles in the complex x-plane.

Set of poles appearing in the ODE of ~ 200 functions (dependent on MI basis in use):

$$\frac{s}{m_t^2} = \left\{0,\ 1,\ \frac{4}{3},\ 2,\ \frac{8}{3},\ 4,\ \frac{16}{3},\ 8,\ 16,\ \infty\right\}$$

Solving the ϵ-expanded DE numerically

Prepare high-precision initial values by solving DE with a series ansatz

$$\frac{\mathrm{d}\, I_m(x)}{\mathrm{d}\, x} = \sum_n B_{mn}(x)\, I_n(x)$$

- Boundary condition at $x = 0$:
  Large-Mass Expansion (LME)

$$I_G(\{q_e\}, m \to \infty) = \sum_{\gamma \in G} I_{G/\gamma}(\{q_e\}) \otimes \hat{\mathbf{T}}_e \left[I_\gamma(\{q\}, m) \right]$$

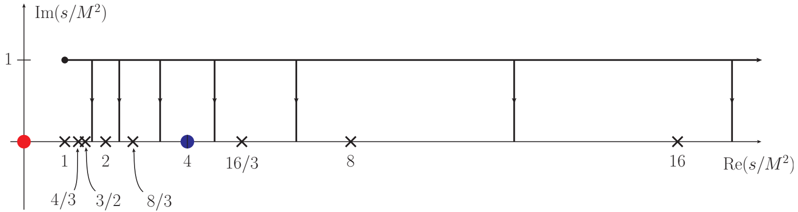
  - massless vertex integrals to 2-L; single-scale vacuum integrals to 3L
- Technical problem: not suitable as the initial point of the **numerical** evolution
  - The $B(x)$ is *singular* at $x = 0$
  - Only first few LME terms, **not accurate** enough at near-by x
- Resolution: a deep series expansion by solving DE around $x = 0$ [Czakon 08]

$$I(x) = \sum_{a \in S} \sum_{b=0}^{b_a} x^a \ln^b(x) \left(\sum_{n=0}^{\infty} x^n C_{a,b,n} \right)$$

In practice, the series is truncated at $\mathcal{O}(x^{100})$, with integration constants fixed by LME.

Solving the ϵ-expanded DE numerically

Evolve ODE numerically between regular points in the *complex x-plane*



Technical settings: [Czakon, Niggetiedt 20]

- Used the *bulirsch_stoer_dense_out* method
- Used multi-precision numbers of 100 decimal digits
- Required a *local* step-error of $\mathcal{O}(10^{-40})$
- Collected 2×10^5 numerical samples with at least 20 correct digits

Use the *odeint* C++ library:

[Ahnert,Mulansky' 2011]

Solving the ϵ-expanded DE numerically

Series expansion around the (genuine) singular points and match

- Pseudo singular points
  - Interpolation of data within a small vicinity (~ 0.001)
  - A (deep) Taylor power series expansion
- Genuine singular points (e.g. the pair threshold)
  - A power-logarithm (asymptotic) series expansion

$$\vec{I}(x_0) = \hat{U}(x_0, 0) \cdot \vec{I}_c = \sum_{a \in S} \sum_{b=0}^{b_a} x_0{}^a \ln^b(x_0) \left(\sum_{n=0}^{\infty} x_0{}^n C_{a,b,n} \right)$$

    with $\vec{I}_c$ completely determined by $\vec{I}(x_0)$ at the *matching* point.
  - Self-contained: **No** need to appeal to any external result!

A (complete) numerical answer for master integrals

Deep series expansion around (true) singular points
+ High-precision numerical results on a dense grid

Successively applied in the last 15 years in Czakon's group: [Boughezal, Czakon, Schutzmeier 07; Czakon 08; Bärnreuther, Czakon, Fiedler 13; LC, Czakon, Poncelet 17; Czakon, Niggetiedt 20]

Mission Impossible?

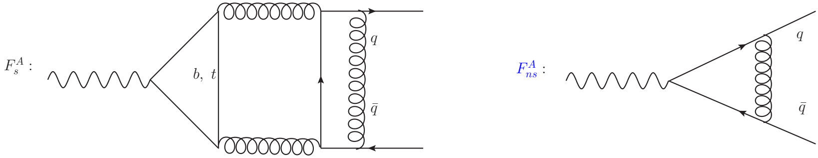
Restore numerically $M(\epsilon, x) = \sum_{k=p}^{\infty} \epsilon^k I_k(x)$ at $x = 0$ despite singular $I_k(0)$?

- Laurent ϵ-expansion does not commute with $x \to 0$ if $I_k(x = 0)$ are singular.
- According to the *Expansion-by-Region* (EbR) prescription [Beneke, Smirnov 98; Smirnov 02]: $M(\epsilon, x{=}0)$ **=** the **leading** term of the ***hard**-region* (x^0-scaling) contribution to $M(\epsilon, x \to 0)$ [Henn, Smirnov, Smirnov 13].
- The EbR ansatz agrees with the one from applying the well-known Frobenius method [Kniehl, Pikelner, Veretin 17; Liu, Ma, Wang 17],

$$M(\epsilon, x \to 0) = \sum_{a=r\epsilon}^{r\in S} \sum_{b=0}^{b_a} x^a \ln^b(x) \left(\sum_{n=0}^{\infty} x^n C_{a,b,n}(\epsilon) \right)$$

  $C_{0,0,0}(\epsilon) = M(\epsilon, x = 0)$ according to EbR.
- **NOT** necessary to keep the full ϵ dependence, just those with $\ln(x)$ resummed.
- This (numerical) matching strategy from $M(\epsilon, x \neq 0)$ has found many impressive applications recently, e.g. [Kniehl, Pikelner, Veretin 17; Lee, Smirnov, Smirnov 17; Liu, Ma, Wang 17; Liu, Ma, Tao, Zhang 20; Liu, Ma 22; Baranowski, Delto, Melnikov, Wang 21-22; Fael, Lange, Schönwald, Steinhauser 22; Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser 22]

Finite remainders of singlet FFs: UV-renormalization



UV renormalization for individual singlet contributions [Chetyrkin, Kühn 93; LC, Czakon, Niggetiedt 21]

$$\mathbf{F}^A_{s,b}(a_s, m_t, \mu) \;=\; Z_{ns}\, Z_2\, F^A_{s,b}(\hat{a}_s, \hat{m}_t) \;+\; Z_s\, Z_2 \Big(F^A_{ns}(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F^A_{s,i}(\hat{a}_s, \hat{m}_t) \Big)\,,$$

$$\mathbf{F}^A_{s,t}(a_s, m_t, \mu) \;=\; Z_{ns}\, Z_2\, F^A_{s,t}(\hat{a}_s, \hat{m}_t) \;+\; Z_s\, Z_2 \Big(F^A_{ns}(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F^A_{s,i}(\hat{a}_s, \hat{m}_t) \Big)\,,$$

- $\hat{a}_s\ S_\epsilon = Z_{a_s}(\mu^2)\, a_s(\mu^2)\, \mu^{2\epsilon}$ and $\hat{m}_t = Z_m\, m_t$
- A *non-anticommuting* γ_5 [tHooft, Veltman 71; Breitenlohner, Maison 77] in Larin's prescription [Larin 93]
  $Z_s \equiv \frac{1}{n_f}\left(Z_S - Z_{ns}\right)$ and $\mu^2 \frac{\mathrm{d}Z_s}{\mathrm{d}\mu^2} = \gamma_s\left(Z_{ns} + n_f\, Z_s\right)$.

For the **total** *non-anomalous* combination:

$$\mathbf{F}^A_{s,b}(a_s, m_t) - \mathbf{F}^A_{s,t}(a_s, m_t) \;=\; Z_{ns}\, Z_2\, \left(F^A_{s,b}(\hat{a}_s, \hat{m}_t) - F^A_{s,t}(\hat{a}_s, \hat{m}_t) \right),$$

Finite remainders of singlet FFs: IR-subtraction

The **finite remainders** after pulling out the IR singularities:

$$\begin{aligned}
\mathcal{F}^{A}_{s,b}(a_s, m_t, \mu) &= I_{q\bar{q}}\, \mathbf{F}^{A}_{s,b}(a_s, m_t, \mu) \\
&= a_s^2\, \mathcal{F}^{A,2}_{s,b}(\mu) + a_s^3\, \mathcal{F}^{A,3}_{s,b}(m_t, \mu) + \mathcal{O}(a_s^4)\,, \\
\mathcal{F}^{A}_{s,t}(a_s, m_t, \mu) &= I_{q\bar{q}}\, \mathbf{F}^{A}_{s,t}(a_s, m_t, \mu) \\
&= a_s^2\, \mathcal{F}^{A,2}_{s,t}(m_t, \mu) + a_s^3\, \mathcal{F}^{A,3}_{s,t}(m_t, \mu) + \mathcal{O}(a_s^4)\,.
\end{aligned}$$

- The $I_{q\bar{q}}$ needed reads (Catani's convention [Catani 98])

$$I_{q\bar{q}} = 1 - 2\, a_s \left(\frac{\mu^2}{-s - i0^+} \right)^{\epsilon} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)}\, C_F \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) + \mathcal{O}(a_s^2).$$

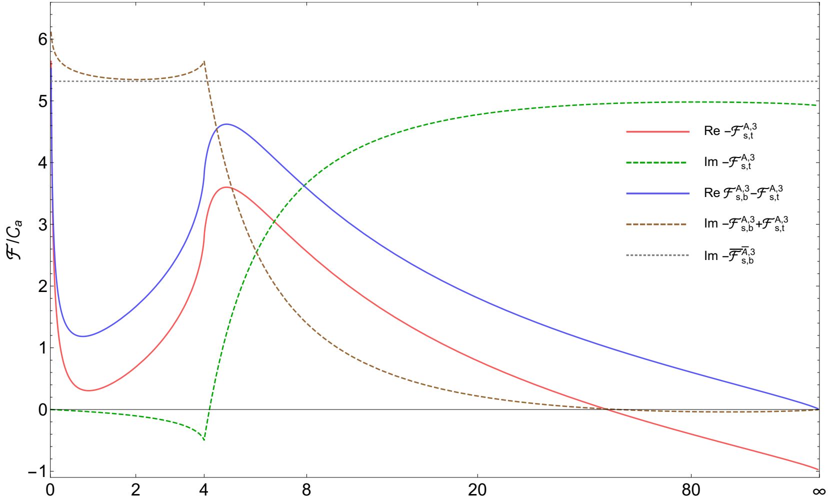
  The alternative $\overline{\mathrm{MS}}$-factorization convention [Becher, Neubert 09]:

$$\mathcal{F}^{A}_{s,b}(a_s, m_t, \mu) = I_{q\bar{q}}\, \mathbf{F}^{A}_{s,b}(a_s, m_t, \mu) = I_{q\bar{q}}\, Z_{q\bar{q}}\, \mathcal{F}'^{A}_{s,b}(a_s, m_t, \mu)$$

- Finiteness: cancellation of poles at a numerical precision better than 20 digits.

Results for the singlet FF: the axial case

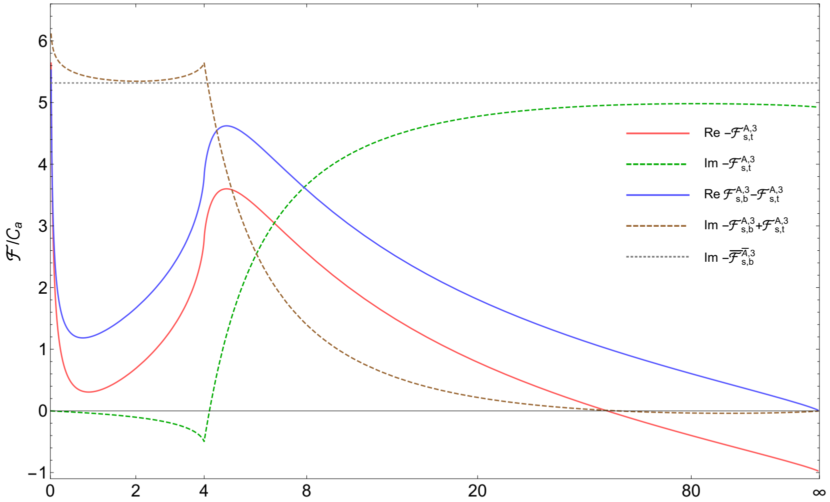
The exact result for finite remainder $\mathcal{F}^{A}_{s,b/t}(x)$ at 3-loop order:



- $x = s/m_t^2$ at $\mu^2 = s$
- $\mathcal{C}_a =$ the real part of the 5-flavor massless result [Gehrmann, Primo 21]

Results for the singlet FF: the axial case

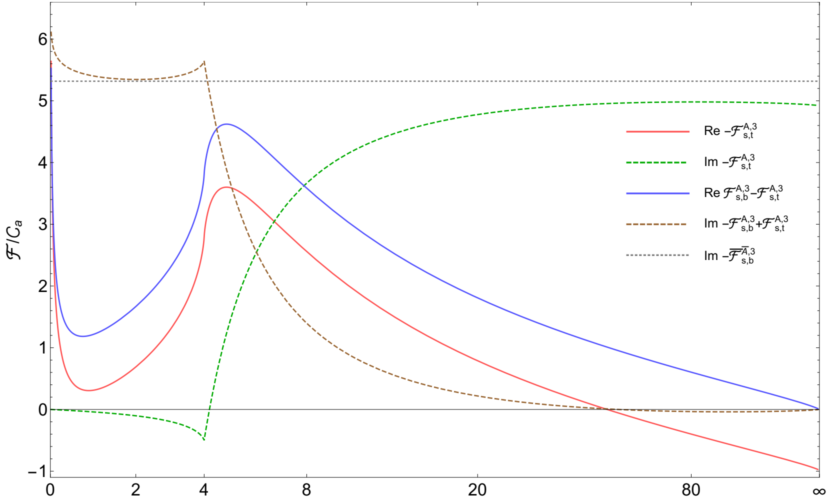
The exact result for finite remainder $\mathcal{F}^{A}_{s,b/t}(x)$ at 3-loop order:



- Strong check: $\mathcal{F}^{A}_{s,t}(a_s, x) \to \mathcal{F}^{A}_{s,b}(a_s, x)$ [Gehrmann, Primo 21] in the high-energy limit ($x \to \infty$)
- Typical threshold behavior due to Coulomb effect (but no divergence here!)

Results for the singlet FF: the axial case

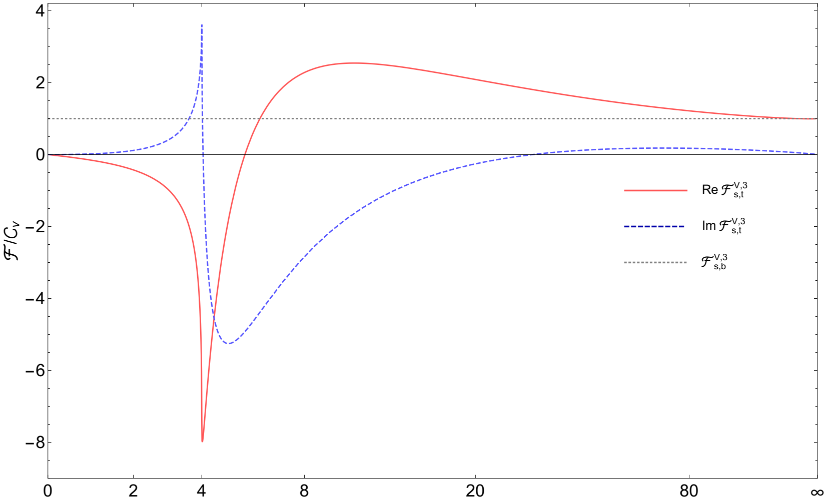
The exact result for finite remainder $\mathcal{F}^{A}_{s,b/t}(x)$ at 3-loop order:



- Strong check: $\mathcal{F}^{A}_{s,t}(a_s, x) \to \mathcal{F}^{A}_{s,b}(a_s, x)$ [Gehrmann, Primo 21] in the high-energy limit ($x \to \infty$)
- The axial massless quark FF diverge in $x \to 0$: *non-decoupling m_t-logarithm*!

Results for the singlet FF: the vector case

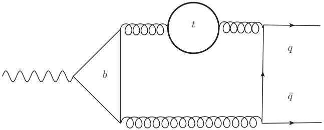
The exact result for UV and IR finite $\mathcal{F}^{V}_{s,t}(x)$ at 3-loop order:



- Strong check: $\mathcal{F}^{V}_{s,t}(a_s, x)\big|_{x\to\infty} \to \mathcal{F}^{V}_{s,b}(a_s, x)$ [Vermaseren *et al* 05; Baikov *et al* 09; Gehrmann *et al* 10]
- The top-loop contribution is power-suppressed in the low-energy $x \to 0$ limit.

Light quark form factors in the heavy top limit

Appearance of the non-decoupling m_t-logarithms [Collins, Wilczek, Zee 78; Chetyrkin, Kühn 93]



$$\mathcal{F}^{A,3}_{s,b}(m_t \to \infty) = \bar{\mathcal{F}}^{\bar{A},3}_{s,b}(\mu) + \bar{\mathcal{F}}^{A_{\text{nDC}},3}_{s,b}(m_t,\mu)\,,$$
$$= \bar{\mathcal{F}}^{\bar{A},3}_{s,b}(\mu) - \frac{85}{9}C_F + \frac{4}{3}C_F L_\mu - \frac{1}{4}C_F L_\mu^2 + \mathcal{O}(1/m_t^2)$$

where $L_\mu \equiv \ln\frac{\mu^2}{m_t^2}$.

A Wilson coefficient function $C_w(\bar{a}_s, \mu/m_t)$

$$\mathcal{F}^A_{s,b} - \mathcal{F}^A_{s,t}\Big|_{m_t\to\infty} = \bar{\mathcal{F}}^{\bar{A}}_{s,b}(\bar{a}_s,\mu) + \bar{\mathcal{F}}^{A_{\text{nDC}}}_{s,b}(\bar{a}_s,m_t,\mu) - \bar{\mathcal{F}}^{A}_{s,t}(\bar{a}_s,m_t,\mu)\Big|_{m_t\to\infty}$$
$$= \bar{\mathcal{F}}^{\bar{A}}_{s,b}(\bar{a}_s,\mu) - C_w(\bar{a}_s,\mu/m_t)\left(\bar{\mathcal{F}}^{A}_{ns}(\bar{a}_s,\mu) + \sum_{i=1}^{n_l}\bar{\mathcal{F}}^{\bar{A}}_{s,i}(\bar{a}_s,\mu)\right) + \mathcal{O}(1/m_t^2)\,.$$

The renormalized low-energy effective Lagrangian [Chetyrkin, Kühn 93; LC, Czakon, Niggetiedt 21]

$$\delta\mathcal{L}^R_{\text{eff}} = \Big(Z_{ns}\sum_{i=1}^{n_l} a_i\,\bar{\psi}^B_i\,\gamma^\mu\gamma_5\,\psi^B_i + a_b\,Z_s\,[J^\mu_5]_B \quad \longrightarrow n_l\text{-flavor massless part}$$
$$+ a_t\,C_w(a_s,\mu/m_t)\,(Z_{ns} + n_l\,Z_s)\,[J^\mu_5]_B\Big)Z_\mu\,,$$

with $J^\mu_5 = \sum_{i=1}^{n_l}\bar{\psi}_i\gamma^\mu\gamma_5\psi_i$.

Resuming the non-decoupling m_t logarithms

RG equation of the Wilson coefficient $C_w(\bar{a}_s, \mu/m_t)$ [Chetyrkin, Kühn 93; LC, Czakon, Niggetiedt 21]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} C_w(\bar{a}_s, \mu/m_t) = \bar{\gamma}_s - n_l\, \bar{\gamma}_s\, C_w(\bar{a}_s, \mu/m_t)\,,$$

$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \left[J^{\mu}_{5,q}\right]_R = \bar{\gamma}_s \left[J^{\mu}_5\right]_R$ with $J^{\mu}_5 = \sum_{i=1}^{n_l} = \bar{\psi}_i \gamma^{\mu} \gamma_5 \psi_i$ and $\mu^2 \frac{\mathrm{d}Z_s}{\mathrm{d}\mu^2} = \bar{\gamma}_s \left(Z_{ns} + n_l\, Z_s\right)$.

The solution for $C_t \equiv -1/n_l + C_w$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} C_t(\bar{a}_s, \mu/m_t) = n_l\, \bar{\gamma}_s\, C_t(\bar{a}_s, \mu/m_t)\,,$$

$$C_t(\bar{a}_s(\mu), \mu/m_t) = C_t(\bar{a}_s(m_t), 1) \exp\left(\int_{\bar{a}_s(m_t)}^{\bar{a}_s(\mu)} \frac{-n_l\, \bar{\gamma}_s(a_s)}{\beta(a_s)} \frac{\mathrm{d}a_s}{a_s} \right),$$

in Larin's scheme.

The solution can also be done by numerically solving the RGE.

The γ_s at $\mathcal{O}(a_s^5)$ in $\overline{\mathrm{MS}}$-scheme

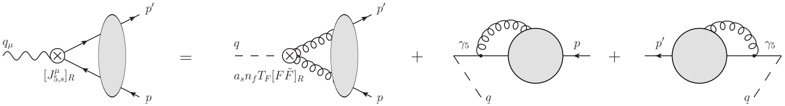
The Adler-Bell-Jackiw (ABJ) equation in terms of renormalized operators:

$$\left[\partial_\mu J^\mu_{5,s}\right]_R = a_s\, n_f\, \mathrm{T}_F\, \left[F\tilde{F}\right]_R$$
$$Z_s\left[\partial_\mu J^\mu_{5,s}\right]_B = a_s\, n_f\, \mathrm{T}_F\left(Z_{FJ}\left[\partial_\mu J^\mu_{5,s}\right]_B + Z_{F\tilde{F}}\left[F\tilde{F}\right]_B\right)$$

with $Z_s \equiv Z_s^f\, Z_s^{ms}$.

Results for Z_s^{ms}:

- $\mathcal{O}(a_s^3)$ from UV-poles in Zqq-vertex (Z_s^f at $\mathcal{O}(a_s^2)$ from 3L AVV-amplitude) [Larin 93]
- $\mathcal{O}(a_s^4)$ in the calculation of Ellis-Jaffe sum rule [Larin, Ritbergen, Vermaseren 97]
- $\mathcal{O}(a_s^5)$ from 4-loop calculations by combining [LC, Czakon 22]
  - The anomalous Ward-Takahashi identity (with a NAC-γ_5):

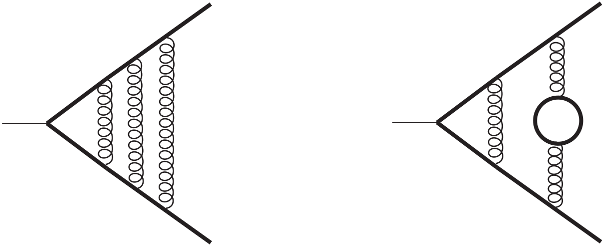


  - Consequence of the ABJ equation with the proved $Z_{F\tilde{F}} = Z_{a_s}$:

$$\gamma_s^{ms} \equiv \frac{\mathrm{d}\,\ln Z_s^{ms}}{\mathrm{d}\,\ln \mu^2} = a_s\, n_f\, \mathrm{T}_F\, \gamma_{FJ} - \beta\,\frac{\mathrm{d}\,\ln Z_s^f}{\mathrm{d}\,\ln a_s}\,.$$

Outline

Non-singlet QCD corrections to massive FFs



- Two-loop QCD corrections fully known analytically [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi 04-06]
- Feynman diagrams at 3-loop order: 227 non-singlet, 114 singlet
- Partial 3-loop analytic results
  - Non-singlet contribution at the large-Nc limit [Henn, Smirnov, Smirnov, Steinhauser 16-18; Ablinger, Marquard, Rana, Schneider 18]
  - those with closed fermion loops [Lee, Smirnov, Smirnov, Steinhauser 18; Blümlein, Marquard, Rana, Schneider 19]
- The full set of 3-loop non-singlet master integrals were evaluated recently in [Fael, Lange, Schönwald, Steinhauser 22] by solving DE in terms of a sequence of series expansions.

Basis of MIs and their DE w.r.t s

- 427 non-singlet MIs from IdSolver [Czakon]
  A *D-factorizing* basis [Simirnov, Simirnov 20; Usovitsch 20] found using **ImproveMasters.m** [Simirnov, Simirnov 20]

- Derive the DE in $x = s$ (with $m_t = 1$)

$$\frac{\mathrm{d}\, M_i(\epsilon, x)}{\mathrm{d}\, x} = \sum_j A_{ij}(\epsilon, x)\, M_j(\epsilon, x)\,,$$

  A_{ij}: **no** irreducible denominator factors mixing D with s.

- Insert $M_i(\epsilon, x) = \sum_{l=\underline{i}}^{\bar{i}} \epsilon^l\, I_{i,l}(x)$, and obtain

$$\frac{\mathrm{d}\, I_m(x)}{\mathrm{d}\, x} = \sum_n B_{mn}(x)\, I_n(x)$$

  for 2166 I_n.

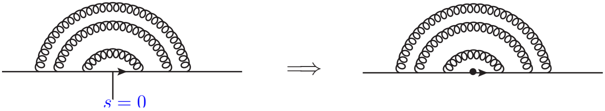
- Set of poles appearing (dependent on MI basis in use):

$$\frac{s}{m_t^2} = \{-4,\ -2,\ -\frac{1}{2},\ 0,\ 1,\ 2,\ 3,\ 4,\ \frac{9}{2} \pm \frac{3}{2}\sqrt{3},\ \frac{16}{3},\ 16,\ \infty\}$$

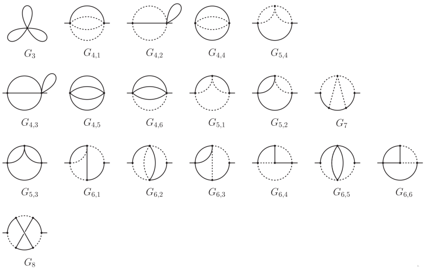
Solving the ϵ-expanded DE in s

Boundary condition taken at $s = 0$ where all non-singlet MIs are **regular**!

- Using **asy.m** [Pak, Simirnov 10]$\Rightarrow$ non-singlet MIs have only the *hard-region* in $s \to 0$.
- 3-point Vertex $\Rightarrow$ Quark Propagator



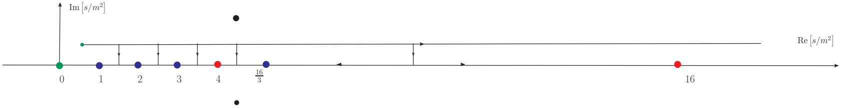
- All masters in massive quark propagators known in QCD to 3-loop [Lee, Simirnov 10]:

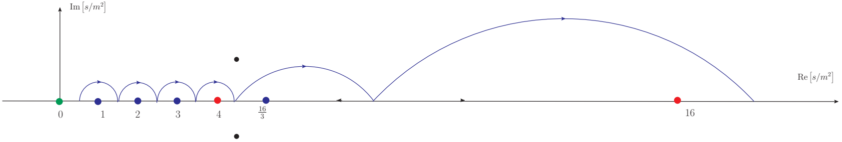


Solving the ϵ-expanded DE in s

Evolving numerically DE and deep-series expansion around (physical) singularities.

Two choices of integration contours:

- 

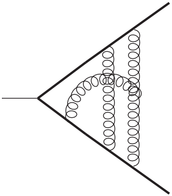
- 

Technical settings:

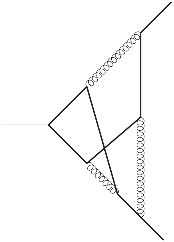
- Used multi-precision numbers of 500 decimal digits
- Required a *local* step-error of $\mathcal{O}(10^{-40})$
- Collected $\sim 10^5$ numerical samples with at least 20 correct digits

A sample of the high-precision numerical results

At $\frac{s}{m_t^2} = 20$ (above the four-top threshold):

- 

$$\text{PR486}[1,1,1,1,1,1,1,1,1,0,0,0] = \frac{1}{\epsilon}(-0.020647440796693996305 + 0.015082900459951443810\, i) + (0.62590765632704470189 - 0.61859104529847674513\, i) + \epsilon\,(1.5919548374820074115 + 3.7600381842246118352\, i)$$

- 

$$\text{PR453}[1,1,1,1,1,1,1,1,1,0,0,0] = \frac{1}{\epsilon}(0.095706261149592678 10 + 0.33537685751829638912\, i) + (0.62590765632704470189 - 0.61859104529847674513\, i)$$

Precision: $\sim$ 20 correct digits

The next step(s) ...

- Validate the numerical results for the non-singlet contributions at hand
- Provide deep-series expansions around a set of (singular) points to encode the (full) result, alternative to an interpolation table
- Transforming into the *normalized Fuchsian form* [Lee, 15] could help to make the DE **less stiff** when approaching singular points
- A better control of the precision on amplitudes/form-factors
- Singlet contributions? $\mathcal{O}(\alpha\,\alpha_s^2)$ corrections? ......

Outline

Summary and Outlook

- A brief description of an efficient high-precision numerical method for computing loop integrals :"numerically solving DE supplemented by deep series expansion around singular points", and its successful applications.

- Massless quark FFs: the 3-loop QCD corrections with **exact quark-mass dependence** + the *resummation* of non-decoupling top-mass logarithms.

- 3-loop non-singlet contributions to massive quark FFs are now known **numerically** (by two groups).

- These are among the ingredients needed for computing N^3LO QCD corrections to heavy-quark pair productions in e^+e^- collisions.

- Further improvements to the approach are to be undertaken, applicable to other multi-loop corrections to processes at e^+e^- colliders.

Summary and Outlook

THANK YOU

Backup Slides

From [Baikov, Chetyrkin, Kühn, Rittinger, arXiv:1201.5804]

The decay rate of the Z-boson into hadrons in massless QCD up to $\mathcal{O}(\alpha_s^4)$:

$$
\begin{aligned}
\Gamma_Z =& \Gamma_0 \, R^{\text{nc}} = \frac{G_F \, M_Z^3}{24\pi\sqrt{2}} \, R^{\text{nc}} \\
R^{\text{nc}} =& 20.1945 + 20.1945 \, \alpha_s \\
& + (28.4587 - 13.0575 + 0) \, \alpha_s^2 \\
& + (-257.825 - 52.8736 - 2.12068) \, \alpha_s^3 \\
& + (-1615.17 + 262.656 - 25.5814) \, \alpha_s^4 \; ,
\end{aligned}
$$

The three terms in the brackets display separately non-singlet, axial singlet and vector singlet contributions.