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Mixed QCD-EW corrections to NC Drell-Yan

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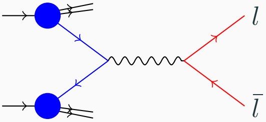
Precision calculations for future e+e− colliders: targets and tools

CERN, 16th June 2022

Goal of this talk

In this talk we discuss the technology we used to obtain the mixed QCD-EW corrections to Drell-Yan and how they can be exploited to obtain precise predictions for FCC-ee processes.

DRELL-YAN



Standard model precision studies

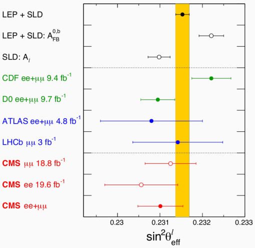
Precise measurement of m_W ($<$ 10 MeV)!

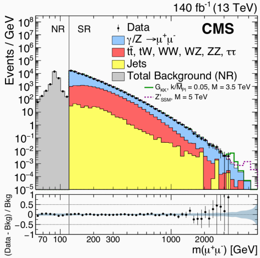
Determination of $\sin^2\theta_W$ is becoming competitive with LEP result.

BSM studies

Precise determination of the SM background is crucial for BSM studies!

Requires control of the SM prediction at the $\mathcal{O}(0.5\%)$ level in the TeV region.





Perturbative expansion

Parton model

$$\sigma_{tot}(z) = \sum_{i,j\in q,\bar{q},g,\gamma} \int \mathrm{d}x_1 \mathrm{d}x_2 \ f_i(x_1,\mu_F) f_j(x_2,\mu_F) \sigma_{ij}(z,\varepsilon,\mu_F)$$

In the full QCD-EW SM, we have a double series expansion of the partonic cross sections in the electromagnetic and strong coupling constants, α and α_s, respectively:

$$\begin{aligned}
\sigma_{ij}(z) &= \sigma_{ij}^{(0)} \sum_{m,n=0}^{\infty} \alpha_s^m \alpha^n \ \sigma_{ij}^{(m,n)}(z) \\
&= \sigma_{ij}^{(0)} \Big[\sigma_{ij}^{(0,0)}(z) \\
&\quad + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) \\
&\quad + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha\alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) \\
&\quad + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha\alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2\alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]
\end{aligned}$$

Perturbative expansion : QCD corrections

$$\sigma_{ij}(z) = \sigma_{ij}^{(0)} \Big[\sigma_{ij}^{(0,0)}(z) + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha \alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha \alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2 \alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]$$

NLO
Altarelli, Ellis, Martinelli (1979);

NNLO
Hamberg, Matsuura, van Neerven (1991);
Anastasiou, Dixon, Melnikov, Petriello (2003);
Catani, Cieri, Ferrera, de Florian, Grazzini (2009);

N^3LO
Duhr, Dulat, Mistlberger (2020); Chen, Gehrmann, Glover, Huss, Yang, Zhu (2021); Camarda, Cieri, Ferrera (2021); Chen, Gehrmann, Glover, Huss, Monni, Re, Rottoli, Torrielli (2022)

Perturbative expansion : EW corrections

$$
\begin{aligned}
\sigma_{ij}(z) = \sigma_{ij}^{(0)} \Big[& \sigma_{ij}^{(0,0)}(z) \\
& + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) \\
& + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha \alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) \\
& + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha \alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2 \alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]
\end{aligned}
$$

NLO

Baur, Brein, Hollik, Schappacher, Wackeroth (2002);
Carloni Calame, Montagna, Nicrosini, Vicini (2007);
Dittmaier, Huber (2010);

NNLO (approximated)

Jantzen, Kühn, Penin, Smirnov (2005);

Perturbative expansion : mixed corrections

$$\sigma_{ij}(z) = \sigma_{ij}^{(0)} \Big[\sigma_{ij}^{(0,0)}(z) + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha \alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha \alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2 \alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]$$

NLO QCD and NLO EW corrections are separately large. What about the mixed corrections, particularly $\sigma_{ij}^{(1,1)}(z)$?

Recent progress in the NNLO mixed QCDxEW corrections

On-shell Z/W production

- Pole approximation : Dittmaier, Huss, Schwinn;
- Analytic QCDxQED corrections : de Florian, Der, Fabre;
- p_T^Z distribution in QCDxQED including p_T resummation : Cieri, Ferrera, Sborlini;
- Differential on-shell Z production including QCDxQED : Delto, Jaquier, Melnikov, Roentsch;
- Total QCDxEW corrections to Z production (fully analytic): Bonciani, Buccioni, NR, Triscari, Vicini; Bonciani, Buccioni, NR, Vicini;
- Differential on-shell Z/W production including QCDxEW : Behring, Buccioni, Caola, Delto, Jaquier, Melnikov, Roentsch;

Technical developments

- Master integrals : Aglietti, Bonciani; Bonciani, Di Vita, Mastrolia, Schubert; Heller, von Manteuffel, Schabinger; Long, Zhang, Ma, Jiang, Han, Li, Wang; Liu, Ma;
- Mixed QCD-QED splitting functions : de Florian, Sborlini, Rodrigo;
- Renormalisation : Degrassi, Vicini; Dittmaier, Schmidt, Schwarz; Dittmaier;

Complete Drell-Yan

- neutrino pair production in QCDxQED : Cieri, de Florian, Der, Mazzitelli;
- $pp \to l\nu_l + X$ in QCDxEW : Buonocore, Grazzini, Kallweit, Savoini, Tramontano;
- two-loop amplitudes: Heller, von Manteuffel, Schabinger; $\Longleftarrow$ Talk by A. von Manteuffel
  Armadillo, Bonciani, Devoto, NR, Vicini; $\Longleftarrow$ This Talk
- Complete NNLO QCDxEW corrections to neutral current Drell-Yan: Bonciani, Buonocore, Grazzini, Kallweit, NR, Tramontano, Vicini; Buccioni, Caola, Chawdhry, Devoto, Heller, von Manteuffel, Melnikov, Röntsch, Signorile-Signorile;

Why $\sigma_{ij}^{(1,1)}(z)$ is important?

$\alpha_s(m_Z) \simeq 0.118$	$\alpha(m_Z) \simeq 0.0078$	$\frac{\alpha_s(m_Z)}{\alpha(m_Z)} \simeq 15.1$	$\frac{\alpha_s^2(m_Z)}{\alpha(m_Z)} \simeq 1.8$

1. From naive argument of coupling strength, N^3LO QCD $\sim$ mixed NNLO QCD$\otimes$EW.
2. However, in specific phase-space points, fixed order EW corrections can become very large because of logarithmic (weak and QED Sudakov type) enhancement. These effects are large for W mass measurements. On the other hand, these corrections suffer from large uncertainties coming from unphysical scales.
3. N^3LO QCD corrections control the uncertainties arising from the unphysical scales, but they lack the large EW effects.
4. The appearance of photon induced processes $\Rightarrow$ photon PDFs.

The NNLO mixed QCD-EW corrections

- have similar magnitude as N^3LO QCD,
- contain the large EW effects,
- reduce the theoretical uncertainties.

NNLO QCD$\otimes$EW corrections extremely important for high ($\mathcal{O}(10^{-4})$) precision pheno.

Another motivation : Electroweak scheme dependence

The Lagrangian has 3 inputs (g, g', v). More observables (like $G_\mu, \alpha, m_W, m_Z, \sin\theta_W$) are experimentally measured and can be considered as input parameters in different schemes. Such two schemes are

1. G_μ-scheme : where (G_μ, m_W, m_Z) are considered as input
2. $\alpha(0)$-scheme : where (α, m_W, m_Z) are considered as input

The relation between G_μ and α gets EW and mixed QCD⊗EW corrections.

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi\alpha}{2\sin^2\theta_W \cos^2\theta_W m_Z^2}(1+\Delta r)$$

At LO, $\alpha(G_\mu)$ and $\alpha(0)$ differs by 3.53%. Example for onshell Z production

order	G_μ-scheme	$\alpha(0)$-scheme	$\delta_{G_\mu-\alpha(0)}$ (%)
LO	48882	47215	3.53
NLO QCD (LO + Δ_{10})	55732	53831	3.53
NNLO QCD (LO + Δ_{10} + Δ_{20})	55651	53753	3.53
NLO EW (LO + Δ_{01})	48732	48477	0.53
LO + Δ_{10} + Δ_{01}	55582	55093	0.89
LO + Δ_{10} + Δ_{20} + Δ_{01} + Δ_{11}	55469	55340	0.23

NNLO contributions to neutral current Drell-Yan

Pure Virtual

Real-Virtual

Double Real

Each individual contribution is divergent : $\frac{1}{\epsilon}$ in dimensional regularization

NNLO contributions to neutral current Drell-Yan

Pure Virtual

$- \; S^{(1,1)}$

Real-Virtual

Double Real

$\Big\} + d\sigma_{CT}^{(1,1)}$

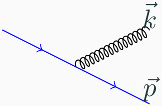
Subtraction : $S^{(1,1)} \sim \int d\sigma_{CT}^{(1,1)} \quad \Rightarrow$ The two sets are separately finite!

NNLO contributions to neutral current Drell-Yan

Pure Virtual

The two-loop virtual amplitudes contain divergences of two types

(a) **Ultraviolet divergences** : UV renormalization of fields and couplings
(b) **Infrared divergences** : Soft (soft gluons & photons) & collinear (collinear partons)



$$\frac{1}{(k+p)^2} = \frac{1}{2k.p} = \frac{1}{2k^0 p^0 (1-\cos\theta)}$$

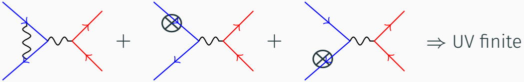
$k^0 \to 0$ Soft divergence
$\theta \to 0$ Collinear divergence

The infrared structure of scattering amplitudes is universal!

Ultraviolet renormalization

⊛ The Born contribution is zeroth order in α_s, hence no α_s renormalization is needed.

⊛ Renormalization of quark wave function receives one-loop EW and two-loop mixed QCD⊗EW contributions in the on-shell scheme.



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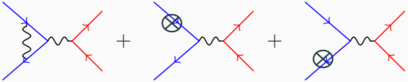
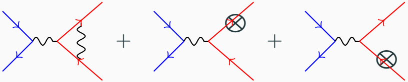


$\Rightarrow$ UV finite

⊛ Renormalization of lepton wave function receives one-loop EW contributions.



$\Rightarrow$ UV finite

We consider massive leptons, but small mass limit. In that case, the QED part of the renormalization constant is with massive lepton. On the other hand, the weak part can be computed using massless lepton.

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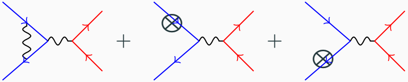

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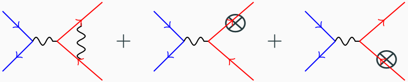
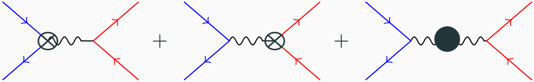


$\Rightarrow$ UV finite

⊛ The computation is performed in background field gauge, with the advantage that the vertex and propagator contributions are separately UV finite.



$\Rightarrow$ UV finite

The infrared divergences and lepton mass

The infrared structure of scattering amplitudes is universal!

$$\mathcal{M}_{\rm fin}^{(1,1)} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}_{\rm fin}^{(1,0)} - \mathcal{I}^{(1,0)}\mathcal{M}_{\rm fin}^{(0,1)}$$

The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

Divergence regulator	massless lepton : $\frac{1}{\epsilon}$	massive lepton : $\log m_l$

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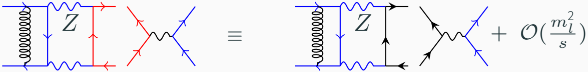
Divergence regulator massless lepton : $\frac{1}{\epsilon}$ massive lepton : $\log m_l$

(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.



The infrared divergences and lepton mass

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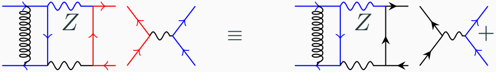
(b) In a single box diagram, where lepton is attached to one photon and one Z boson, it generates a collinear singularity. However, thanks to [Frenkel, Taylor], once all diagrams are summed up, the collinear divergences cancel.



$$+ \mathcal{C}_l \log\left(\frac{m_l^2}{s}\right) + \mathcal{O}\left(\frac{m_l^2}{s}\right)$$

It is also reflected in the subtraction formula e.g. for the QED box part

$$\left[H(-1, y_l) - H(-1, z_l)\right]\big|_{m_l \to 0} \equiv \log(t/u)$$

The infrared divergences and lepton mass

The infrared structure of scattering amplitudes is universal!

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Divergence regulator	massless lepton : $\frac{1}{\epsilon}$	massive lepton : $\log m_l$

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(b) In a single box diagram, where lepton is attached to one photon and one Z boson, it generates a collinear singularity. However, thanks to [Frenkel, Taylor], once all diagrams are summed up, the collinear divergences cancel.

(c) Hence, the collinear singularities from leptons ($\log m_l$) come from only the QED-type corrections to the lepton vertex, which we compute with full lepton mass dependence.

Sub-grouping Feynman diagrams

To achieve IR subtraction in a smooth & efficient way, we arrange all diagrams in four subsets : AA, AZ, ZZ, W. We further subdivide them according to the topology : form-factor/box/factorized.

- AA : Diagrams with two photon propagators
- AZ : Diagrams with one photon and one Z propagators
- ZZ : Diagrams with two Z propagators
- W : Diagrams with at least one W propagators

Accordingly, the subtraction term is also arranged.

$$
\begin{aligned}
\mathcal{I}^{(0,1)} &= \mathcal{I}_A^{(0,1)}\,, \\
\mathcal{I}^{(1,1)} &= \mathcal{I}^{(1,1),non-fact} + \mathcal{I}^{(1,1),fact} \\
&= \mathcal{I}_{AA}^{(1,1),non-fact} + \mathcal{I}_{AZ}^{(1,1),non-fact} + \mathcal{I}^{(1,0)} * \mathcal{I}_A^{(0,1)}
\end{aligned}
$$

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For AA and AZ subsets : True non-factorizable two-loop QCD-QED IR contributions

- We compute the box contributions for AA subset with exact lepton mass, and we explicitly show the cancellation of $\log m_l$ after summing t and u channel diagrams.
- We compute the box contributions for AZ subset with massless lepton. The leading collinear pole also cancels after summing t and u channel diagrams.

For ZZ and W subsets : The IR structure is basically one-loop QCD ($\mathcal{I}^{(1,0)}$)

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The subdivision allows us to obtain the finite remainder for each subset, leading to an efficient and smooth numerical evaluation.

Computational procedure

$d = 4 - 2\epsilon$

- Diagrammatic approach -> `QGRAF`/`FeynArts` to generate Feynman diagrams
- In-house `FORM`/`Mathematica` routines for algebraic manipulation :

  Lorentz, Dirac and Color algebra
- Decomposition of the dot products to obtain scalar integrals

$$\frac{2l.p}{l^2(l-p)^2} = \frac{l^2 - (l-p)^2 + p^2}{l^2(l-p)^2} = \frac{1}{(l-p)^2} - \frac{1}{l^2} + \frac{p^2}{l^2(l-p)^2}$$

- Identity relations among scalar integrals : IBPs, LIs & SRs
- Algebraic linear system of equations relating the integrals

$$\Downarrow$$

Master integrals (MIs)

- Computation of MIs : Method of differential equation & SeaSyde
- Ultraviolet renormalization
- Subtraction of the universal infrared poles ($S^{(1,1)}$).
- Numerical evaluation of the hard function to prepare the grid.

Computational procedure : γ_5

γ_5 is inherently a four-dimensional object.
How can we use it in dimensional regularization?

	Anti-commutation $\{\gamma_\mu, \gamma_5\} = 0$	Cyclicity of the trace
't Hooft and Veltmann	X	✓
Kreimer et al.	✓	X

For the mixed QCD-EW corrections to the NCDY, the two prescriptions yield

- Different one- and two-loop scattering amplitudes
- Same finite remainder after subtraction

[Heller, von Manteuffel, Schabinger, Spiesberger]

$\Leftarrow$ talk by A. von Manteuffel

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Our approach :

- Consider a fixed point to start the Dirac trace.
- Use anti-commutation relation, bring all γ_5 at the end and use $\gamma_5^2 = 1$.
- Use $\gamma_5 = \frac{i}{24!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ for the single leftover γ_5.

The method of differential equations

A Feynman integral is a function of spacetime dimension d and kinematic invariant x, y.

$$J_i \sim \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{1}{l_1^2 l_2^2 ((l_1 - l_2)^2 - m^2)(l_1 - p_1 - p_2)^2 (l_2 - p_3)^2} \equiv f(d,\, x,\, y)$$

The idea is to obtain differential eqns. for the integral $w.r.t.\ x,\ y$ and solve it.

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 0 & 0 & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

To solve such a system, we need to perform series expansion in ϵ and to organize the matrix in each order of ϵ in such a way that it diagonalizes, or at least it takes a block-triangular form. Now, it can be solved using bottom-up approach.

The homogeneous solutions are in general $\log$ or Li_2. Because of the ϵ expansion, the non-homogeneous solutions are recursive integral over the homogeneous solutions.

The results are obtained in terms of iterated integrals (GPLs).

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Direct numerical integration of Feynman integrals is tedious, unstable and challenging to obtain precise results. $\Leftarrow$ talk by V. Maheria

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Iterated integrals are one-dimensional. They can be computed with great precision in a short amount of time. Besides, they have the following properties:

(a) **Shuffle algebra** : Allows to obtain a basis for a set of iterated integrals. Reduction to such a basis is extremely effective to reduce the computation time by few times.

(b) **Scaling invariance** : Allows to convert the limit of these integrals from kinematical variables (z) to constants (1). This makes the integration really precise.

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- Form factor type MIs : Aglietti, Bonciani; Bonciani, Buccioni, NR, Vicini;
- Box type ($\gamma\gamma$ with massive lepton) : Bonciani, Ferroglia, Gehrmann, Maitre, Studerus;
- Box type (γZ & ZZ with massless lepton) :

Bonciani, Di Vita, Mastrolia, Schubert; Heller, von Manteuffel, Schabinger

Five among the 36 two-mass MIs of Bonciani *et al.* contain Chen iterated integrals!

The 36 two-mass master integrals

Fully analytic

- Most MIs are solved in GPLs.
- Five MIs are solved in terms of Chen's iterated integrals! Numerical evaluation possible only in the non-physical region.

Fully numerical

- Evaluation of the MIs in physical region is demanding! (using Fiesta/pySecDec)
- Specially for those five MIs, achieving a single digit precision in the physical region is extremely challenging!

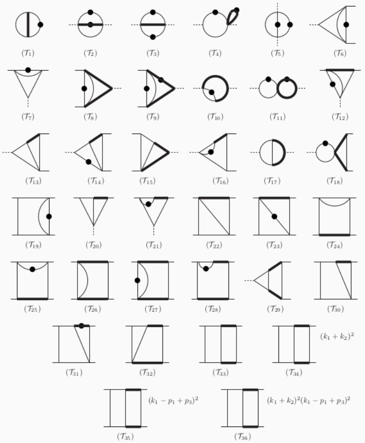

Fig from Roberto et al.

Can we find a mixed approach?

Our semi-analytic approach

What do we need for the two-loop virtual amplitudes?

Our semi-analytic approach

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) A formula for the finite part which should be numerically stable and precise.

(i) The universal subtraction operator indicates that the singular part of the amplitude contains only simple GPLs.

(ii) The individual contribution from the five MIs to the single pole of the matrix element contains the Chen iterated integrals, which cancel after summing them.

(iii) Certain internal combinations of the MIs (at the lowest order in ϵ) can be found which can be solved in terms of simple GPLs.

So, only simple GPLs in the singular part! SOLVED!

Our semi-analytic approach

What do we need for the two-loop virtual amplitudes?

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Most of the MIs are known in terms of GPLs. Few MIs (32-36), which contain Chen iterated integrals, we solve them using series expansion through `SeaSyde`.

Implemented in the Mathematica package `DiffExp` for real kinematic variables.
[F. Moriello (2019), M. Hidding (2020)] $\Leftarrow$ talk by M. Hidding

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Most of the MIs are known in terms of GPLs. Few MIs (32-36), which contain Chen iterated integrals, we solve them using series expansion through `SeaSyde`.

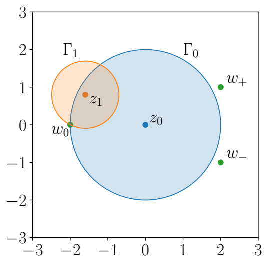
(i) We consider the system of differential equations for all the 36 MIs. Given a boundary point, the system can be solved using series expansion for a nearby point.
(ii) The solution in this new point can now be considered as boundary and thus we can go forward along a path to obtain solution in any phase space point.

SEASYDE

Series Expansion Approach for SYstems of Differential Equations

We have implemented the series expansion method generalizing it with complex variables $\Rightarrow$ complex plane!

- The radius of convergence of the series is limited by the presence of poles.
- Transport from one point to another needs to consider branch-cuts.



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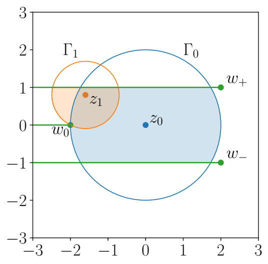

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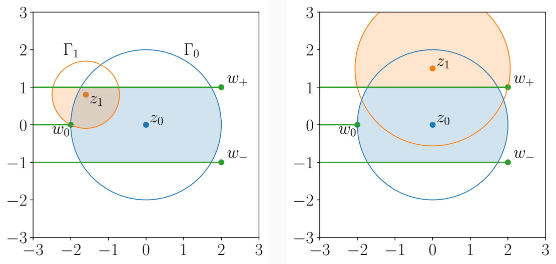

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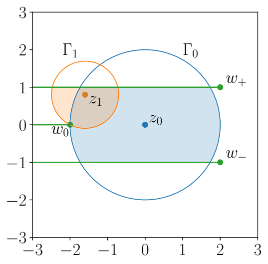

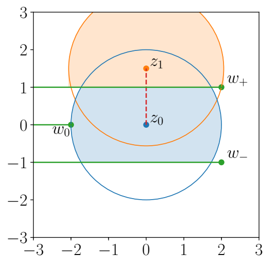

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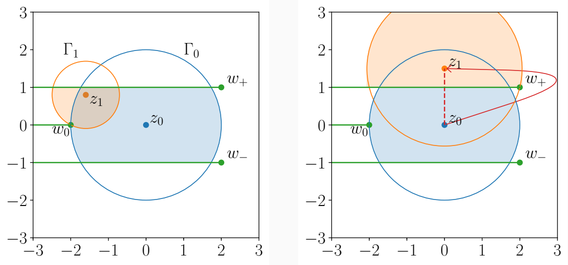
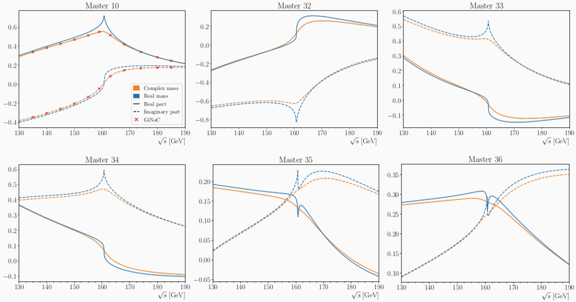


* We transport solution along the red solid line with several steps (The corresponding circles are not drawn.)

SEASYDE

Series Expansion Approach for SYstems of Differential Equations

- We consider the system of diff. eqns. for the 36 MIs and solve it with SeaSyde.
- The solution can be obtained with **arbitrary number of significant digits**.
- 31 MIs are known analytically and their numerical evaluations using `GiNaC` provide a crucial check. The results for 5 MIs are new predictions : checks with `Fiesta`, `pySecDec`, `DiffExp`.
- Because of using complex variable in `SeaSyde`, it is possible to use complex mass scheme which smoothens the behaviour at threshold.



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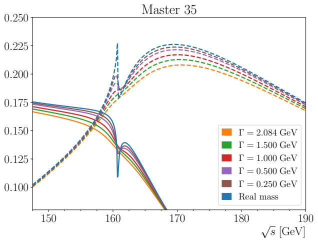
The small width limit reproduces the result in real mass with Feynman prescription



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Time required for transporting the boundary conditions from the Euclidean region to a test point

Number of terms	Precision	Execution time
50 terms	10^{-14}	$\sim$ 14 min
75 terms	10^{-19}	$\sim$ 26 min
100 terms	10^{-25}	$\sim$ 50 min
125 terms	10^{-33}	$\sim$ 75 min
150 terms	10^{-40}	$\sim$ 90 min

Finally

We obtain the two-loop virtual amplitude:
(a) The singular part is analytic and contains GPLs. This allows us to successfully check with the universal infrared behaviour of the scattering amplitudes.

(b) The finite part after performing the infrared subtraction contains GPLs and few MIs 'symbolically' which have been computed using our semi-analytic approach.

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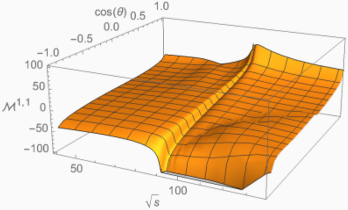
Next?

We need to evaluate the subtracted finite part numerically for few thousand phase-space points. Although evaluation of a single GPL is fast, there are ~ 11000 GPLs in the full expression. Also the expression is extremely large.

Numerical evaluation and the grid

To obtain a fast compilation and successful numerical evaluation, we divide the contributions from various Feynman diagrams in a gauge invariant way by the presence of different EW vector bosons (γ, Z, W), and further by different topologies.

These subdivisions allow us to parallelize the computation. Production of the grid (3250 points) for the MIs using SeaSyde required $\mathcal{O}$(12h) on a 32-cores machine. Evaluation of the GPLs on a single phase-space point, for 40 digits precision, ranges from few minutes to $\sim$ 20 minutes, depending on the phase-space point. Evaluation time substantially goes down for smaller number of digits.



Results!

σ [pb]	σ_{LO}	$\sigma^{(1,0)}$	$\sigma^{(0,1)}$	$\sigma^{(2,0)}$	$\sigma^{(1,1)}$
$q\bar{q}$	809.56	191.85	-33.76	49.9	-4.8
qg	—	-158.08	—	-74.8	8.6
$q(g)\gamma$	—	—	-0.839	—	0.084
$q(\bar{q})q'$	—	—	—	6.3	0.19
gg	—	—	—	18.1	—
$\gamma\gamma$	1.42	—	-0.0117	—	—
total	810.98	33.77	-34.61	-0.5	4.0

$\frac{\sigma^{(i,j)}}{\sigma_{LO}}$		$+4.2\%$	-4.3%	$\sim 0\%$	$+0.5\%$

* The size of the NNLO QCD corrections depends on the chosen setup!

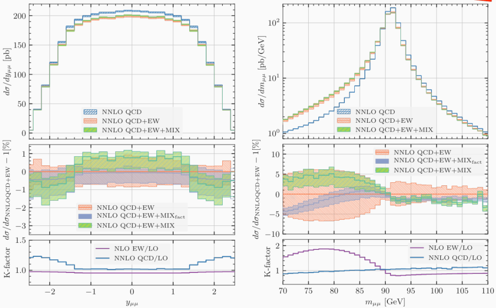
Phenomenology of mixed QCD-EW corrections for NC-DY

PRELIMINARY

SETUP (LHC @ $\sqrt{s} = 13.6$ TeV)

- `NNPDF31_nnlo_as_0118_luxqed`
- $p_{T,\mu} > 25$ GeV, $|y_\mu| < 2.5$, 66 GeV $< m_{\mu^+\mu^-} <$ 116 GeV
- massive muons (no photon lepton recombination)
- G_μ scheme, complex mass scheme
- fixed scale $\mu_F = \mu_R = m_Z$

G_μ-scheme	σ [pb]	$\sigma^{(i,j)}$ [pb]	$\sigma^{(i,j)}/\sigma_{\mathrm{LO}}$
LO	$763.40(2)^{+12.7\%}_{-13.6\%}$	–	–
NLO QCD	$802.26(6)^{+2.7}_{-4.2\%}$	38.86(6)	5.1%
NNLO QCD	$802.5(7)^{+0.4}_{-0.8\%}$	0.2(7)	0.0%
NLO EW	$730.76(2)^{+12.7}_{-13.6\%}$	−32.65(3)	−4.3%
NNLO QCD+EW	$769.8(7)^{+0.5}_{-0.6\%}$	–	–
NNLO QCD+EW+MIX$_{\mathrm{fact}}$	$768.2(7)^{+0.3}_{-0.7\%}$	−2.0(1)	−0.2%
NNLO QCD+EW+MIX	$772.4(8)^{+0.3}_{-0.7\%}$	2.6(2)	0.3%



- Mixed QCD-EW corrections are smaller in this setup, but **non-trivial $\mathcal{O}(1\%)$ shape distortion** in the distributions
- Stabilisation of theory uncertainties

Uncertainities: 7-point scale variation
NNLO QCD+EW+MIX$_{\mathrm{fact}}$: NNLO QCD+EW+ **factorised** approximation of mixed corrections

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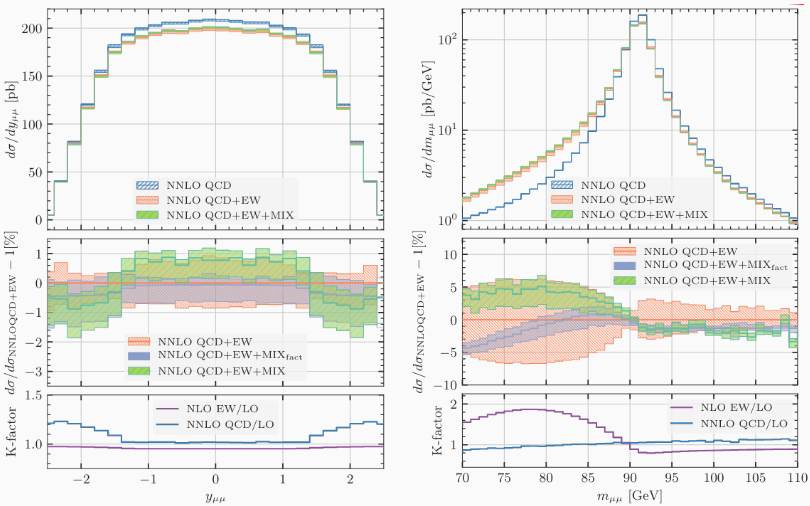

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Phenomenology of mixed QCD-EW corrections for NC-DY

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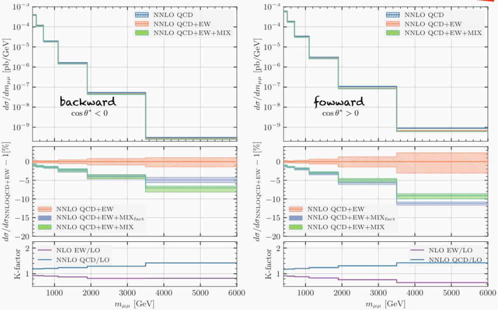
SETUP (LHC @ $\sqrt{s}$ = 13 TeV) CMS 2103.02708

- `NNPDF31_nnlo_as_0118_luxqed`
- $p_{T,\mu} > 53$ GeV, $|y_\mu| < 2.4$, $m_{\mu^+\mu^-} > 150$ GeV
- massive muons (no photon lepton recombination)
- G_μ scheme, complex mass scheme
- dynamic scale $\mu_F = \mu_R = m_{\mu^+\mu^-}$



- Negative corrections of several percents in the tails with respect to NNLO QCD+EW
- The **factorised approximation** catches the bulk of QCD-EW corrections pointing towards a factorisation of NLO QCD corrections and EW Sudakov logarithms
  as observed in [Buccioni et al (2022)]
- Small residual non-factorisable effects at (sub) percent level

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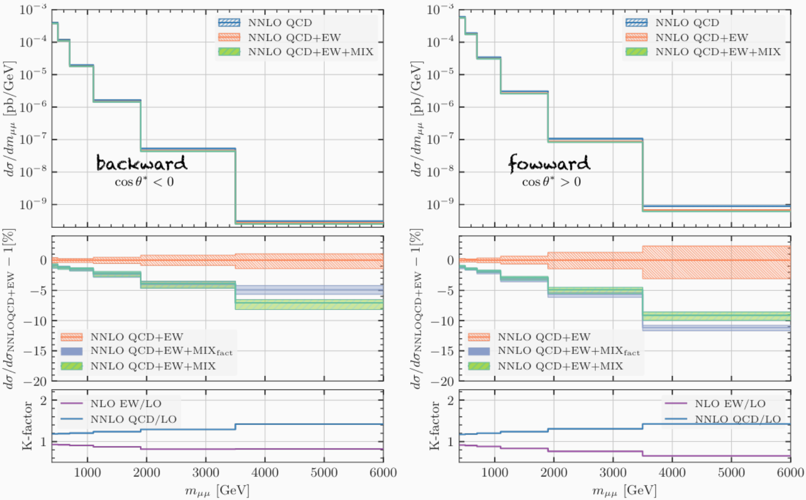

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Summarizing

- One of the bottleneck is the computation of two-loop virtual amplitudes. We have computed it using small lepton mass limit in the complex mass scheme.
- Our semi-analytic approach allows us to achieve analytic cancellation of the universal subtraction term, as well as fast and stable numerical evaluation of the finite hard function.
- To evaluate the two-mass MIs, we wrote `SeaSyde` : a Mathematica package that generalizes the series expansion method to complex variables.
- The phenomenological impact of mixed QCD-EW corrections is crucial.

- These computations are challenging! Although ingredients at several stages are well studied, a continuous workflow is necessary to obtain the numerical grid from the Feynman diagrams. The automation of the intermediate steps are in progress!

Prospects for e+e- precision predictions

- Precision corrections to FCC-ee processes will involve multi-scale loop computation.
- Primary difficulty would be to perform the reduction to MIs & solving them.
- Progress in reduction technology looks very promising and hence, computation of the MIs seems to be the main bottleneck.

- Any MIs can be solved in series expansion, and `SeaSyde` can provide the semi-analytical solution given a boundary condition. Use of `AMFlow` can substantially help in computing the boundary conditions. ⇐ talk by Xiao Liu

An immediate use of our technology could be

NNLO mixed QCD-EW corrections to $e^+ + e^- \to b + \bar{b}/(t + \bar{t})$

(considering massless electron) ⇐ talk by S. Frixione

Thank you for your attention!