Feynman parametrization and numerical integration

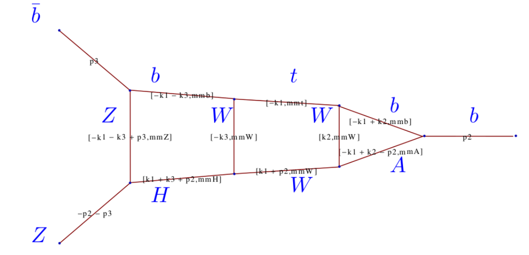
Janusz Gluza,
University of Silesia in Katowice

Precision calculations for future e^+e^- colliders:
targets and tools

16 June 2022, CERN

Collider Physics at the Precision Frontier G. Heinrich, 2009.00516

	analytic	numerical
pole cancellation	exact	with numerical uncertainty
control of integrable singularities	analytic continuation	less straightforward
fast evaluation	yes	depends
extension to more scales/loops	difficult	promising
automation	difficult	less difficult



Four scales :

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{s+i\varepsilon}{M_Z^2} \right\}$$

Direct numerical approaches (beyond 1-loop)

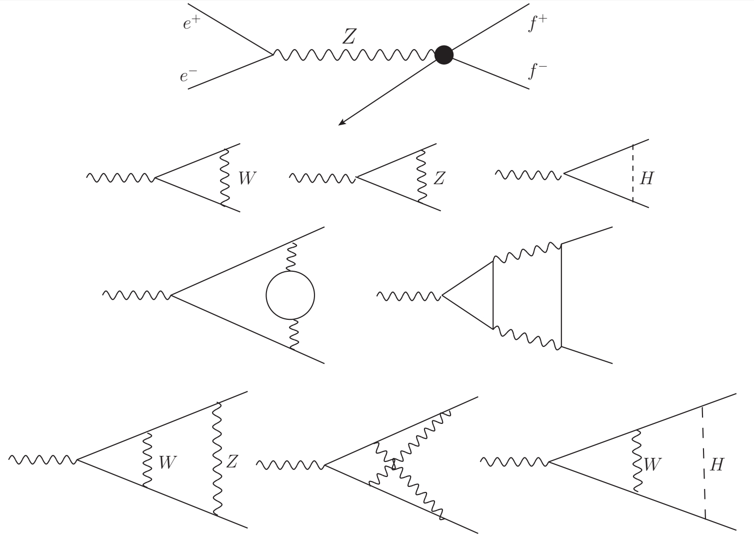
- Sector decomposition (SD) method: *Talk by Vitalii Maheria*
  - FIESTA [2016], [A.V.Smirnov]
  - pySecDec [2022], Expansion by regions with pySecDec],
- The Mellin-Barnes (MB) method:
  - MB [M.Czakon, 2006]
  - MBnumerics [J.Usovitsch, I.Dubovyk, T.Riemann, 2015] – Minkowskian kinematics
- Differential equations (DEs) method: *(Talk by Long Chen)*
  - `DiffExp` [F. Moriello, 2019; M. Hidding, 2021], *Talk by Martijn Hidding*
  - `AMFlow` [X. Liu, Y.-Q. Ma, 2022], *Talk by Xiao Liu*
  - `SeaSyde` [T. Armadillo, R. Bonciani, S. Devoto, N. Rana, A. Vi, 2022] , *Talk by Narayan Rana*

SE: `TVID 2` (S. Bauberger, A. Freitas, D. Wiegand), `BOKASUN` (M. Caffo, H. Czyz, M. Gunia, E. Remiddi),

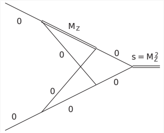
+ `DREAM` (dimensional recurrence relations solutions, R. Lee, K. Mingulov), αLoop loop-tree duality *Talk by Valentin Hirschi*, HPL, GPL, MPL, eMPL, integrand subtraction (≤ 2 loops: `NICODEMOS` - A. Freitas, *talk by Charalampos Anastasiou*),

Four-Dimensionally Regularized/Renormalized (FDR) integrals (R. Pittau), dispersion relations, ...

Context: Extracting the $Zf\bar{f}$ vertex and EW corrections



Substantial progress for critical cases



Euclidean results (constant part, $(p_1+p_2)^2 = m^2 = 1$:):

$$
\begin{array}{ll}
\text{Analytical}: & -0.49661983060 57021 \\
\text{MB(Vegas)}: & -0.4969417442183914 \\
\text{MB(Cuhre)}: & -0.4966198313219404 \\
\text{FIESTA}: & -0.4966184488196595 \\
\text{SecDec}: & -0.4966192150541896
\end{array}
$$

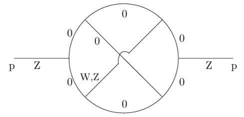
Minkowskian results (constant part, $-(p_1+p_2)^2 = m^2 = 1$:):

$$
\begin{array}{lc}
\text{Analytical}: & -0.778599608979684 - 4.123512593396311 \cdot i \\
\text{MBnumerics}: & -0.778599608324769 - 4.123512600516016 \cdot i \\
\text{MB + thresholds}: & -0.7785242512636401 - 4.123512600516016 \cdot i \\
\text{SecDec}: & \text{big error } [\mathbf{2016}], -0.77 - \mathrm{i} \cdot 4.1 \ [\mathbf{2017}], -0.778 - \mathrm{i} \cdot 4.123 \ [\mathbf{2019}] \\
\text{pySecDec + rescaling}: & -0.778598 - i \cdot 4.123512 \ [\mathbf{2020}]
\end{array}
$$

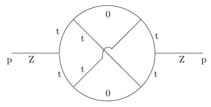
SD and MB are independent of IBPs (at 2-loops SM we haven't used IBPs)

MIs with high accuracy, results*

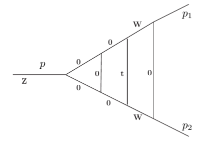
* *Results for 3-loop EWPOs at the e^+e^- Z-resonance peak,*
I. Dubovyk, A. Freitas, JG, K. Grzanka, M. Hidding, J. Usovitsch, 'Evaluation of multi-loop multi-scale Feynman integrals for precision physics', *2201.02576*



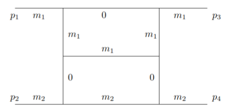
lhNp1



taNPl1



vtwPl



box2l

$$
\begin{aligned}
I_{\text{box2l}}[2,1,1,1,1,1,1,0,0,s,t,m_1^2,m_2^2] \quad = \quad & +0.000328707579/\epsilon^2 \\
- \quad & (0.0014129475 - 0.0020653306\ i)/\epsilon \\
- \quad & (0.005702737 - 0.000485980\ i) + \mathcal{O}(\epsilon), \\
& 55 \text{ MIs}, s = 2, t = 5, m_1^2 = 4, m_2^2 = 16.
\end{aligned}
$$

MB used so far, some examples

- Evaluation of MIs (Tausk, Smirnov, ...)
- Bhabha massive QED 2-loop (M. Czakon, JG, T. Riemann, S. Actis) (MB & expansions), (MB & dispersion relations)
- "On the Numerical Evaluation of Loop Integrals With Mellin-Barnes Representations", A. Freitas, Yi-Cheng Huang, JHEP, 2010
- "Angular integrals in d dimensions", Gabor Somogyi, J.Math.Phys, 2011
- "Soft triple-real radiation for Higgs production at N3LO", C. Anastasiou, C. Duhr, F. Dulat, B. Mistlberger, JHEP, 2013
- "Evaluating multi-loop Feynman diagrams with infrared and threshold singularities numerically", C. Anastasiou, S. Beerli, A. Daleo, JHEP, 2007
- High-Energy Expansion of Two-Loop Massive Four-Point Diagrams, G. Mishima, JHEP 02 (2019) 08, (Higgs pair production cross section)

Scenery

$$G_L[1] = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L}{(q_1^2 - m_1^2)^{n_1} \ldots (q_i^2 - m_i^2)^{n_j} \ldots (q_N^2 - m_N^2)^{n_N}}$$

$$D_i = q_i^2 - m_i^2 + i\delta = \left[\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e \right]^2 - m_i^2 + i\delta,$$

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1 + \ldots + n_N)}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{n_1-1} \ldots x_N^{n_N-1} \delta(1 - x_1 - \ldots - x_m)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}}$$

$$m^2(\vec{x}) = x_1 D_1 + \ldots + x_i D_i + \ldots + x_N D_N = k_i M_{ij} k_j - 2Q_j k_j + J$$

$$m^2(\vec{x}) = kMk - 2Qk + J \Leftrightarrow U = \det M, \quad F = -\det M \; J + QM^T Q$$

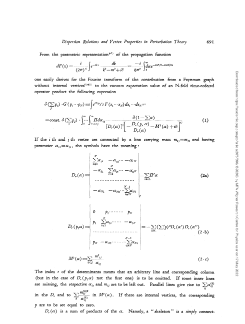
$$G_L[1] = \frac{(-1)^{N_\nu} \Gamma\left(N_\nu - \frac{d}{2} L\right)}{\prod_{i=1}^{N} \Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

Scenery

$$G_L[1] = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j-1}\delta(1-\sum_{i=1}^{N} x_i)\frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

U, F - Symanzik polynomials, K. Symanzik, Dispersion Relations and Vertex Properties in Perturbation Theory, Progress of Theoretical Physics 20(5) (1958) 690–702, https://doi.org/10.1143/PTP.20.690

N. Nakanishi, Graph Theory and Feynman Integrals, Gordon and Breach, 1971.



Dispersion Relations and Vertex Properties in Perturbation Theory 691

From the parametric representation[4,3)] of the propagation function

$$\Delta F(x) = \frac{i}{(2\pi)^4}\int e^{-ikx}\frac{dk}{k^2-m^2+i\varepsilon} = \frac{-i}{8\pi^2}\int_0^\infty d\alpha e^{-i(x^2/\alpha-\alpha m^2)/2\alpha}$$

one easily derives for the Fourier transform of the contribution from a Feynman graph without internal vertices[(1)(2)] to the vacuum expectation value of an N-fold time-ordered operator product the following expression

$$\delta(\sum_j p_j)\cdot G(p_1\cdots p_N) = \int e^{i\sum p_j x_j}F(x_1\cdots x_N)dx_1\cdots dx_N =$$

$$= \text{const.}\ \delta(\sum_j p_j)\cdot\int_0^\infty\cdots\int_0^\infty \prod_{i<j} d\alpha_{ij}\frac{\delta(1-\sum\alpha)}{[D_r(\alpha)]^2\left[-\frac{D_r(p,\alpha)}{D_r(\alpha)}-M^2(\alpha)+i\varepsilon\right]^C} \quad (1)$$

If the i th and j th vertex are connected by a line carrying mass $m_{ij}=m_{ji}$ and having parameter $\alpha_{ij}=\alpha_{ji}$, the symbols have the meaning:

$$D_r(\alpha) = \begin{vmatrix} \sum_{i\neq 1}^{N}\alpha_{1i} & -\alpha_{12}\cdots & -\alpha_{1N} \\ -\alpha_{21} & \sum_{i\neq 2}^{N}\alpha_{2i}\cdots & -\alpha_{2N} \\ \cdots & \cdots & \cdots \\ -\alpha_{N1} & -\alpha_{N2}\cdots & \sum_{i\neq N}^{N-1}\alpha_{Ni}\end{vmatrix}_r = \sum_{\text{skel.}}\Pi'\alpha \quad (2a)$$

$$D_r(p,\alpha) = \begin{vmatrix} 0 & p_1\cdots\cdots & p_N \\ p_1 & \sum_{i\neq 1}^{N}\alpha_{1i}\cdots\cdots & -\alpha_{1N} \\ \cdots & \cdots & \cdots \\ p_N & -\alpha_{N1}\cdots\cdots & \sum_{i\neq N}^{N-1}\alpha_{Ni}\end{vmatrix} = -\sum_{\text{div.}}(\sum' p)^2 D_r(\alpha')D_r(\alpha'') \quad (2\cdot b)$$

$$M^2(\alpha) = \sum_{i<j}\frac{m^2_{ij}}{\alpha_{ij}} \quad (2\cdot c)$$

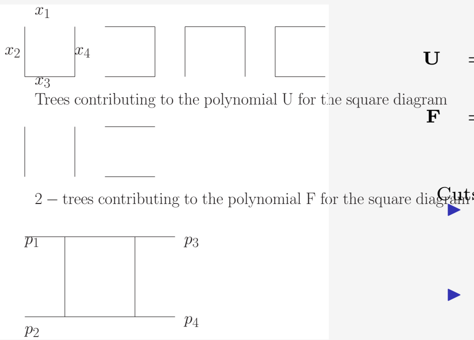
The index r of the determinants means that an arbitrary line and corresponding column (but in the case of $D_r(p,\alpha)$ not the first one) is to be omitted. If some inner lines are missing, the respective α_{ij} and m_{ij} are to be left out. Parallel lines give rise to $\sum_k \alpha_{ij}^{(k)}$ in the D_r and to $\sum_k \frac{m_{ij}^{(k)2}}{\alpha_{ij}^{(k)}}$ in $M^2(\alpha)$. If there are internal vertices, the corresponding p are to be set equal to zero.

$D_r(\alpha)$ is a sum of products of the α. Namely, a "skeleton" is a *simply* connect-

Multiloop Feynman diagrams, general MB integrals

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} \to \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

$$N_\nu = n_1 + \dots + n_N$$



$$\mathbf{U} \quad = \quad \mathbf{x_1 + x_2 + x_3 + x_4} \quad ! \; \textit{1-loop} \longrightarrow 1$$

$$\mathbf{F} \quad = \quad \mathbf{t \cdot x_1 x_3 + s \cdot x_2 x_4}$$

Cuts of internal lines such that:

- U: (i) every vertex is still connected to every other vertex by a sequence of uncut lines; (ii) no further cuts without violating (i)
- F: (iii) divide the graph into two disjoint parts such that within each part (i) and (ii) are obeyed and such that at least one external momentum line is connected to each part;

Dimension of MB integrals depends on factorizations of F and U!

Mellin-Barnes representations in HEP - method

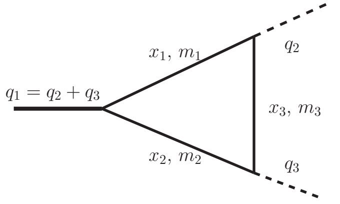
- "Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895),
"The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$
\begin{aligned}
mathematics \longrightarrow \frac{1}{(A+B)^\lambda} &= \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}} \\
physics \longrightarrow \frac{1}{(p^2-m^2)^a} &= \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z)\frac{(m^2)^z}{(p^2)^{a+z}}
\end{aligned}
$$

It is recursive $\Longrightarrow$ multidimensional complex integrals.

$$
\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1}
\times \prod_{i=1}^{n-1} A_i^{z_i}\; A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\;\Gamma(\lambda+z_1+\ldots+z_{n-1})
$$

"One-loop" example:



$$U = x_1 + x_2 + x_3 \equiv 1$$
$$F_0 = -(q_2+q_3)^2 x_1 x_2 - q_2^2 x_1 x_3 - q_3^2 x_2 x_3$$
$$F = F_0 + U(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2)$$

$$G(X) \sim \int dz_1 dz_2 dz_3 \, (-s x_1 x_2)^{z_1} (-q_2^2 x_1 x_3)^{z_2} (-q_3^2 x_2 x_3)^{z_3} \times \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)^{-z_1-z_2-z_3-N_\nu+d/2}$$

Beyond one-loop:

- $U(\vec{x}) \neq 1$
- complexity/dimensionality starts to depend on $U(\vec{x})$ structure
- nontrivial simplification of graph polynomials is needed

$x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$	3-dim representation
$(x_1 + x_2)(x_3 + x_4)$	2-dim representation
$(x_1 + x_2)(x_3 + x_4) \rightarrow$ $[x_1 \rightarrow v_1\xi_{11}, x_2 \rightarrow v_1\xi_{12}, \delta(1 - \xi_{11} - \xi_{12});$ $x_3 \rightarrow v_2\xi_{21}, \ldots] \rightarrow v_1v_2$	0-dim representation
$(x_1 + x_2)(x_3 + x_4)$ + BL	0-dim representation *)

*)

$$(x_1 + x_2)^p \rightarrow \int dx_1\ dx_2\ dz_1\ \delta(1 - x_1 - x_2) x_1^{z_1} x_2^{p-z_1} \Gamma(-z_1)\Gamma(-p + z_1)$$

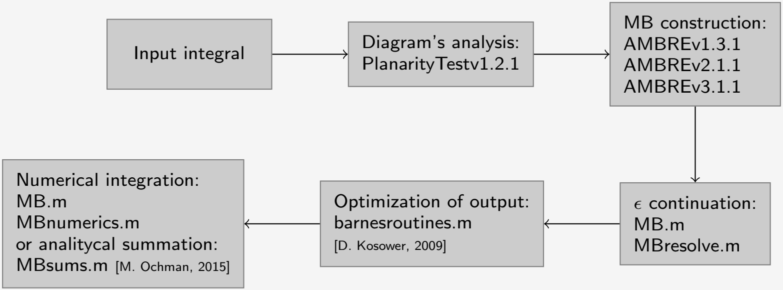
$$\rightarrow \int dz_1 \Gamma(-z_1)\Gamma(-p + z_1)\Gamma(z_1 + 1)\Gamma(p - z_1 + 1)/\Gamma(p + 2)$$

BL can be also applied without factorization, but this requires special transformation of z_i variables, see e.g., barnesroutines.m [D. Kosower, 2009]

$$\int_{-i\infty}^{i\infty} dz \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

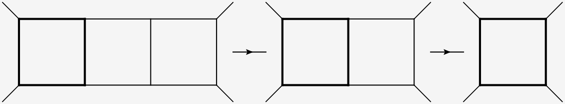
Computation of Feynman integrals with Mellin-Barnes (MB) method

Operational sequence of the MB-suite:

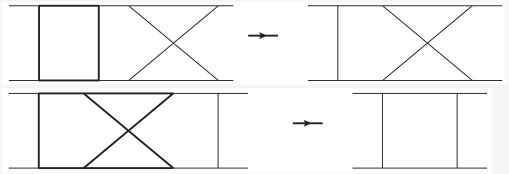


AMBRE versions overview:

- iteratively to each subloop – loop-by-loop approach (LA): mostly for planar (AMBREv1.3.1 & AMBREv2.1.1)



- in one step to the complete U and F polynomials – global approach (GA): general (AMBREv3.1.1)
- combination of the above methods – Hybrid approach (HA) (AMBREv4, coming soon)



Examples, description, links to basic tools and literature:
https://jgluza.us.edu.pl/ambre/

Limitations of GA approach

U polynomial for non-planar 3-loop box (64 terms) - *How to deal with that?*

```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

Cheng–Wu Theorem

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod\limits_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j\; x_j^{n_j-1} \delta(1-\sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The Cheng–Wu theorem states that the same formula holds with the delta function

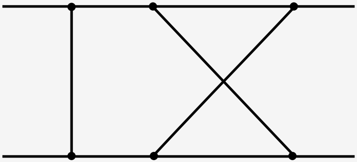
$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1,\ldots,L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.
One can prove this theorem in a simple way starting from the alpha representation using

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i=1}^{N}\alpha_i\right) \Leftrightarrow 1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i\in\Omega}\alpha_i\right)$$

and change variables from α_i to $\alpha_i = \lambda x_i$ as shown above.

Non–Planar DoubleBox

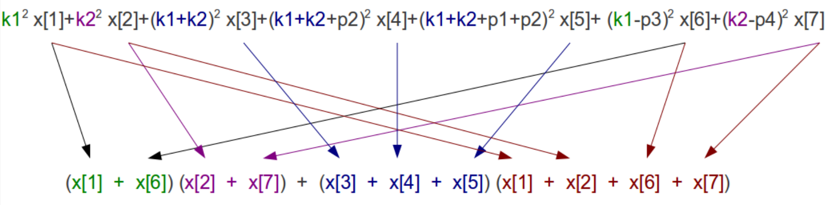


$$B_7^{NP} = \iint d^dk_1 d^dk_2 \frac{1}{[(k_1+k_2+p_1+p_2)^2]^{n_1}[(k_1+k_2+p_2)^2]^{n_2}[(k_1+K_2)^2]^{n_3}}$$
$$\frac{1}{[(k_1-p_3)^2]^{n_4}[(k_1)^2]^{n_5}[(k_2-p_4)^2]^{n_6}[(k_2)^2]^{n_7}}$$

$$U(x) = x[1]x[2] + x[1]x[4] + x[2]x[4] + x[1]x[5] + x[2]x[5] + x[2]x[6] + x[4]x[6] + x[5]x[6] + x[1]x[7] + x[4]x[7] + x[5]x[7] + x[6]x[7] \quad \mathit{11d}$$

$$F(x) = -s\ x[1]x[2]x[5] - s\ x[1]x[3]x[5] - s\ x[2]x[3]x[5] - u\ x[2]x[4]x[6] - s\ x[3]x[5]x[6] - t\ x[1]x[4]x[7] - s\ x[3]x[5]x[7] - s\ x[3]x[6]x[7] \quad \mathit{7d}$$

In this case F, U polynomials are the following



Factorization scheme

$$U(x) = (x[1]+x[6])(x[2]+x[7]) + (x[3]+x[4]+x[5])(x[1]+x[2]+x[6]+x[7])$$

$$F(x) = -t\ x[1]x[4]x[7] - u\ x[2]x[4]x[6] - s\ x[1]x[2]x[5] - s\ x[3]x[6]x[7] - s\ x[3]x[5](x[1]+x[2]+x[6]+x[7])$$

Now we can apply the Cheng-Wu theorem and integrations will look as follows

$$B_7^{NP} = \frac{(-1)^{N_\nu}\Gamma(N_\nu - d)}{\Gamma(n_1)\dots\Gamma(n_N)} \int_0^\infty dx_3 dx_4 dx_5 \int_0^1 dx_1 dx_2 dx_6 dx_7 \delta(1-(x_1+x_2+x_6+x_7)) \frac{((x_1+x_6)(x_2+x_7)+x_3+x_4+x_5)^{N_\nu - \frac{3d}{2}}}{(-t\ x_1x_4x_7 - u\ x_2x_4x_6 - s\ x_1x_2x_5 - s\ x_3x_6x_7 - s\ x_3x_5)^{N_\nu - d}}$$

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\dots\Gamma(n_N)} \int_{-i\infty}^{i\infty} dz_1\dots dz_4 \int dx_1\dots dx_7\ (-s)^{-N_\nu+d-z_2-z_3}(-t)^{z_2}(-u)^{z_3} \times \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(N_\nu - d + z_1+z_2+z_3+z_4) \times x_1^{-N_\nu+d-z_1-z_2-z_3} x_2^{z_2+z_3} x_3{}^{-N_\nu+d-z_2-z_3-z_4} x_4{}^{z_1+z_3} x_5{}^{z_2+z_4} x_6^{z_1+z_2} x_7^{z_3+z_4} \times (x_3+x_4+x_5+(x_1+x_6)(x_2+x_7))^{N_\nu-\frac{3d}{2}}$$

Integration over Cheng–Wu variables

$$\int_0^\infty dx\ x^{N_1}(x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)}$$

4-dim result:

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_7)} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_4 (-s)^{4-2\epsilon-N_\nu-z_{23}} (-t)^{z_3} (-u)^{z_2}$$

$$\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(2-\epsilon-n_{45})\Gamma(2-\epsilon-n_{67})}{\Gamma(4-2\epsilon-n_{4567})\Gamma(n_{45}+z_{1234})\Gamma(n_{67}+z_{1234})\Gamma(6-3\epsilon-N_\nu)}$$

$$\Gamma(n_2+z_{23})\Gamma(n_4+z_{24})\Gamma(n_5+z_{13})\Gamma(n_6+z_{34})\Gamma(n_7+z_{12})\Gamma^3(-2+\epsilon+n_{4567}+z_{1234})$$

$$\Gamma(4-2\epsilon-n_{124567}-z_{123})\Gamma(4-2\epsilon-n_{234567}-z_{234})\Gamma(-4+2\epsilon+N_\nu+z_{1234})$$

---

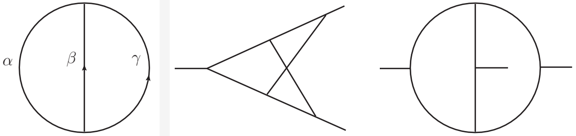
with notations $z_{i\ldots j\ldots k} = z_i + \ldots + z_j + \ldots + z_k$
and $n_{i\ldots j\ldots k} = n_i + \ldots + n_j + \ldots + n_k$

In general: $\Gamma[\Lambda_i] = \Gamma[\sum_l \alpha_{ij} z_j + \beta_i]$, *massless cases:* $\alpha_{ij} = \pm 1$

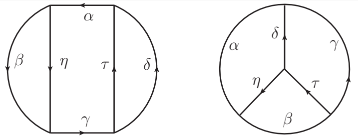
`AMBREv3.m`:

- topology based factorization - chain diagrams, Kinoshita '74

2-loop:



3-loop:



Transformation/rescaling of Feynman parameters:

$$\{\vec{x}\}_i : \;\; x_k \to v_i \xi_{ik} \;\; \times \delta\left(1 - \sum_{k=1}^{\eta_i} \xi_{ik}\right),$$

where i denotes chain index and $k \in [1, \eta_i]$, with η_i - number of propagators in chain. δ-function keeps number of variables unchanged.

For any 2-loop diagram:

$$U_{\text{2-loop}} = v_1 v_2 + v_2 v_3 + v_1 v_3$$

For any "ladder" 3-loop diagram (7-dim):

$$U_{\text{3-loop(I)}} = v_1v_2v_3 + v_1v_2v_4 + v_2v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_2v_3v_5 + v_1v_4v_5 + v_3v_4v_5$$

For any "mercedes" 3-loop diagram (15-dim):

$$U_{\text{3-loop(II)}} = v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_2v_3v_5 + v_2v_4v_5 + v_3v_4v_5$$
$$+ v_1v_2v_6 + v_2v_3v_6 + v_1v_4v_6 + v_2v_4v_6 + v_3v_4v_6 + v_1v_5v_6 + v_3v_5v_6 + v_4v_5v_6$$

- 2-loop: $\delta(1 - v_1 - v_2)$, $U(\vec{v}) = v_3 + v_1 v_2$
  no additional MB integrations from U, similar to 1-loop cases
- 3-loop: $\delta(1 - v_1 - v_2 - v_3)$
  - "ladder" - 2 additional MB integrations *64-dim → 2-dim (!)*
  - "mercedes" - 4 additional MB integrations

To get minimal dimensionality:

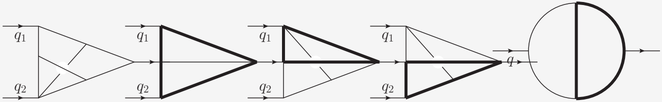
- 1-loop: $U(\vec{x}) \equiv 1$ whenever it's possible
- 2- and 3-loop: expression for F polynomial is not expanded

$$F = F_0 + U \sum_{i=1}^{N} x_i m_i^2$$

- Barnes lemmas

Methods of brackets (Schwinger parametrization)

M. Prausa, Mellin-Barnes meets Method of Brackets: a novel approach to Mellin-Barnes representations of Feynman integrals, Eur. Phys. J. C77 (9) (2017) 594. arXiv:1706.0985



(a) (b) (c) (d) (e)

diagram	Method of Brackets	AMBRE	planarity	AMBRE 4*/method
fig.(a)	**7**	13	NP	**4** $(2 \to 1)$
fig.(b)	**1**	2	P	**1**
fig.(c)	**7**	9	NP	**5**
fig.(d)	**7**	8	NP	8
fig.(e)	5	**3**	P	**3**

The number of MB integrations of the representation constructed by the Method of Brackets and AMBRE

Possible improvements

- Decoupling of Feynman variables

$$M_\Gamma Z \quad = \quad \begin{bmatrix} \alpha_{ij}(\text{numerator}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \alpha_{ij}(\text{denominator}) \end{bmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}.$$

$$\Gamma[\sum_j \alpha_{ij} z_j + \beta_i]$$

Any linear variable transformation can be represented as

$$M_\Gamma Z \quad = \quad M_\Gamma U U^{-1} Z = M_\Gamma U Z', \;\; Z' = U^{-1} Z,$$

U - non-singular $r \times r$ transformation matrix . M_Γ encodes a new z structure of gamma functions for applying `BL` or decoupling:

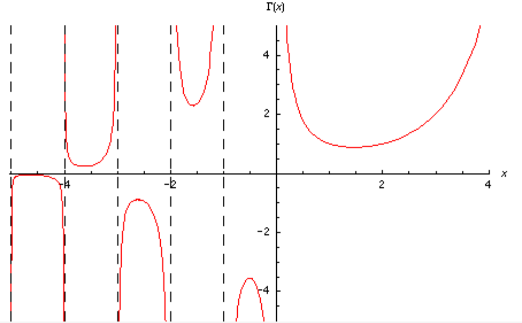
$$M_\Gamma \longrightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

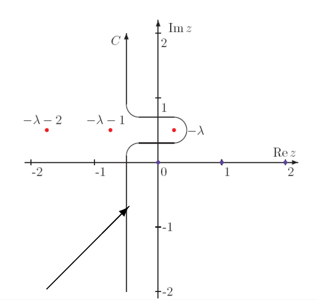
Numerical integration of MB integrals

Gamma function: Singularities in the complex plane

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad \int dz \Gamma[z+\lambda]$$

SINGULARITIES REGULAR





Contours: shifts, deformations

Asymptotic behavior: $\Gamma(z)_{|z|\to\infty} = \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\right]$

- core: ("smooth" function)

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \xrightarrow{|z|\to\infty} e^{z(\ln z - \ln(-z)) + \frac{1}{2}\ln z - \frac{5}{2}\ln(-z)}.$$

$\ln z - \ln(-z) = i\pi\, \mathrm{sign}(\Im m\, z)$

$z = z_0 + i\,t,\;\; t \in (-\infty, \infty),\; |z| \to \infty \Leftrightarrow t \to \pm\infty$

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \longrightarrow \mathbf{e}^{-\pi|\mathbf{t}|} \frac{1}{|t|^2} \;\text{(nice suppression)}$$

- kinematics: (oscillations)
  in Minkowskian case $s \to s + i\delta\, (s > 0)$ $\quad \to \frac{1}{\pm p^2 - m^2 + i\delta}$

$$\left(\frac{M_Z^2}{-s}\right)^z = e^{z\ln(-\frac{M_Z^2}{s} + i\,\delta)} \longrightarrow e^{i\,t\ln\frac{M_Z^3}{s}} \mathbf{e}^{-\pi \mathbf{t}}, s > 0$$

$e^{-\pi|t|}$ and $e^{-\pi t}$ cancel each other when $t \to -\infty$ and oscillations are **NOT** damped any more by an exponential factor

Types of contour deformations

$$V(s) = \frac{e^{\epsilon\gamma_E}}{i\pi^{d/2}} \int \frac{d^d k}{[(k+p_1)^2 - m^2][k^2][(k-p_2)^2 - m^2]} = \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots,$$

$$V_{-1}(s) = -\frac{1}{2s} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dz}{2\pi i} \underbrace{(-s)^{-z}}_{\textbf{Problem I}} \overbrace{\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)}}^{\textbf{Problem II}}$$

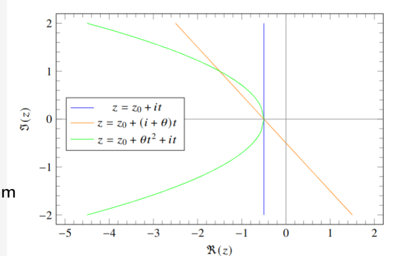
$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{2\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}},$$

$$z = \Re[z] + i\, y, \quad y \in (-\infty, +\infty),$$

$z(t) = x_0 + \theta t + it$

$\int_{-\infty}^{+\infty} (\theta + i)\ dt\ I[z(t)]$

high accuracy, no problem



Numerical integration approaches

- integration over contours parallel to imaginary axis
  - requires combination of different types of transformation to finite integration region $(-\infty, +\infty) \to [0,1]$

$$t_i \to \ln\left(\frac{x_i}{1-x_i}\right), \quad t_i \to \tan\left(\pi(x_i - \frac{1}{2})\right)$$

  - low numerical stability
  - can be improved by new integration methods/libraries
- contours deformation (restoring of the exponential damping factor)

```
In[1]: INT = -((-s)^(-z1) Gamma[-z1]^3 Gamma[1+z1])/(2*s*Gamma[-2*z1])/(2*Pi*I);
In[2]: NIntegrate[ D[-1/2+theta*t+I*t, t]*INT
       /. s->1 /. z1-> -1/2 + theta t + I t /. theta-> -1, {t,-Infinity,Infinity},
       Method -> DoubleExponential]
```

  - steepest descent method - $z_i = z_{i0} + f_i(t_1, ..., t_n) + it_i$ (JG, Jeliński, Kosower '17), only one-dimensional cases
  - rotation of integration contours - $z_i = z_{i0} + (i + \theta)t_i$ (Freitas '10) Works well for certain integrals, but is not general The core of the MB integral (gammas) becomes non-smooth

Contour shifts (MBnumerics)

PhD thesis by Johann Usovitsch,

https://edoc.hu-berlin.de/handle/18452/20256

Related and auxiliary Software

MBnumerics

Project: I. Dubovyk, T. Riemann, J. Usovitsch (jusovitsch,googlemail.com)
Software: Johann Usovitsch
Publications: https://doi.org/10.18452/19484 , https://doi.org/10.1016/j.cpc.2006.07.002, https://doi.org/10.1
To be cited by users in publications, for details see README_copyright in the downloaded tarball.
Features: MBnumerics is a software for evaluation of MB integrals in the Minkowski kinematics
Download: http://us.edu.pl/~gluza/ambre/packages/mbnumerics.tgz

- gives high accuracy results up to certain dimensionality of MB integrals
- can produce huge cascade of lower-dimensional integrals

https://jgluza.us.edu.pl/ambre

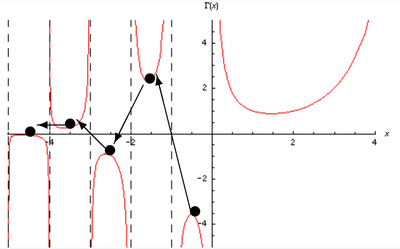
Basic observations for shifting z follows

$$\int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + n + Im[z_k], \ldots) \qquad I_{orig}$$

$$= \quad Residue[\int dz_1 \ldots \cancel{dz_k} \ldots I]_{Re[z_k]+n} \qquad I_{Res}$$

$$+ \quad \int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + (n+1) + Im[z_k], \ldots) \qquad I_{new}$$

1. Residues **lower** dimensionality of original MB integrals.
2. Integral after passing a pole (proper shifts) **can be made smaller**.



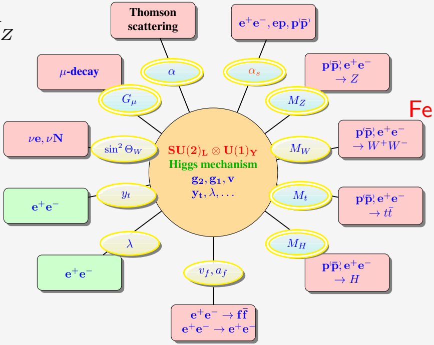
EWPOs: Needs for N^xLO corrections

(i) Input parameters and renormalization schemes
(ii) Extraction of EWPOs at the Z-pole

Input and calculated/measured parameters

Schemes: G_μ vs M_W,...

G_μ, $\sin^2\theta^\ell_{eff}$, M_Z



Fermilab!

Fig. from the FCC-ee report 'α_{QED}' by F. Jegerlehner in 1905.05078

Introduction to Precision Electroweak Analysis by J. Welss, 0512342

Input and calculated/measured parameters

Experimental values:

$$\begin{aligned}
\hat{\alpha} &= 1/137.0359895(61),\ \gamma^* \to e^+e^- \\
\hat{G}_F &= 1.16639(1) \times 10^{-5}\,\text{GeV}^{-2}\ \text{muon decay} \\
\hat{m}_Z &= 91.1875 \pm 0.0021\,\text{GeV} \\
\hat{m}_W &= 80.426 \pm 0.034\,\text{GeV} \\
\hat{s}^2_{\text{eff}} &= 0.23150 \pm 0.00016, \text{effective} \sin^2\theta_{\text{W}}, A_{LR} \equiv \frac{(1/2 - \hat{s}^2_{\text{eff}})^2 - \hat{s}^4_{\text{eff}}}{(1/2 - \hat{s}^2_{\text{eff}})^2 + \hat{s}^4_{\text{eff}}} \\
\hat{\Gamma}_{l+l-} &= 83.984 \pm 0.086\,\text{MeV}
\end{aligned}$$

$$\left\{\begin{array}{l} \mathbf{g}(= e/s_W)\ SU(2) \\ \mathbf{g}'(e/c_W)\ U(1)_Y \\ \mathbf{v}\ \text{VEV}, \end{array}\right. \longrightarrow \left\{\begin{array}{l} \hat{\alpha} = \frac{e^2}{4\pi} \\ \hat{G}_F = \frac{1}{\sqrt{2}v^2} \\ \hat{m}_Z^2 = \frac{e^2 v^2}{4s^2c^2} \\ \hat{m}_W^2 = \frac{e^2v^2}{4s^2} \\ \hat{s}^2_{\text{eff}} = s^2 \\ \hat{\Gamma}_{l+l-} = \frac{v}{96\pi}\frac{e^3}{s^3c^3}\left[\left(-\frac{1}{2} + 2s^2\right)^2 + \frac{1}{4}\right] \end{array}\right.$$

$$m_f = y_f v$$

Shaping the SM, tree level estimates

In terms of $\hat{\alpha}, \hat{G}_F$ and $\hat{m}_Z$

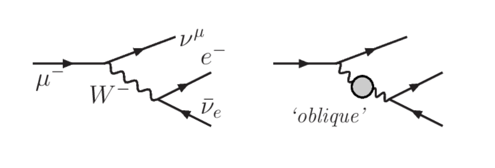
$$\hat{m}_W^2 = \pi\sqrt{2}\hat{G}_F^{-1}\hat{\alpha}\left(1-\sqrt{1-\frac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{m}_Z^2}}\right)^{-1}$$

$$\hat{s}_{\rm eff}^2\hat{c}_{\rm eff}^2 = \frac{\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{m}_Z^2} \equiv \hat{s}_{\rm eff}^2 = \frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{m}_Z^2}}$$

$$\hat{\Gamma}_{l^+l^-} = \frac{\sqrt{2}\hat{G}_F\hat{m}_Z^3}{12\pi}\left\{\left(\frac{1}{2}-\sqrt{1-\frac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{m}_Z^2}}\right)^2+\frac{1}{4}\right\}$$

$$\begin{aligned} Prediction : \hat{m}_W &= 80.939 \pm 0.003\,\mathrm{GeV}\ 15\sigma\ \text{away} \\ Prediction : \hat{s}_{\rm eff}^2 &= 0.21215 \pm 0.00003\ 120\sigma\ \text{away} \\ Prediction : \hat{\Gamma}_{l^+l^-} &= 84.843 \pm 0.012\,\mathrm{MeV}\ 10\sigma\ \text{away} \end{aligned}$$

Shaping SM, oblique corrections also not sufficient



$$\tau_\mu^{-1} = \frac{\hat{G}_F^2 m_\mu^5}{192\pi^3} K(\alpha, m_e, m_\mu, m_W)$$

$$\begin{aligned}\frac{(\hat{G}_F)^{\rm th}}{\sqrt{2}} &= \frac{g^2}{8m_W^2}\left[1 + i\Pi_{WW}(q^2)\left(\frac{-i}{q^2 - m_W^2}\right)\right]_{q\to 0} \\ &= \frac{1}{2v^2}\left[1 - \frac{\Pi_{WW}(0)}{m_W^2}\right].\end{aligned}$$

Primary role of SM radiative corrections, F. Jegerlehner, in 1905.05078

$$\sin^2\Theta_i\,\cos^2\Theta_i = \frac{\pi\,\alpha}{\sqrt{2}\,G_\mu\,M_Z^2}\,\frac{1}{1-\Delta r_i} \quad \Delta r_i = \Delta r_i(\alpha, G_\mu, M_Z, m_H, m_{f\neq t}, m_t)\,,$$

$$\Delta r_i = -\frac{c_W^2}{s_W^2}\,\Delta\rho + \Delta r_{i\,\mathrm{reminder}}\,,$$

$$\Delta\rho \equiv \left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} + 2\frac{s_W}{c_W}\frac{\Pi_{\gamma Z}}{M_Z^2}\right)_{q^2=0} = \frac{3\,m_t^2\,\sqrt{2}G_\mu}{16\,\pi^2}$$

$$\hat{\alpha}(m_Z) = \frac{\hat{\alpha}}{1-\Delta\alpha(m_Z)} = \frac{e^2}{4\pi}\left[1+\frac{\Pi_{\gamma\gamma}(m_Z)}{m_Z^2}\right] \sim 128\ (137\text{ at the Thomson limit})$$

Still, well visible disagreement between SM prediction and experiment for EWPOs without subleading SM corrections, and only with the leading corrections $\Delta\alpha(m_Z)$ and $\Delta\rho$.

$r_{i\,\mathrm{reminder}}$ **matters!**

s_W^2

The weak mixing angle $s_W^2 \equiv \sin^2\theta_W$ has three potential different meanings or functions in the model-building:

(i) It describes the ratio of the two gauge couplings,

$$g'/g = c_W/s_W,$$

usually in the $\overline{\text{MS}}$ scheme.

(ii) It describes the ratio of two gauge boson (on-shell) masses,

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2}.$$

(iii) It describes the ratio of the vector and axial-vector couplings of an (on-shell) Z boson to fermions,

$$\frac{v_f}{a_f} = 1 - 4|Q_f|s_W^2.$$

This definition is called the effective weak mixing angle, denoted as $\sin^2\theta_W^{f,\mathrm{eff}}$.

(iv) or ... LHC $(\alpha/G_\mu, \sin^2\theta^f_{\text{eff}}, M_Z)$

M. Chiesa, F. Piccinini, A. Vicini, Direct determination of $\sin^2\theta^\ell_{eff}$ at hadron colliders, PRD, 1906.11569

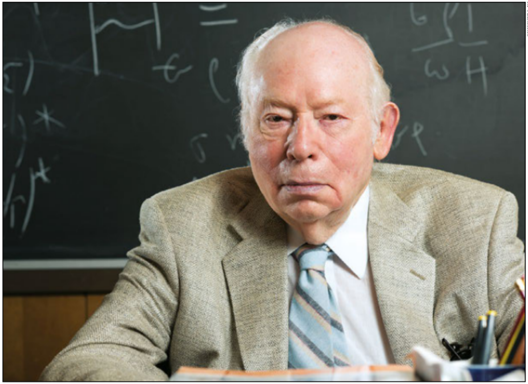
$\sin^2\theta^f_{\text{eff}}$ fixed at measured leptonic $\sin^2\theta^f_{\text{eff}}$ requiring v_l/a_l does not get radiative corrections. Procedure independent of QED corrections (both couplings get the same QED corrections and we have a ratio).

STEVEN WEINBERG 1933–2021

A mind to rank with the greatest

Steven Weinberg, one of the greatest theoretical physicists of all time, passed away on 23 July, aged 88. He revolutionised particle physics, quantum field theory and cosmology with conceptual breakthroughs that still form the foundation of our understanding of physical reality.

Weinberg is well known for the unified theory of weak and electromagnetic forces, which earned him the Nobel Prize in Physics in 1979, jointly awarded with Sheldon Glashow and Abdus Salam, and led to the prediction of the Z and W vector bosons, later discovered at CERN in 1983. His breakthrough was the realisation that some new theoretical ideas, initially believed to play a role in the description of nuclear strong interactions, could instead explain the nature of the weak force. "Then it suddenly occurred to me that this was a perfectly good sort of theory, but I was applying it to the wrong kind of interaction. The right place to apply these ideas was not to the strong interactions, but to the weak and electromagnetic interactions," as he later recalled. With his work, Weinberg had made the next step in the unification of physical laws, after Newton understood that the motion of apples on Earth and planets in the sky are governed by the same gravitational force, and Maxwell understood that electric and magnetic phenomena are the expression of a single force.



M Valentine

Steven Weinberg radically changed the way we look at the universe.

In my life, I have built only *one* model

physicists, and will certainly continue to serve future generations.

Steven Weinberg is among the very few individuals who, during the course of the history

F. Jegerlehner, in 1905.05078

Example: the W and Z mass from $\alpha(M_Z)$, G_μ *and* $\sin^2\Theta_{\ell\,\mathrm{eff}}$:
(i) $\sin^2\theta_{\ell,\mathrm{eff}}(M_Z) = \left(1+\frac{\cos^2\Theta_W}{\sin^2\Theta_W}\,\Delta\rho\right)\sin^2\Theta_W\,, \qquad \sin^2\Theta_W = 1 - M_W^2/M_Z^2$

$$\Delta\rho = \frac{3\,M_t^2\,\sqrt{2}G_\mu}{16\,\pi^2}\ ;\ \ M_t = 173 \pm 0.4\ GeV$$

The solution with exp. input $\sin^2\theta_{\ell,\mathrm{eff}}(M_Z) \equiv (1 - v_\ell/a_\ell)/4 = 0.23148$ is

$$\sin^2\Theta_W = 0.22426.$$

(ii) Predicting then the masses we have

$$M_W = \frac{A_0}{\sin^2\Theta_W}\ ;\ \ A_0 = \sqrt{\frac{\pi\alpha}{\sqrt{2}G_\mu}}\ ;\ \ M_Z = \frac{M_W}{\cos\Theta_W}$$

where, including photon VP correction $\alpha^{-1}(M_Z) = 128.953 \pm 0.016$. For the W, Z mass we get

$$\begin{aligned} M_W^{\mathrm{TH}} &= 81.1636 \pm 0.0346\ ;\ \ M_Z^{\mathrm{TH}} = 92.1484 \pm 0.0264\ . \\ M_W^{\mathrm{exp}} &= 80.379 \pm 0.012\ ;\ \ M_Z^{\mathrm{exp}} = 91.1876 \pm 0.0021\ \mathrm{GeV}, \end{aligned}$$

Deviavions (errors added in quadrature): $W:\ \ 23\,\sigma\ ;\ \ Z:\ \ 36\,\sigma$
Adding 1-loop and leading 2-loop we go down below 2 σ.

Input, theoretical and parametric errors,

A. Freitas et al., "Theoretical uncertainties for electroweak and Higgs-boson precision measurements at FCC-ee", https://arxiv.org/abs/1906.05379

Quantity	FCC-ee	Current intrinsic error		Projected intrinsic error (at start of FCC-ee)
M_W [MeV]	0.5–1 [‡]	4	$(\alpha^3, \alpha^2\alpha_s)$	1
$\sin^2\theta^\ell_{\rm eff}$ $[10^{-5}]$	0.6	4.5	$(\alpha^3, \alpha^2\alpha_s)$	1.5
Γ_Z [MeV]	0.1	0.4	$(\alpha^3, \alpha^2\alpha_s, \alpha\alpha_s^2)$	0.15
R_b $[10^{-5}]$	6	11	$(\alpha^3, \alpha^2\alpha_s)$	5
R_l $[10^{-3}]$	1	6	$(\alpha^3, \alpha^2\alpha_s)$	1.5

[‡] The pure experimental precision on M_W is $\sim 0.5\,\mathrm{MeV}$.

Quantity	FCC-ee	future parametric unc.	Main source
M_W [MeV]	$0.5 - 1$	1 (0.6)	$\delta(\Delta\alpha)$
$\sin^2\theta^\ell_{\rm eff}$ $[10^{-5}]$	0.6	2 (1)	$\delta(\Delta\alpha)$
Γ_Z [MeV]	0.1	0.1 (0.06)	$\delta\alpha_s$
R_b $[10^{-5}]$	6	< 1	$\delta\alpha_s$
R_ℓ $[10^{-3}]$	1	1.3 (0.7)	$\delta\alpha_s$

Important input parameter errors are $\delta(\Delta\alpha) = 3 \cdot 10^{-5}$, $\delta\alpha_s = 0.00015$.

E.g. the bosonic 2-loop corrections shift the value of Γ_Z by 0.51 MeV when using M_W as input and 0.34 MeV when using G_μ as input.

Reminder: $\delta\Gamma_{Z,\mathrm{FCC-ee}} = 0.1$ MeV

I. Dubovyk, A. Freitas, JG, T. Riemann, J. Usovitsch,
https://doi.org/10.1016/j.physletb.2018.06.037

Γ_i [MeV]	$\Gamma_e, \Gamma_\mu, \Gamma_\tau$	$\Gamma_{\nu_e}, \Gamma_{\nu_\mu}, \Gamma_{\nu_\tau}$	Γ_d, Γ_s	Γ_u, Γ_c	Γ_b	Γ_Z
Born	81.142	160.096	371.141	292.445	369.56	2420.2
$\mathcal{O}(\alpha)$	2.273	6.174	9.717	5.799	3.857	60.22
$\mathcal{O}(\alpha\alpha_\mathrm{s})$	0.288	0.458	1.276	1.156	2.006	9.11
$\mathcal{O}(N_f^2\alpha^2)$	0.244	0.416	0.698	0.528	0.694	5.13
$\mathcal{O}(N_f\alpha^2)$	0.120	0.185	0.493	0.494	0.144	3.04
$\mathcal{O}(\alpha^2_\mathrm{bos})$	**0.017**	**0.019**	**0.058**	**0.057**	**0.167**	**0.505**
$\mathcal{O}(\alpha_\mathrm{t}\alpha_\mathrm{s}^2,\ \alpha_\mathrm{t}\alpha_\mathrm{s}^3,\ \alpha_\mathrm{t}^2\alpha_\mathrm{s},\ \alpha_\mathrm{t}^3)$	0.038	0.059	0.191	0.170	**0.190**	1.20

* Fixed values of M_W
(α, G_μ, M_Z) or $(\mathrm{M_W}, \mathrm{G}_\mu, \mathrm{M_Z})$ or $(\mathrm{G}_\mu, \mathrm{s_W^2}, \mathrm{M_Z})$, ...?

We have to describe

$$e^+e^- \longrightarrow (\gamma, Z) \longrightarrow f^+f^-(\gamma),$$

S-matrix Ansatz in the complex energy plane

$$\mathcal{A}^{e^+e^-\to b\bar{b}} = \underbrace{\frac{R_Z}{s-s_Z} + \overbrace{\frac{R_\gamma}{s}}_{\gamma - Z\ interference} + S + (s-s_Z)S' + \ldots}^{Background},$$

$$s_Z = \overline{M}_Z^2 - i\overline{M}_Z\overline{\Gamma}_Z$$

- $R, S, S', \ldots$ are individually gauge-invariant and UV-finite - unitarity and analyticity of the S-matrix. IR-finite, when soft and collinear real photon emission is added. [Willenbrock, Valencia,1991] [Sirlin,1991] [Stuart,1991] [Riemann, 1991, 1992] [H. Veltman,1994] [Passera, Sirlin, 1998] [Gambino, Grassi, 2000] [Awramik, Czakon, Freitas, 2006].

The term $R_\gamma(s)/s$ is part of the the background

- The poles of $\mathcal{A}$ have complex residua R_Z and R_γ.
- There is only ONE pole in mathematics, while in physics we observe two of them: photon exchange at $s = 0$, Z exchange at $s_0 = s_Z$. Mathematicaly, the appearance of the photon pole is result of summing of part of background around Z pole, $s_0 = s_Z$

[Tera-Z report 2019]

$$
\begin{aligned}
\frac{R_\gamma(s)}{s} &= \frac{\sum_{n=0}^{\infty} R_n (s-s_0)^n}{s} \\
&= \frac{\sum_{n=0}^{\infty} R_n (s-s_0)^n}{s_0 - (s_0 - s)} \\
&= \sum_{n=0}^{\infty} R_n (s-s_0)^n \; \frac{1}{s_0} \; \frac{1}{1 - \frac{s_0 - s}{s_0}} \\
&= \sum_{n=0}^{\infty} R_n (s-s_0)^n \; \frac{1}{s_0} \; \left[1 + \frac{s_0 - s}{s_0} + \left(\frac{s_0 - s}{s_0} \right)^2 \cdots \right];
\end{aligned}
$$

Born level and beyond

$$\mathcal{M}_Z^{(0,B)}(e^-e^+ \to f^-f^+) = 4ie^2 \frac{\chi_Z(s)}{s} \left(v_e^B - a_e^B\gamma_5\right)\gamma_\alpha \otimes \left(v_f^B - a_f^B\gamma_5\right)\gamma^\alpha,$$

$$\mathcal{M}_\gamma^{(0,B)}(e^-e^+ \to f^-f^+) = \frac{ie^2}{s} Q_e Q_f \, \gamma_\alpha \otimes \gamma^\alpha,$$

$$\mathcal{M}_\gamma^{(0)}(e^-e^+ \to f^-f^+) = \frac{4\pi i\alpha_{em}(s)}{s} Q_e Q_f \, \gamma_\alpha \otimes \gamma^\alpha,$$

$$\mathcal{M}_Z^{(0)}(e^-e^+ \to f^-f^+) = 4ie^2 \frac{\chi_Z(s)}{s} \Big[M_{vv}^{ef} \, \gamma_\alpha \otimes \gamma^\alpha - M_{av}^{ef} \, \gamma_\alpha\gamma_5 \otimes \gamma^\alpha - M_{va}^{ef} \, \gamma_\alpha \times \gamma^\alpha\gamma_5 + M_{aa}^{ef} \, \gamma_\alpha\gamma_5 \otimes \gamma^\alpha\gamma_5 \Big].$$

Matching Born

$$M_{vv}^{ef,B} = v_e^B v_f^B, \quad M_{va}^{ef,B} = v_e^B a_f^B, \quad M_{av}^{ef,B} = a_e^B v_f^B, \quad M_{aa}^{ef,B} = a_e^B a_f^B.$$

This factorization is spoiled at 10^{-4}

Born level and beyond

$$\mathcal{M}_Z^{(0)}(e^-e^+\to f^-f^+) = 4ie^2\frac{\chi_Z(s)}{s}\big[M_{vv}^{ef}\,\gamma_\alpha\otimes\gamma^\alpha - M_{av}^{ef}\,\gamma_\alpha\gamma_5\otimes\gamma^\alpha - M_{va}^{ef}\,\gamma_\alpha\times\gamma^\alpha\gamma_5 + M_{aa}^{ef}\,\gamma_\alpha\gamma_5\otimes\gamma^\alpha\gamma_5\big].$$

$$M_{aa}^{ef} = I_e I_f\ \rho_Z, \quad \frac{M_{av}^{ef}}{M_{aa}^{ef}} \equiv 1-4|Q_f|\kappa_f\sin^2\theta_W, \quad \frac{M_{va}^{ef}}{M_{aa}^{ef}} \equiv 1-4|Q_e|\kappa_e\sin^2\theta_W,$$

$$\frac{M_{vv}^{ef}}{M_{aa}^{ef}} \equiv 1-4(|Q_e|\kappa_e+|Q_f|\kappa_f)\sin^2\theta_W+16|Q_eQ_f|^2\sin^4\theta_W\kappa_{ef},$$

$$\mathcal{M}_{\mathbf{Z}}^{(\mathbf{0})}(s,t) \sim 4ie^2\frac{\chi_Z(s)}{s}\ I_eI_f\rho_Z(s,t)\ \big\{\gamma_\alpha(1-\gamma_5)\otimes\gamma^\alpha(1-\gamma_5) -4|Q_e|\sin^2\theta_W\kappa_e(s,t)\gamma_\alpha\otimes\gamma^\alpha(1-\gamma_5)-4|Q_f|\sin^2\theta_W\kappa_f(s,t)\gamma_\alpha(1-\gamma_5)\otimes\gamma^\alpha +16|Q_eQ_f|\sin^4\theta_W\kappa_{ef}(s,t)\gamma_\alpha\otimes\gamma^\alpha\big\}.$$

General prescription, WW, ZZ boxes, photonic, BSM included!

EW SM theory at loops, an example ($\Delta_{ef} \neq 0$)

$$\begin{cases} \Gamma_Z, \Gamma_{partial} \\ A^{eff.,Born}_{FB,peak}, A^{eff.,Born}_{LR,peak} \\ R_b, R_\ell, \ldots \end{cases} \longrightarrow \begin{cases} v^{eff}_{l,\nu,u,d,b} \\ a^{eff}_{l,\nu,u,d,b} \\ \sin^2\theta^b_{\mathrm{eff}}, \sin^2\theta^{lept}_{\mathrm{eff}} \end{cases}$$

e.g. : improvements needed for subtle corrections $\Delta_{1,2}$ *(e.g. boxes, 5PF)*

$$\begin{aligned} A^{eff.,Born}_{FB,peak} &= \frac{\sigma_f\left[\theta < \frac{\pi}{2}\right] - \sigma_f\left[\theta > \frac{\pi}{2}\right]}{\sigma_f\left[\theta < \frac{\pi}{2}\right] + \sigma_f\left[\theta > \frac{\pi}{2}\right]} \\ &= \frac{2\Re e\left[\frac{v_e a_e^*}{|a_e|^2}\right] 2\Re e\left[\frac{v_f a_f^*}{|a_f|^2}\right]}{\left(1+\frac{|v_e|^2}{|a_e|^2}\right)\left(1+\frac{|v_f|^2}{|a_f|^2}\right)} + \Delta_1 - \Delta_2 \simeq \frac{3}{4} A_e A_f, \\ \Delta_1 &= 2\Re e\left[\Delta_{ef}\right],\ \Delta_2 = |\Delta_{ef}|^2 + 2\Re e\left[\frac{v_e a_e^*}{|a_e|^2}\frac{v_f a_f^*}{|a_f|^2}\Delta^*_{ef}\right], \\ \Delta_{ef} &= 16|Q_e Q_f| s_W^4(\kappa_{ef} - \kappa_e\kappa_f), \text{factorization} : \kappa_{\mathrm{ef}} = \kappa_{\mathrm{e}}\kappa_{\mathrm{f}}, \Delta_{ef} \to 0. \end{aligned}$$

Summary and Outlook[1,*]

1. Challenges at Z-pole:
   1.1 3-loop EW and mixed EW-QCD, leading 4-loop corrections for $Z \to 2f$ vertices
   1.2 QED interference effects, non-factorizable corrections
   1.3 Adjusting MC generators at NNLO and beyond (Bhabha (!), exclusive NNLO $e^+e^- \to f\bar{f}$).
2. Challenge to improve input parameters (α, α_s, physics at ZH, WW, tt)
3. Challenge to optimize/understand paths towards BSM discovery (RHNs, DM, CP effects,...)
4. Challenge: SM(BSM)EFT, precision physics for concrete BSM models
5. Challenge: Tools (MC generators, ~~multiloop numerical~~, analytical programs)

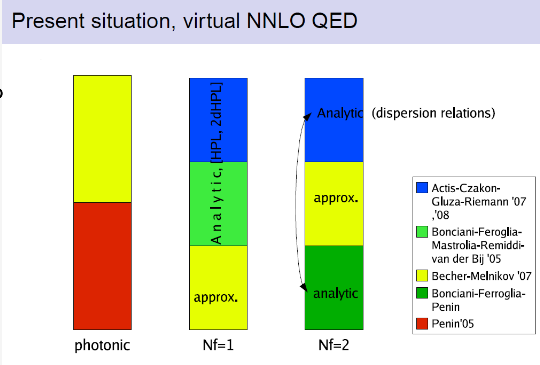
* *'FCC-ee: the challenge for theory', talk at 4th FCC Physics and Experiments Workshop, link*

[1] 'At each meeting it always seems to me that very little progress is made. Nevertheless, if you look ever any reasonable length of time, a few years say, you find a fantastic progress and it is hard to understand how that can happen at the same time that nothing is happening in anyone moment (zeno's paradox).' - R.P. Feynman

BACKUP

Waves of changes (in methods efficiency)

$e^+e^- \to e^+e^-$, ∼15 years ago



+ J. Henn, V. Smirnov, 2013 - analytic solutions for planar cases.

It is reasonable to keep developing different methods, complementarity, cross-checks etc.

Other directions (1)

K.H. Phan and T. Riemann, Phys. Lett. B791 (2019) 257 (The general d-dependence of 1-loop Feynman integrals) + numerics,

(a) ${}_2F_1$ Gauss hypergeometric functions are needed for self-energies;
(b) F_1 Appell functions are needed for vertices;
(c) F_S Lauricella-Saran functions are needed for boxes.

New approach to Mellin–Barnes integrals for massive one-loop Feynman integrals, Johann Usovitsch, Tord Riemann Tera-Z report, section E.6., arXiv:1809.01830, doi:10.23731/CYRM-2019-003

`MBOneLoop` package.

$$J_n = (-1)^n \Gamma(n-d/2) \int_0^1 \prod_{i=1}^n dx_i \delta\left(1 - \sum_{j=1}^n x_j\right) \frac{1}{F_n(x)^{n-d/2}}$$

F-function rewritten with $\delta(1-\sum x_i)$ which makes the n-fold x-integration to be an integral over an $(n-1)$-simplex.

$$J_n(d,\{q_i,m_i^2\}) = \frac{-1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s)\Gamma(\frac{d-n+1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} \left(\frac{1}{R_n}\right)^s \times \sum_{k=1}^n \left(\frac{1}{R_n}\frac{\partial r_n}{\partial m_k^2}\right) \mathbf{k}^- J_n(d+2s;\{q_i,m_i^2\}).$$

- *Recursion formula which gives the minimal integration dimension for 1-loop Mellin-Barnes integrals compared to following the U and F polynomial aproach (e.g. 9dim box → 3-dim).* *We would like to see such recursion formulas at multi-loop level*

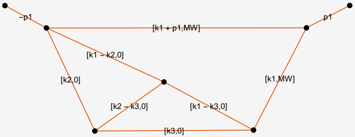
Other directions (2)

Summations, asymptotics, hypergeometric functions

- J. Davies, G. Mishima, M. Steinhauser, D. Wellmann, JHEP 03 (2018) 048;

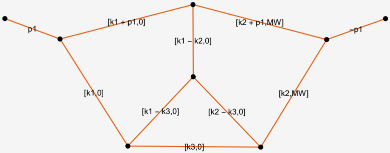
$$\int_C \frac{dz}{2\pi i} \frac{\Gamma[a_1 - z, a_2 - z, a_3 + z, a_4 + z, a_5 + z]}{\Gamma(-a_6 + z)}$$
$$= \frac{\Gamma[a_{13}, a_{23}, a_{14}, a_{24}, a_{15}, a_{25}]}{\Gamma[a_{1235}, a_{1245}, -a_{56}]} \, {}_3F_2 \left(\begin{matrix} a_{15}, a_{25}, a_{123456} \\ a_{1235}, a_{1245} \end{matrix} ; 1 \right),$$

- B. Ananthanarayan, S. Banik, S. Friot, S. Ghosh, Multiple Series Representations of N-fold Mellin-Barnes Integrals, Phys. Rev. Lett. 127 (15) (2021);
  B. Ananthanarayan, Souvik Bera, S. Friot, T. Pathak, Olsson.wl : a Mathematic package for the computation of linear transformations of multivariable hypergeometric functions, 2201.01189;



1-dim

$-18.779406962 - 6.390785027i$



4-dim

$-22.5213 + 4.74442i \pm (0.001 + 0.001i)$

$$I = -\frac{1}{(-s)^{1+3\epsilon}} \int_{-i\infty}^{+i\infty} \prod_{i=1}^{4} dz_i \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-\epsilon - z_1)\Gamma(-z_1)\Gamma(1+2\epsilon+z_1)}{\Gamma(1-2\epsilon)\Gamma(1-3\epsilon-z_1)}$$
$$\times \frac{\Gamma(-2\epsilon - z_{12})\Gamma(1-\epsilon+z_2)\Gamma(1+z_{12})\Gamma(1+\epsilon+z_{12})\Gamma(1+3\epsilon+z_3)\Gamma(1-\epsilon-z_4)}{\Gamma(1-z_2)\Gamma(2+\epsilon+z_{12})}$$
$$\times \frac{\Gamma(-\epsilon-z_2)\Gamma(-z_2)\Gamma(1+z_3-z_4)\Gamma(-z_4)\Gamma(-z_3+z_4)\Gamma(-3\epsilon-z_3+z_4)}{\Gamma(1-4\epsilon-z_3)\Gamma(2+2\epsilon+z_3-z_4)}.$$

$$I = \frac{3}{s} \int_{-i\infty-\frac{17}{28}}^{+i\infty-\frac{17}{28}} dz_3 \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-1-z_3)\Gamma(-z_3)(\Gamma(1-z_3)\Gamma(-z_3) - \Gamma(-2z_3))\Gamma(1+z_3)\psi^{(2)}(z_1)}{\Gamma(1+z_3)\Gamma(-2z_3)}.$$

Numerical integration of MB integrals

In the most general form MB integral can be represented as follows:

$$I = \frac{1}{(2\pi i)^r} \int_{-i\infty+z_{10}}^{+i\infty+z_{10}} \cdots \int_{-i\infty+z_{r0}}^{+i\infty+z_{r0}} \prod_i^r dz_i \; f_S(Z) \, \frac{\prod_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod_{k=1}^{N_d} \Gamma(\Lambda_k)} f_\psi(Z)$$

$f_S(Z)$ depends on: Z – some subset of integration variables
S – kinematic parameters and masses

Λ_i : linear combinations of z_i, e.g., $\Lambda_i = \sum_l \alpha_{il} z_l + \gamma_i$

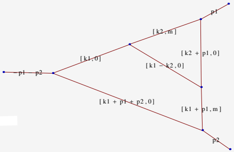
An example:

$$I^{0h0w}_{5,\epsilon^{-2}} = \frac{1}{2s} \frac{1}{2\pi i} \int_{-i\infty-\frac{1}{2}}^{+i\infty-\frac{1}{2}} dz \left(\frac{M_Z^2}{-s} \right)^z \frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)}$$

Progress for critical cases (quasi-Monte Carlo).

https://www.actaphys.uj.edu.pl/R/50/11/1993/pdf

With QMC, we can approach MB integrals with dim > 5.



$$I = \frac{1}{(2\pi i)^3}\frac{1}{s^2}\int_{-i\infty}^{i\infty} dz_1 \int_{-i\infty}^{i\infty} dz_2 \int_{-i\infty}^{i\infty} dz_3 \left(\frac{m^2}{-s}\right)^{z_1} \frac{\Gamma(-1-z_1)...\Gamma(-z_1-z_2+z_3)}{\Gamma(-z_1)\Gamma(1-z_2)\Gamma(1-z_1+z_3)}.$$

Overlaped integrals

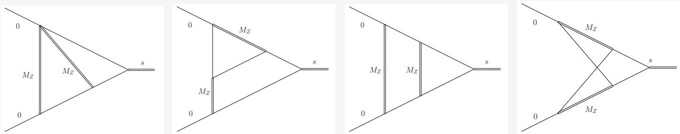
Numerical results for I with $s = m^2 = 1$.

Analytical	$-1.\mathbf{199526183135}$	$+5.\mathbf{567365907880}i$	
MB	$-1.\mathbf{199526183}168$	$+5.\mathbf{567365907}904i$	Cuhre, 10^7, 10^{-8}
MB	$-1.\mathbf{20}4597845834$	$+5.\mathbf{567}518701898i$	Vegas, 10^7, 10^{-3}
MB	$-1.\mathbf{1995}16455248$	$+5.\mathbf{5673}76681167i$	QMC, 10^6, 10^{-5}
MB	$-1.\mathbf{19952}7580305$	$+5.\mathbf{56736}7345229i$	QMC, 10^7, 10^{-6}

MB and SD methods are very much complementary!

- MB works well for hard threshold, on-shell cases, not many internal masses (more IR);
  SD more useful for integrals with many internal masses

10^{-8} accuracy achieved for **any** self-energy and vertex Feynman integral with one of the methods - in **Minkowskian region**.



The MBnumerics.m package

```
gluza@gluza-x1:~/calculations/MBnumerics/MBnumericsv2/MBnumerics_gi
libcuba4.a          README              res_zbb_fig1c_mink
libkernlib.a        README_copyright    run_script_1loop_QED_vertex_
libmathlib.a        res_1loop_QED_eucl  run_script_1loop_QED_vertex_
MB.m                res_1loop_QED_mink  run_script_zbb_fig1a_example
MBnumericsv2.m      res_zbb_fig1a_eucl  run_script_zbb_fig1a_example
MBsplits.m          res_zbb_fig1a_mink  run_script_zbb_fig1c_example
plb16_examples.nb   res_zbb_fig1c_eucl  run_script_zbb_fig1c_example
```

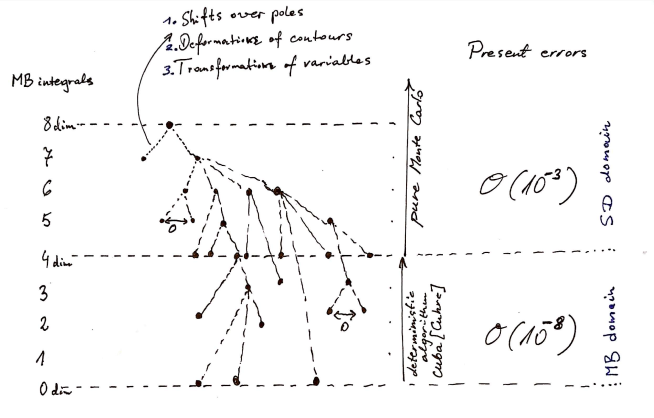
Needs:

1. MB.m
2. Cuba/Cuhre library
3. CERNlib

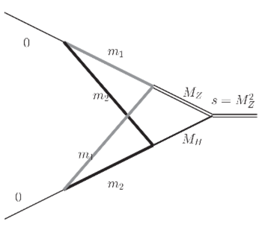
hands-on examples on-line.

Top-bottom approach to evaluation of multidimensional MB integrals

MBnumerics.m - I. Dubovyk, J. Usovitsch, T. Riemann



2-loops $\longrightarrow$ 3-loops

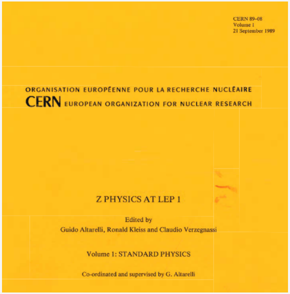


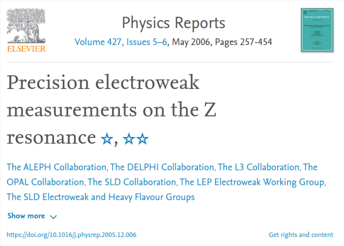
$$m_1 = M_t, m_2 = M_W$$

The integrals contain up to three dimensionless parameters

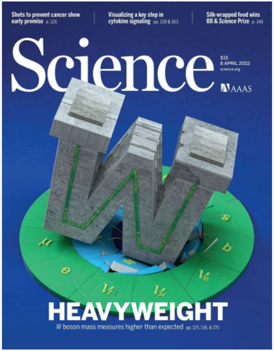
$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{(M_Z + i\varepsilon)^2}{M_Z^2} \right\}$$

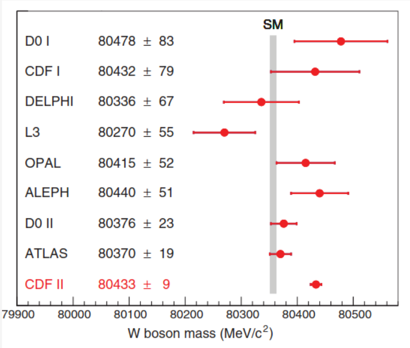
LEP and Tera-Z,





Link: 1989, Z Physics at LEP1 : vol. 1 : Standard Physics
Link: 2006, The Physics Report from the LEP & SLD EW WG
LEP2 (W-physics) Link: 2000, Reports of the WGs on Precision Calculations for LEP2 Physics
D. Bardin, M. Grunewald, G. Passarino, Precision calculation project report, Link: 1999, hep-ph/9902452
FCC-ee report, Link: 2019, Standard Model Theory for the FCC-ee Tera-Z stage





Science 376 (2022) 6589, 170-176

$$\begin{aligned} \text{SM} : M_W &= 80.357 \pm 6 \text{ MeV}, \ (\text{PDG2020}) \\ \text{Global} : M_W &= 80.379 \pm 12 \text{ MeV}, \ (\text{PDG2020}) \\ \text{CDFII} : M_W &= 80433.5 \pm 9.4 \text{ MeV} \end{aligned}$$

$$\texttt{FCC-ee}\ \text{forecast} : M_W = X \pm \mathbf{0.4}\ \text{MeV!}$$

Conclusion?

EWPOs - refers to $|M|^2$; EWPPs - refers to M

Beyond Born level, one can write

$$\mathcal{M}_{\gamma}^{(0)}(e^-e^+ \to f^-f^+) = \frac{4\pi i \alpha_{em}(s)}{s} Q_e Q_f \, \gamma_\alpha \otimes \gamma^\alpha,$$

$$\mathcal{M}_{Z}^{(0)}(e^-e^+ \to f^-f^+) = 4ie^2 \frac{\chi_Z(s)}{s} \Big[M_{vv}^{ef} \gamma_\alpha \otimes \gamma^\alpha - M_{av}^{ef} \gamma_\alpha \gamma_5 \otimes \gamma^\alpha$$
$$- M_{va}^{ef} \gamma_\alpha \times \gamma^\alpha \gamma_5 + M_{aa}^{ef} \gamma_\alpha \gamma_5 \otimes \gamma^\alpha \gamma_5 \Big].$$

In the pole scheme, where $\bar{M}_Z$ is defined as the real part of the pole of the S matrix, one has

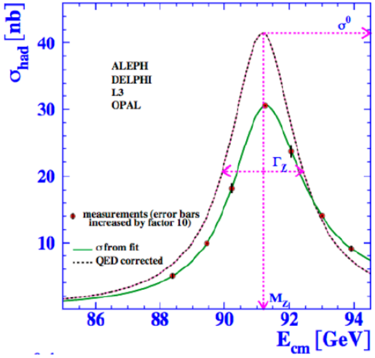
$$\chi_Z(s) = \frac{G_F M_Z^2}{\sqrt{2}\, 8\pi\alpha_{em}} K_Z(s) \simeq \frac{1}{1 + i\frac{\bar{\Gamma}_Z}{\bar{M}_Z}} \frac{s}{s - \bar{M}_Z^2 + i\bar{M}_Z\bar{\Gamma}_Z} \simeq \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z(s)},$$

$$\Gamma_Z(s) = \frac{s}{M_Z^2}\, \Gamma_Z$$

QED unfolding

Altogether $17 \cdot 10^6$ Z-boson decays at LEP

- Cross section : Z mass and width



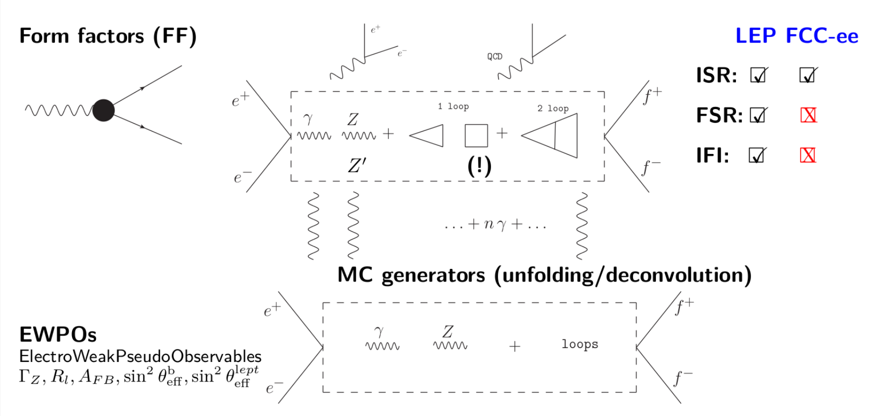
- ~30% QED corrections (ISR)

General remarks on usefulness of EWPOs,

1. EWPOs encapsulate experimental data after *extraction of well known and controllable QED and QCD effects*, in a model-independent manner.
2. They provide a convenient *bridge between real data and the predictions* of the SM (or SM plus New Physics).
3. Archived EWPO scan be exploited over long periods of time *for comparisons with steadily improving theoretical calculations of the SM predictions, and for validations of the New Physics models beyond the SM*.
4. They are also *useful for comparison and combination of results from different experiments*.

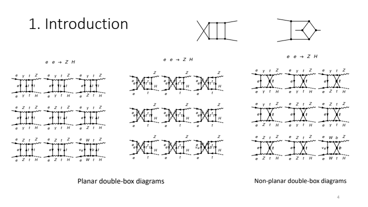
EWPOs at the Z-pole

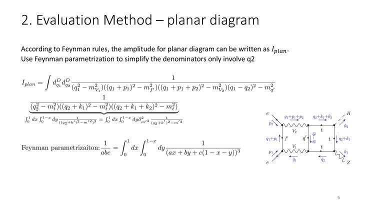
Experimental measurements at the Z-pole: after unfolding



One example - Qian Song, "NNLO EW corrections in HZ production",

https://indico.cern.ch/event/995644/





FCC-ee [exp]: 0.3%, present: $\delta_{TH} \sim 1\%$, full 2-loop $\sim 0.3\%$

6-dim $\rightarrow$ 3-dim integrals

1. Z-resonance and $\gamma, Z', \ldots \longrightarrow$ Laurent series,

$$\mathcal{M} = \frac{R}{s - s_0} + \sum_{n=0}^{\infty} (s - s_0)^n \; B^{(n)}, \quad s_0 = \bar{M}_Z^2 + i \bar{M}_Z \bar{\Gamma}_Z .$$

2. We want to extract EW Z-vertex couplings and definitions like $\sin^2 \theta_{\text{eff}}^f$, but in reality, we deal with complicated process

$$e^+ e^- \to f^+ f^- \quad + \text{invisible} \; (n \; \gamma + \mathrm{e}^+ \mathrm{e}^- \text{pairs} + \cdots)$$

$$\sigma^{e^+ e^- \to f^+ f^- + \cdots}(s) = \int dx \; \overbrace{f(x)} \; \underbrace{\sigma^{e^+ e^- \to f^+ f^-}(s')} \; \delta(x - s'/s)$$

$\longrightarrow$ form factors, QED separation/deconvolution, non-factorizations, ...

To determine the structure function/flux function kernels and hard scattering ansatz for data preparation or for unfolding is one of the challenges of FCC-ee-Z physics.

EWPOs - refers to $|M|^2$; EWPPs - refers to M

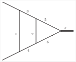
Definitions are related:

$$\bar{M}_Z \approx M_Z - \frac{1}{2}\,\frac{\Gamma_Z^2}{M_Z} \;\approx\; M_Z - 34\ \mathrm{MeV},$$
$$\bar{\Gamma}_Z \approx \Gamma_Z - \frac{1}{2}\,\frac{\Gamma_Z^3}{M_Z^2} \;\approx\; \Gamma_Z - 0.9\ \mathrm{MeV}.$$

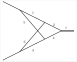
- Known from LEP. One of examples why changing frameworks/assumptions/simplifications of calculations matter (!).
- However, at FCC-ee $\delta\Gamma_Z \sim 0.1$ MeV. Non-facotrization effects must be added properly beyond 1-loop.
- Is it necessary for FCC-ee accuracy to implement MC with radiative corrections calculated at the amplitudes level?
- At this precision it is important which parameters are taken as input parameters in schemes.

EWPOs and Form Factors

$$V_\mu^{Zb\bar{b}} = \gamma_\mu [v_b(s) + a_b(s)\gamma_5] = \cdots + \overbrace{\underbrace{\quad}_{\text{planar,non-planar}}}^{\text{fermionic,bosonic}} + \cdots$$



$+$

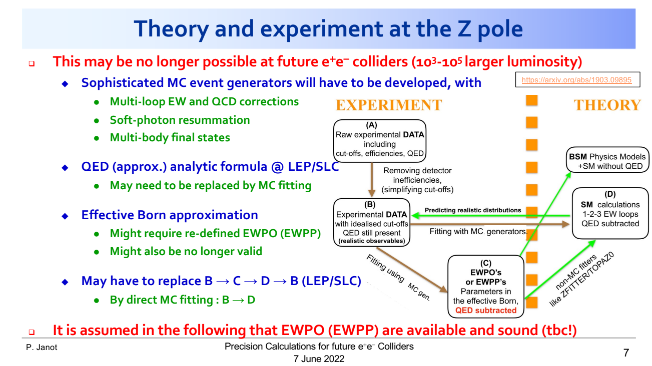


$$A_{FB} = \frac{\left[\int_0^1 dcos\theta - \int_{-1}^0 dcos\theta\right] \frac{d\sigma}{dcos\theta}}{\sigma_T} \sim \overbrace{\frac{2a_e v_e}{a_e^2 + v_e^2}}^{A_e} \overbrace{\frac{2a_f v_f}{a_f^2 + v_f^2}}^{A_f} + \text{corrections}$$

$$A_f = \frac{2\Re e \frac{v_f}{a_f}}{1 + \left(\Re e \frac{v_f}{a_f}\right)^2} = \frac{1 - 4|Q_f| \sin^2\theta_{\rm eff}^{\rm f}}{1 - 4|Q_f| \sin^2\theta_{\rm eff}^{\rm f} + 8(Q_f \sin^2\theta_{\rm eff}^{\rm f})^2},$$

$$\sin^2\theta_{\rm eff}^{\rm f} = F\left(\Re e \frac{v_f}{a_f}\right)$$

Talk by Patrick Janot, last week



Talk by Ayres Freitas, last week

Pole expansion

Express R_ij in terms of $\sin^2\theta^f_{\text{eff}}$ and F^f_A (with NNLO corrections):

$$R_{ij} = 4I_e^3 I_f^3 \sqrt{F_A^e F_A^f}\Big[Q_i^e Q_j^f\big(1 + i\,r^I_{AA} - \tfrac{1}{2}(r^I_{AA})^2 + \tfrac{1}{2}\delta\overline{X}_{(2)}\big) + (Q_i^e I_{j,f} + Q_j^f I_{i,e})(i - r^I_{AA}) - I_{i,e}I_{j,f}\Big] + M_Z\Gamma_Z\, Z_{ie(0)}Z_{jf(0)}\, x^I_{ij}\,,$$

$$Q_V^f = 1 - 4|Q_f|\sin^2\theta^f_{\text{eff}}, \qquad Q_A^f = 1$$

$$I_{V,f} = \frac{1}{(a^Z_{f(0)})^2}\Big[a^Z_{f(0)}\,\text{Im}\,Z_{Vf(1)} - v^Z_{f(0)}\,\text{Im}\,Z_{Af(1)}\Big], \qquad I_{A,f} = 0$$

$$\delta\overline{X}_{(2)} = -(\text{Im}\,\Sigma'_{Z(1)})^2 + 2\,\overline{b}^R_{\gamma Z(1)},$$

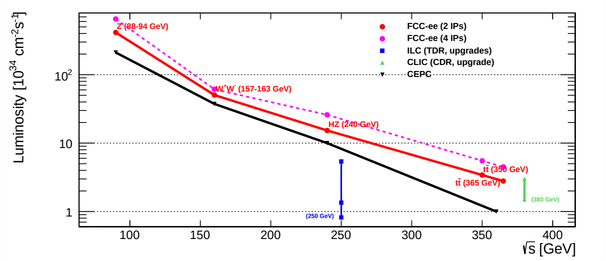
$$r^I_{ij} = \frac{\text{Im}\,Z_{ie(1)}}{Z_{ie(0)}} + \frac{\text{Im}\,Z_{jf(1)}}{Z_{jf(0)}} - \text{Im}\,\Sigma'_{Z(1)},$$

$$x^I_{ij} = \frac{\text{Im}\,Z'_{ie(1)}}{Z_{ie(0)}} + \frac{\text{Im}\,Z'_{jf(1)}}{Z_{jf(0)}} - \frac{1}{2}\,\text{Im}\,\Sigma''_{Z(1)},$$

C++ library GRIFFIN (in preparation)

Chen, Freitas '22

FCC-ee: Z,W,H,t and flavour electroweak factories



https://arxiv.org/abs/2203.06520 [The Future Circular Collider: a Summary for the US 2021 Snowmass Process]

Phase	Run duration (years)	Center-of-mass Energies (GeV)	Integrated Luminosity (ab^{-1})	Event Statistics
FCC-ee-Z	4	88-94	150	$5 \cdot 10^{12}$ **Z decays**
FCC-ee-W	2	157-163	10	10^8 WW events
FCC-ee-H	3	240	5	10^6 ZH events 25k WW $\to H$
FCC-ee-tt	5	340-365	$0.2 \div 1.5$	10^6 $t\bar{t}$ even ts 200k ZH 50k WW $\to H$

A. Blondel, P. Janot, 2106.13885

Table 3 Measurement of selected precision measurements at FCC-ee, compared with present precision. Statistical errors are indicated in boed phase. The systematic uncertainties are initial estimates, aim is to improve down to statistical errors. This set of measurements, together with those of the Higgs properties, achieves indirect sensitivity to new physics up to a scale Λ of 70 TeV in a description with dim 6 operators, and possibly much higher in specific new physics (non-decoupling) models

Observable	Present value ± error	FCC-ee stat.	FCC-ee syst.	Comment and leading exp. error
m_Z (keV)	91186700 ± 2200	**4**	100	From Z line shape scan
				Beam energy calibration
Γ_Z (keV)	2495200 ± 2300	**4**	25	From Z line shape scan
				Beam energy calibration
$\sin^2\theta_W^{eff}$ ($\times 10^6$)	231480 ± 160	**2**	2.4	from $A_{FB}^{\mu\mu}$ at Z peak
				Beam energy calibration
$1/\alpha_{QED}(m_Z^2)(\times 10^3)$	128952 ± 14	**3**	Small	From $A_{FB}^{\mu\mu}$ off peak
				QED&EW errors dominate
R_ℓ^Z ($\times 10^3$)	20767 ± 25	**0.06**	0.2–1	Ratio of hadrons to leptons
				Acceptance for leptons
$\alpha_s(m_Z^2)$ ($\times 10^4$)	1196 ± 30	**0.1**	0.4–1.6	From R_ℓ^Z above
σ_{had}^0 ($\times 10^3$) (nb)	41541 ± 37	**0.1**	4	Peak hadronic cross section
				Luminosity measurement
$N_\nu (\times 10^3)$	2996 ± 7	**0.005**	1	Z peak cross sections
				Luminosity measurement
R_b ($\times 10^6$)	216290 ± 660	**0.3**	< 60	Ratio of $b\bar{b}$ to hadrons

Gluza

Future: W, t, H

- $e^+e^- \to W^+W^-$ at 161 GeV: $\delta m_W^{exp} = 0.5 \div 1$ MeV.
  Challenge to get the same TH error:
  NNLO $e^+e^- \to 4f$.

- $e^+e^- \to t\bar{t}$ at 350 GeV: $\delta {m_t}^{exp} =$ 17 MeV
  Big challenge for theory, today > 100 MeV, future projection $\leq$ 50 MeV:
  $\sim$ 10 MeV unc. from mass def.;
  $\sim$ 15 MeV from α_s unc. to threshold mass def.;
  ~ 30 MeV - h. orders resummation

- $e^+e^- \to HZ$ at 240 GeV: Kinematic constraint fits with $Z \to ll$ and $H \to bb$, ...,
  $m_H = 125.35$ GeV ± 150 MeV [link CMS], $\Gamma_H = 4.1^{5.1}_{4.0}$ MeV, $\Gamma_H <$ 13 MeV at 95 % C.L., 1901.00174
  $\delta m_H^{exp} =$ 10 MeV; Theory errors subdominant.

Monte Carlo generators (**not discussed!**) 'QED challenges at FCC-ee precision measurements', S. Jadach and M. Skrzypek, Eur.Phys.J.C 79 (2019) 9, 756 1903.09895

E. Torrente-Lujan, 1209.0474v2

If

$$\rho_t = \frac{m_Z m_t}{m_H^2},$$

then (for ATLAS, CMS combined $m_H = 125.6 \pm 0.4 \pm 0.5$)

$$\rho_t^{(exp)} = 1.0022 \pm 0.007 \pm 0.009$$

Separately,

$$\rho_t^{(exp)} = 1.0077 \pm 0.007 \pm 0.009 \qquad (m_{h,ATLAS}),$$
$$\rho_t^{(exp)} = 0.9965 \pm 0.007 \pm 0.007 \qquad (m_{h,CMS})$$

Observable	present value ± error	FCC-ee **Stat.**	FCC-ee Syst.	Comment and leading exp. error
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R_ℓ^Z $(\times 10^3)$	20767 ± 25	**0.06**	0.2-1	ratio of hadrons to leptons **acceptance for leptons**
$\alpha_s(m_Z^2)$ $(\times 10^4)$	1196 ± 30	**0.1**	0.4-1.6	from R_ℓ^Z above
σ_{had}^0 $(\times 10^3)$ (nb)	41541 ± 37	**0.1**	4	peak hadronic cross section luminosity measurement
$N_\nu(\times 10^3)$	2996 ± 7	**0.005**	1	Z peak cross sections Luminosity measurement
R_b $(\times 10^6)$	216290 ± 660	**0.3**	< 60	ratio of $b\bar{b}$ to hadrons stat. extrapol. from SLD
$A_{FB}^b, 0$ $(\times 10^4)$	992 ± 16	**0.02**	1-3	b-quark asymmetry at Z pole from jet charge
$A_{FB}^{pol,\tau}$ $(\times 10^4)$	1498 ± 49	**0.15**	<2	τ polarization asymmetry τ decay physics
τ lifetime (fs)	290.3 ± 0.5	**0.001**	0.04	radial alignment
τ mass (MeV)	1776.86 ± 0.12	**0.004**	0.04	momentum scale
τ leptonic $(\mu\nu_\mu\nu_\tau)$ B.R. (%)	17.38 ± 0.04	**0.0001**	0.003	e/μ/hadron separation
m_W (MeV)	80350 ± 15	**0.25**	0.3	From WW threshold scan Beam energy calibration
Γ_W (MeV)	2085 ± 42	1.2	0.3	From WW threshold scan Beam energy calibration
$\alpha_s(m_W^2)(\times 10^4)$	1170 ± 420	**3**	small	from R_ℓ^W
$N_\nu(\times 10^3)$	2920 ± 50	**0.8**	small	ratio of invis. to leptonic in radiative Z returns
m_{top} (MeV/c^2)	172740 ± 500	**17**	small	From $t\bar{t}$ threshold scan QCD errors dominate
Γ_{top} (MeV/c^2)	1410 ± 190	45	small	From $t\bar{t}$ threshold scan QCD errors dominate
$\lambda_{top}/\lambda_{top}^{SM}$	1.2 ± 0.3	**0.10**	small	From $t\bar{t}$ threshold scan QCD errors dominate
ttZ couplings	± 30%	0.5 – 1.5%	small	From $\sqrt{s} = 365\,\text{GeV}$ run

QED unfolding, S-matrix approach, slide by A.Freitas, AWLC2020, pdf

Consistent (gauge-invariant) theory setup:

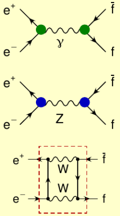
Expansion of $\mathcal{A}[e^+e^- \to \mu^+\mu^-]$ about $s_0 = M_Z^2 - iM_Z\Gamma_Z$:

$$\mathcal{A}[e^+e^- \to f\bar{f}] = \frac{R}{s - s_0} + S + (s - s_0)T + \dots$$

$$R = g_Z^e(s_0)g_Z^f(s_0)$$

$$S = \left[\frac{1}{M_Z^2} g_\gamma^e g_\gamma^f + g_Z^e g_Z^{f\prime} + g_Z^{e\prime} g_Z^f + S_{\text{box}}\right]_{s=s_0}$$

$g_V^f(s)$: effective $Vf\bar{f}$ couplings



At NNLO: Need R at $\mathcal{O}(\alpha^2)$, S at $\mathcal{O}(\alpha)$, etc.

Current state of art: full one-loop for S, T

→ $\mathcal{O}(0.01\%)$ uncertainty within SM (improvements may be needed) — see, e.g., Bardin, Grünewald, Passarino '99

→ Sensitivity to some NP beyond EWPO

QED unfolding, ISR, slide by A.Freitas, Snowmass 2020, pdf

Z lineshape

Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{\text{ini}}(s,s') \otimes \sigma_{\text{hard}}(s')$$

Kureav, Fadin '85
Berends, Burgers, v. Neerven '88
Kniehl, Krawczyk, Kühn, Stuart '88
Beenakker, Berends, v. Neerven '89
Bardin et al. '91; Skrzypek '92
Montagna, Nicrosini, Piccinini '97

Soft photons (resummed) + collinear photons

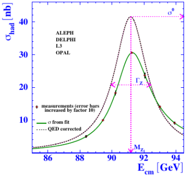
$$\mathcal{R}_{\text{ini}} = \sum_n \left(\frac{\alpha}{\pi}\right)^n \sum_{m=0}^{n} h_{nm} \ln^m\left(\frac{s}{m_e^2}\right)$$

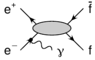
Universal ($m{=}n$) logs known to $n=6$,
also some sub-leading terms

Ablinger, Blümlein, De Freitas, Schönwald '20

Exclusive description: MC tools → talk by Jadach

LEP EWWG '05

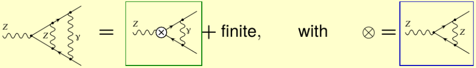




QED unfolding, FSR, slide by A.Freitas, Snowmass 2020, pdf

Factorization of massive and QED/QCD FSR:

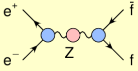
$$\Gamma_f \approx \frac{N_c M_Z}{12\pi}\left[\left(\mathcal{R}_V^f|g_V^f|^2 + \mathcal{R}_A^f|g_A^f|^2\right)\frac{1}{1+\mathrm{Re}\,\Sigma_Z'}\right]_{s=M_Z^2}$$



$\mathcal{R}_V^f$, $\mathcal{R}_A^f$: Final-state QED/QCD radiation;
known to $\mathcal{O}(\alpha_s^4)$, $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha\alpha_s)$

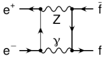
Kataev '92
Chetyrkin, Kühn, Kwiatkowski '96
Baikov, Chetyrkin, Kühn, Rittinger '12

g_V^f, g_A^f, Σ_Z': Electroweak corrections

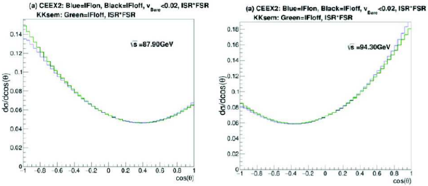


QED unfolding, IFI, slide by A.Freitas, Snowmass 2020, pdf

- Interference between ISR and FSR suppressed by Γ_Z/M_Z on Z resonance



- Still relevant for high precision an off-resonance



Jadach, Yost '18

- Factorization from hard matrix element requires 4-variable convolution
- Soft-photon resummation can be included

Jadach, Yost '18
Greco, Pancheri-Srivastava, Srivastava '75

SM precision parameters determination: $\alpha(M_Z^2)$, F. Jegerlechner, pdf

1. $\alpha(M_Z^2)$ in precision physics (precision physics limitations)

Uncertainties of hadronic contributions to effective α are a problem for electroweak precision physics: besides top Yukawa y_t and Higgs self-coupling λ

α , G_μ, M_Z **most precise input parameters** $\Rightarrow$ **precision predictions** $\sin^2\Theta_f, v_f, a_f, M_W, \Gamma_Z, \Gamma_W, \cdots$

50% non-perturbative

$\alpha(M_Z), G_\mu, M_Z$ **best effective input parameters for VB physics (Z,W) etc.**

$$
\begin{array}{cccccl}
\frac{\delta\alpha}{\alpha} & \sim & 3.6 & \times & 10^{-9} & \\
\frac{\delta G_\mu}{G_\mu} & \sim & 8.6 & \times & 10^{-6} & \\
\frac{\delta M_Z}{M_Z} & \sim & 2.4 & \times & 10^{-5} & \\
\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} & \sim & 0.9 \div 1.6 & \times & 10^{-4} & \text{(present : lost } 10^5 \text{ in precision!)} \\
\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} & \sim & 5.3 & \times & 10^{-5} & \text{(FCC} - \text{ee/ILC requirement)}
\end{array}
$$

LEP/SLD: $\sin^2\Theta_{\rm eff} = (1 - v_l/a_l)/4 = 0.23148 \pm 0.00017$

$\delta\Delta\alpha(M_Z) = 0.00020 \quad \Rightarrow \quad \delta\sin^2\Theta_{\rm eff} = 0.00007$; $\delta M_W/M_W \sim 4.3\times 10^{-5}$

affects most precision tests and new physics searches!!!

$$\frac{\delta M_W}{M_W} \sim 1.5\times 10^{-4}\,,\quad \frac{\delta M_H}{M_H} \sim 1.3\times 10^{-3}\,,\quad \frac{\delta M_t}{M_t} \sim 2.3\times 10^{-3}$$

For pQCD contributions very crucial: precise QCD parameters $\alpha_s,\ m_c,\ m_b,\ m_t \Rightarrow$ **Lattice-QCD**