DiffExp and Feynman parameter integration

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workshop on
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Outline of the talk

1. Introduction
   - Analytic vs. numerical methods
   - Differential equation method

2. Series expansion methods
   - Generalized series expansions (& DiffExp)
   - Analytic continuation & $i\delta$-prescription
   - Segmentation
   - Examples & EW-application

3. Direct integration through differential equations
   - Feynman trick recursion
   - Integration through differential equations
   - Five-point 2-loop example

Introduction and context

Analytic methods for Feynman integrals

- Typical analytic approaches express Feynman integrals in terms of classes of iterated integrals:
  - Multiple polylogarithms
  - Iterated integrals over modular form
  - Elliptic multiple polylogarithms

- Strengths of analytic methods:
  - Branch-cuts and analytic structure is manifest (through the symbol map)
  - Specialized algorithms can be developed for evaluating the relevant classes of functions

- Drawbacks of analytic methods:
  - The analytic continuation may be difficult to perform
  - Many Feynman integrals lie outside the known classes of functions

Numerical methods

- Prototypical example:
  - Sector decomposition and numerical integration (FIESTA, pySecDec)

- Strengths of numerical methods:
  - Numerical integration is fully algorithmic and general purpose
  - Applicable to integrals with many scales

- Drawbacks of numerical methods:
  - Numbers might not expose symmetries and/or structures underlying the integrals
  - Performance can lack behind analytic methods

Semi-numerical methods

- Semi-numerical methods perform as much as possible of the computation analytically, before resorting to numerical approximations
- We may set up differential equations in analytic form, and then solve these differential equations
  - Numerically using finite difference methods
  - Semi-analytically through one-dimensional series expansions

[Talk by Chen]
[Mandal, Zhao, 1812.03060]
[Lee, Smirnov, Smirnov, 1709.07525]
[Liu, Ma, Wang, 1711.09572, 2201.11669]
[Moriello, 1907.13234]
[MH, 2006.05510]

- Strengths of series expansion methods:
  - State of the art performance on many types of Feynman integrals
  - Speed improves as more points are computed
  - Analytic continuation of Feynman integrals becomes simple

- Drawbacks:
  - Simplification of the differential equations is not fully algorithmic
  - Derivation of boundary conditions requires some manual effort

Series expansions

- The main steps of the approach which we discuss in this talk is as follows:

  - Set up a linear system of differential equations
  - Reduce multi-scale problems to a single-scale problem by integrating along a one-dimensional contour
  - Split up the contour into multiple segments such that series expansions converge on each segment
  - Find series solutions of the integrals along each segment, and fix boundary conditions by matching neighbouring segments
  - Cross thresholds by assigning $\pm i\delta$ to logarithms and algebraic roots in the solutions

(History) Series expansions

- This strategy was demonstrated in [F. Moriello, 1907.13234] for the computation of planar integrals relevant to H+j production in QCD at NLO

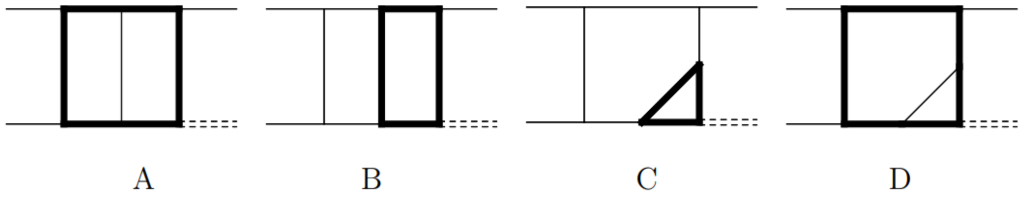
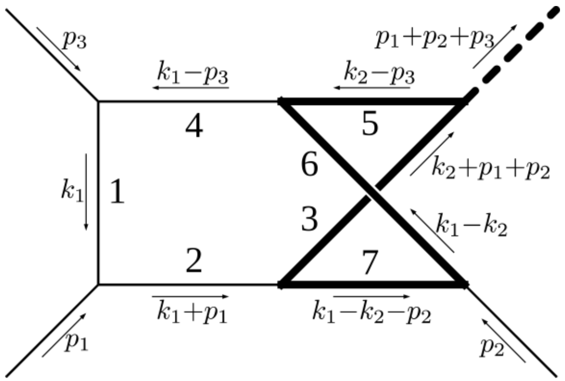


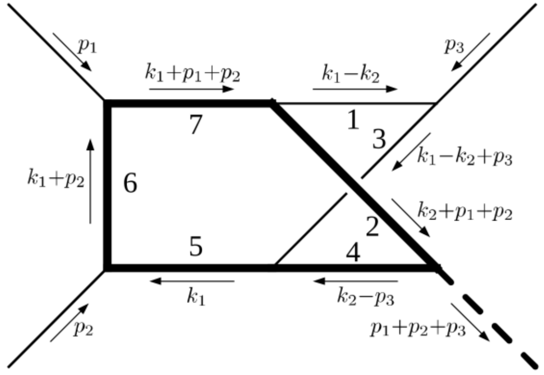
Figure 1: The four planar integral families contributing to two-loop $H+j$-production in QCD.

- Simultaneously, in a larger collaboration, we applied these methods to the computation of non-planar H+j integrals:

[R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, MH, L. Maestri, F. Moriello, G. Salvatori, V. A. Smirnov]



[Bonciani et al, 1907.13156]



[Frellesvig et al, 1911.06308]

Family F

Master integrals

- IBP-reduction:
  - 73 master integrals
  - Default FIRE basis: $\mathcal{O}$(1 GB)
  - More suitable (pre-canonical) basis: $\mathcal{O}$(100 MB)
  - Possible using either FIRE or KIRA

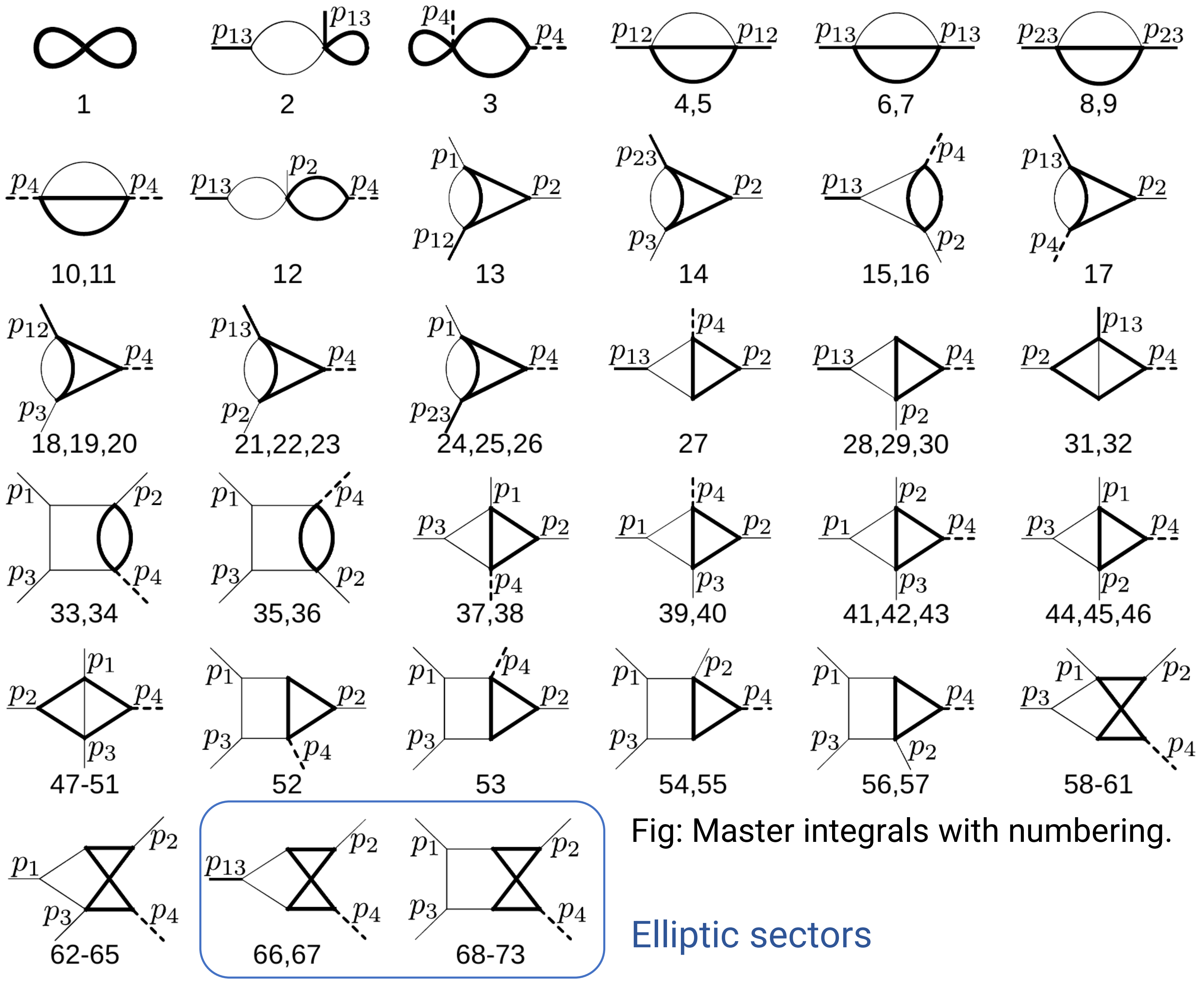


Fig: Master integrals with numbering.

Elliptic sectors

Family F alphabet

- 69 letters in total
- We labeled the following roots:

$$r_1 = \sqrt{-p_4^2}\,, \qquad r_2 = \sqrt{-s}\,,$$
$$r_3 = \sqrt{-t}\,, \qquad r_4 = \sqrt{t - p_4^2}\,,$$
$$r_5 = \sqrt{s + t - p_4^2}\,, \qquad r_6 = \sqrt{4m^2 - p_4^2}\,,$$
$$r_7 = \sqrt{4m^2 - s}\,, \qquad r_8 = \sqrt{4m^2 - t}\,,$$
$$r_9 = \sqrt{4m^2 - p_4^2 + t}\,, \qquad r_{10} = \sqrt{4m^2 - p_4^2 + s + t}\,,$$
$$r_{11} = \sqrt{4m^2(p_4^2 - s - t) + st}\,, \qquad r_{12} = \sqrt{4m^2 t + s(p_4^2 - s - t)}\,,$$
$$r_{13} = \sqrt{4m^2 s + t(p_4^2 - s - t)}\,, \qquad r_{14} = \sqrt{4m^2 t(s + t - p_4^2) - (p_4^2)^2 s}\,,$$
$$r_{15} = \sqrt{-4m^2 st + (p_4^2)^2 (s + t - p_4^2)}\,, \qquad r_{16} = \sqrt{16m^2 t + (p_4^2 - t)^2}\,.$$

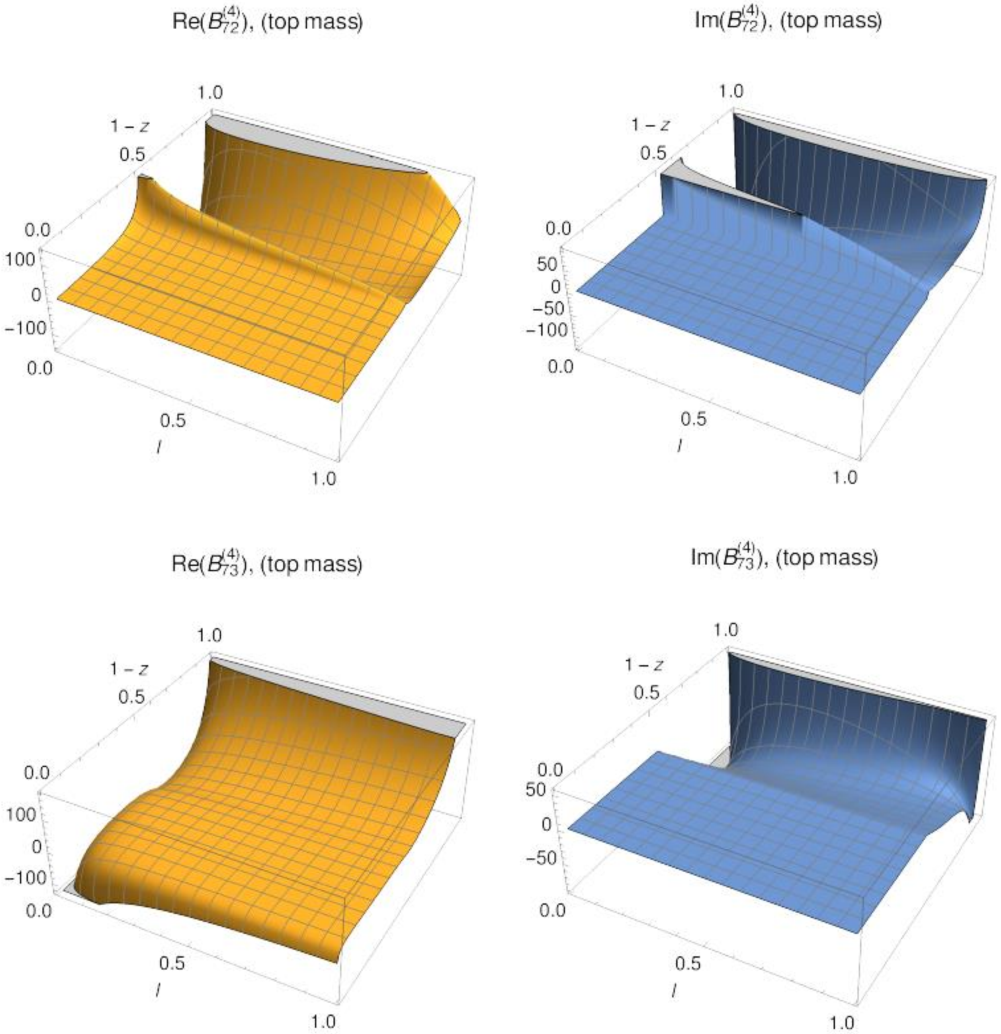
- These roots appear in 10 independent combinations:

$$\{r_1r_6, r_2r_7, r_3r_8, r_4r_9, r_5r_{10}, r_2r_3r_{11}, r_2r_5r_{12}, r_3r_5r_{13}, r_2r_{14}, r_5r_{15}\}\,.$$

- The roots are not simultaneously rationalizable

Plots for family F

The real part of the integrals is in blue, the imaginary part is orange.



DiffExp

- A general implementation of these methods was made into the Mathematica package DiffExp, introduced in arXiv:2006.05510, (available at https://gitlab.com/hiddingm/diffexp)
- DiffExp accepts (any) system of differential equations of the form

$$\frac{\partial}{\partial s}\vec{f}(\{S\},\epsilon) = \mathbf{A}_s\vec{f}(\{S\},\epsilon) \qquad \mathbf{A}_x(x,\epsilon) = \sum_{k=0}^{\infty}\mathbf{A}_x^{(k)}(x)\epsilon^k$$

  for which the matrix entries are combinations of rational and algebraic functions
- It enables one to numerically integrate various multi-scale Feynman integrals at arbitrary points in phase-space, and at precisions of tens of digits (or higher)
- The Feynman integrals do not have to be in canonical form and may also be of “elliptic”-type or associated with more complicated geometries.

Series expansions

- Series expansions have been featured various times in the past literature.

- For single-scale problems, see e.g:

S. Pozzorini and E. Remiddi, *Precise numerical evaluation of the two loop sunrise graph master integrals in the equal mass case*, *Comput. Phys. Commun.* **175** (2006) 381–387, [hep-ph/0505041].

U. Aglietti, R. Bonciani, L. Grassi, and E. Remiddi, *The Two loop crossed ladder vertex diagram with two massive exchanges*, *Nucl. Phys.* **B789** (2008) 45–83, [arXiv:0705.2616].

R. Mueller and D. G. Öztürk, *On the computation of finite bottom-quark mass effects in Higgs boson production*, *JHEP* **08** (2016) 055, [arXiv:1512.08570].

B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028, [arXiv:1802.00833].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Solving differential equations for Feynman integrals by expansions near singular points*, *JHEP* **03** (2018) 008, [arXiv:1709.07525].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Evaluating elliptic master integrals at special kinematic values: using differential equations and their solutions via expansions near singular points*, *JHEP* **07** (2018) 102, [arXiv:1805.00227].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Gröber, *A Numerical Routine for the Crossed Vertex Diagram with a Massive-Particle Loop*, *Comput. Phys. Commun.* **241** (2019) 122–131, [arXiv:1812.02698].

- For multi-scale problems, see for example:

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop $gg \to Hg$ amplitude mediated by a nearly massless quark*, *JHEP* **11** (2016) 104, [arXiv:1610.03747].

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop amplitudes for $qg \to Hq$ and $q\bar{q} \to Hg$ mediated by a nearly massless quark*, *Phys. Rev.* **D95** (2017), no. 5 054012, [arXiv:1702.00426].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Grober, *Analytical Method for Next-to-Leading-Order QCD Corrections to Double-Higgs Production*, *Phys. Rev. Lett.* **121** (2018), no. 16 162003, [arXiv:1806.11564].

R. Bruser, S. Caron-Huot, and J. M. Henn, *Subleading Regge limit from a soft anomalous dimension*, *JHEP* **04** (2018) 047, [arXiv:1802.02524].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double-Higgs boson production in the high-energy limit: planar master integrals*, *JHEP* **03** (2018) 048, [arXiv:1801.09696].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double Higgs boson production at NLO in the high-energy limit: complete analytic results*, *JHEP* **01** (2019) 176, [arXiv:1811.05489].

B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028 [1802.00833].

Additional literature

Samuel Abreu, Harald Ita, Francesco Moriello, Ben Page, Wladimir Tschernow, and Mao Zeng. Two-Loop Integrals for Planar Five-Point One-Mass Processes. 2020.

Xiao Liu, Yan-Qing Ma, Wei Tao, and Peng Zhang. Calculation of Feynman loop integration and phase-space integration via auxiliary mass flow. *Chin. Phys. C*, 45(1):013115, 2021.

Samuel Abreu, Harald Ita, Ben Page, and Wladimir Tschernow. Two-Loop Hexa-Box Integrals for Non-Planar Five-Point One-Mass Processes. 7 2021.

Matteo Fael, Fabian Lange, Kay Schönwald, and Matthias Steinhauser. A semi-analytic method to compute Feynman integrals applied to four-loop corrections to the $\overline{\text{MS}}$-pole quark mass relation. *JHEP*, 09:152, 2021.

Matteo Becchetti, Francesco Moriello, and Armin Schweitzer. Two-loop amplitude for mixed QCD-EW corrections to $gg \to Hg$. 12 2021.

Tommaso Armadillo, Roberto Bonciani, Simone Devoto, Narayan Rana, and Alessandro Vicini. Two-loop mixed QCD-EW corrections to neutral current Drell-Yan. 1 2022.

Xiao Liu and Yan-Qing Ma. Multiloop corrections for collider processes using auxiliary mass flow. 7 2021.

Xiao Liu and Yan-Qing Ma. AMFlow: a Mathematica Package for Feynman integrals computation via Auxiliary Mass Flow. 1 2022.

Zhi-Feng Liu and Yan-Qing Ma. Automatic computation of Feynman integrals containing linear propagators via auxiliary mass flow. 1 2022.

Additional literature

Matteo Fael, Fabian Lange, Kay Schönwald, and Matthias Steinhauser. Massive Vector Form Factors to Three Loops. *Phys. Rev. Lett.*, 128(17):172003, 2022.

Simon Badger, Heribertus Bayu Hartanto, and Simone Zoia. Two-Loop QCD Corrections to Wbb⁻ Production at Hadron Colliders. *Phys. Rev. Lett.*, 127(1):012001, 2021.

Simon Badger, Heribertus Bayu Hartanto, Jakub Kryś, and Simone Zoia. Two-loop leading-colour QCD helicity amplitudes for Higgs boson production in association with a bottom-quark pair at the LHC. *JHEP*, 11:012, 2021.

Simon Badger, Heribertus Bayu Hartanto, Jakub Kryś, and Simone Zoia. Two-loop leading colour helicity amplitudes for $W^{\pm}\gamma + j$ production at the LHC. 1 2022.

Simon Badger, Matteo Becchetti, Ekta Chaubey, Robin Marzucca, and Francesco Sarandrea. One-loop QCD helicity amplitudes for $pp \to t\bar{t}j$ to $O(\epsilon^2)$. 1 2022.

Samuel Abreu, Matteo Becchetti, Claude Duhr, and Melih A. Ozcelik. Two-loop master integrals for pseudo-scalar quarkonium and leptonium production and decay. 6 2022.

Tommaso Armadillo, Roberto Bonciani, Simone Devoto, Narayan Rana, and Alessandro Vicini. Evaluation of Feynman integrals with arbitrary complex masses via series expansions. 5 2022.

Additional literature

- As we have already seen during this workshop, various other public programs and packages are in development, such as AMFlow and SeaSyde.
  - AMFlow: very efficient + automatic determination of boundary conditions.
  - SeaSyde: computations with complex masses and complex contours.

Series expansion methods

Differential equations

- We consider a family of scalar Feynman integrals:

$$I_{a_1,\dots,a_{n+m}} = \int \left(\prod_{i=1}^{l} d^d k_i\right) \frac{\prod_{i=n+1}^{n+m} N_i^{-a_i}}{\prod_{i=1}^{n} D_i^{a_i}} \qquad \begin{aligned} d &= d_{\text{int}} - 2\epsilon \\ D_i &= -q_i^2 + m_i^2 - i\delta \end{aligned}$$

and a basis of master integrals $\vec{I}$. Taking derivatives on kinematic invariants and masses and performing IBP reductions, we obtain:

[Kotikov, 1991], [Remiddi, 1997]
[Gehrmann, Remiddi, 2000]

$$\boxed{\partial_{s_j} \vec{I} = \mathbf{M}_{s_j}\left(\{s_i\}, \epsilon\right) \vec{I}}$$

- We aim to solve these differential equations. Since they are of Fuchsian type, they admit convergent (generalized) power series solutions

(See e.g. [1212.4389], [1411.0911]
[1702.04279])

Canonical differential equations

- In many cases the differential equations can be brought into a canonical form:

[Henn, 2013]
See also:
[Lee, 1411.0911]
[Prausa, 1701.00725]
[Gituliar, Magerya, 1701.04269]
[Meyer, 1705.06252]
[Dlapa, Henn, Yan, 2002.02340]

$$\frac{\partial \vec{B}}{\partial s_i} = \epsilon \frac{\partial \tilde{\mathbf{A}}}{\partial s_i} \vec{B}, \quad d\vec{B} = \epsilon d\tilde{\mathbf{A}} \vec{B}$$

- Along a one-dimensional contour we find: $\gamma : [0,1] \to \mathbb{C}^{|S|}$

$$x \mapsto \left(\gamma_{s_1}(x), \ldots, \gamma_{s_{S|}}(x)\right)$$

$$\partial_x \vec{B} \equiv \varepsilon \mathbf{A}_x \vec{B}$$

$$\vec{B} = \sum_{i \geq 0} \vec{B}^{(i)} \varepsilon^i \quad \vec{B}^{(i)}(x) = \int_0^x \mathbf{A}_{x'} \vec{B}^{(i-1)}(x')\, dx' + \vec{B}^{(i)}(x=0)$$

Series expansions - canonical basis

- Let us expand the matrix as a power series:

$$\mathbf{A}_x = x^r \left[\sum_{p=0}^{n} \mathbf{C}_p x^p + \mathcal{O}\left(x^{n+1}\right) \right]$$

- Using integration-by-parts, we can always write:

$$\int x^m \log(x)^n = x^{m+1} \sum_{j=1}^{n} c_j \log(x)^j$$

- Thus, all the integrations can be performed in terms of (generalized) series expansions:

$$B_j^{(k)}(x) = x^r \sum_{n=0}^{\infty} \sum_{m=0}^{k} c_{mn} x^n \log(x)^m, \quad c_{mn} \in \mathbb{C}, \quad 0 \geq r \in \mathbb{Q}$$

Line segmentation

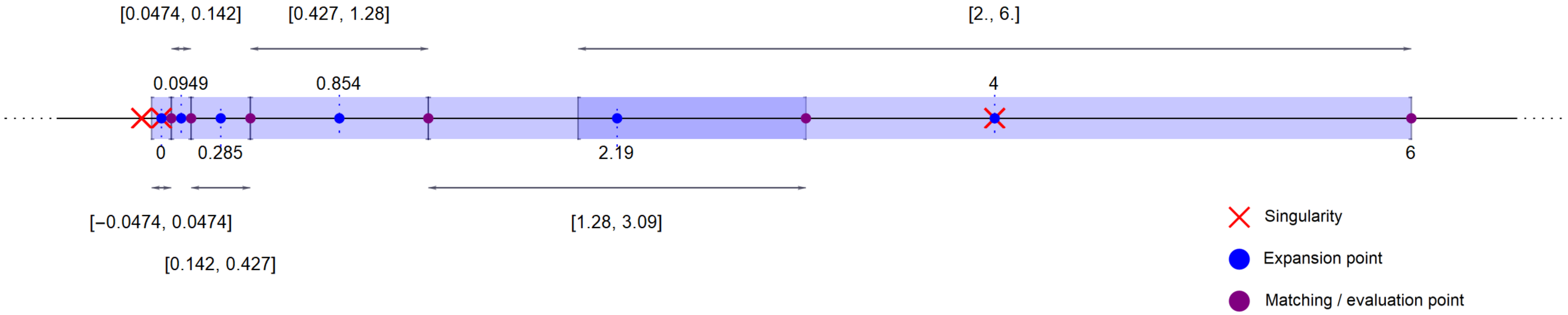
- The series solutions have a finite radius of convergence.
- Rule of thumb: radius of convergence ~ distance to nearest singularity.
- By concatenating series expansions (possibly centered at singularities) we can reach any point in phase-space. How do we choose the center-points?

Line segmentation

- Strategy:
  - Choose the line segments such that each expansion is evaluated at most $1/k$ the distance to the nearest singularity, where $k > 1$.
  - For example, we may choose $k = 2$, and evaluate each expansion at most half the distance to the nearest singularity.
  - To cross singularities, we center expansions at singularities.
  - We give an illustrative example on the next slide.

Line segmentation: example

- Suppose: $X_{\text{sing}} = (\dots, -0.095, 0, 4, 16, \dots)$, is the set of singularities.
- We seek to integrate from: $x_{\text{start}} = 0$, to $x_{\text{end}} = 6$
- Then we may pick the following partitioning into six line segments, such that each evaluation happens at most ½ the distance to the nearest singularity:



Analytic continuation

- The series solutions centered at singularities may contain logarithms and square roots.
  - Logarithms appear after integration of terms $1/x$.
  - Square roots can arise from homogeneous solutions (when the indicial equation has a half-integer root), or from the basis definition.
- By transferring an $i\delta$-prescription to the line parameter, we can perform the analytic continuation of these functions. In particular we can let:

$$\log(x+i\delta) = \log(x), \qquad \sqrt{x+i\delta} = \sqrt{x}$$

$$\log(x-i\delta) = \log(x) - 2\pi i\theta_m, \qquad \sqrt{x-i\delta} = (\theta_p - \theta_m)\sqrt{x}$$

Analytic continuation

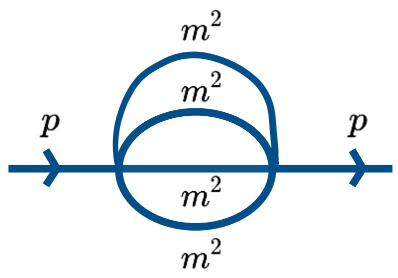
- We don't like to carry theta functions around in the series expansions (for performance reasons), so we may instead use replacement rules.
- For example, if x carries $-i\delta$, and we evaluate at a point $x < 0$, we let:

$$\log(x) \to \log(x) - 2\pi i, \quad \sqrt{x} \to -\sqrt{x}$$

- <u>Additional comments:</u>
  - The $i\delta$-prescriptions can be determined from the Feynman prescription
  - Typically, we should avoid crossing two singular regions at the same time

Examples

3-loop banana graph



- Load DiffExp:

```
Get[FileNameJoin[{NotebookDirectory[], "..", "DiffExp.m"}]];

Loading DiffExp version 1.0.7

For questions, email: martijn.hidding@physics.uu.se

For the latest version, see: https://gitlab.com/hiddingm/diffexp
```

- Set the configuration options and load the matrices

```
EqualMassConfiguration = {
   DeltaPrescriptions → {t - 16 + I δ},
   MatrixDirectory → NotebookDirectory[] <> "Banana_EqualMass_Matrices/",
   UseMobius → True, UsePade → True
  };

LoadConfiguration[EqualMassConfiguration];

DiffExp: Loading matrices.

DiffExp: Found files: {dt_0.m, dt_1.m, dt_2.m, dt_3.m, dt_4.m}

DiffExp: Kinematic invariants and masses: {t}

DiffExp: Getting irreducible factors..

DiffExp: Configuration updated.
```

3-loop banana graph

- Prepare the boundary conditions along an asymptotic limit:

```
EqualMassBoundaryConditions = {
    "?",
    "?",
    ϵ (1 + 3 ϵ) (1 + 4 ϵ) (- (4 ⅇ^(3 EulerGamma ϵ) Gamma[ϵ]^3)/t + (6 ⅇ^(3 EulerGamma ϵ) (-1/t)^(1+ϵ) ϵ Gamma[-ϵ]^2 Gamma[ϵ]^3)/Gamma[-2 ϵ] +
        (8 ⅇ^(3 EulerGamma ϵ) (-1/t)^(1+2 ϵ) ϵ Gamma[-ϵ]^3 Gamma[ϵ] Gamma[2 ϵ])/Gamma[-3 ϵ] + (3 ⅇ^(3 EulerGamma ϵ) (-1/t)^(1+3 ϵ) ϵ Gamma[-ϵ]^4 Gamma[3 ϵ])/Gamma[-4 ϵ]),
    ⅇ^(3 EulerGamma ϵ) ϵ^3 Gamma[ϵ]^3
  } // PrepareBoundaryConditions[#, <|t → -1/x|>] &;
```

```
DiffExp: Integral 1: Ignoring boundary conditions.
DiffExp: Integral 2: Ignoring boundary conditions.
DiffExp: Assuming that integral 3 is exactly zero at epsilon order 0.
DiffExp: Prepared boundary conditions in asymptotic limit, of the form:
```

DiffExp:

?	?	?	?	?
?	?	?	?	?
$O[x]^{51}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$
$(\ldots) + \sqrt{O[x]}$	$\sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$

3-loop banana graph

- Next, we transport the boundary conditions:

```
Transport1 = TransportTo[EqualMassBoundaryConditions, <|t → -1|>];
Transport2 = TransportTo[Transport1, <|t → x|>, 32, True];
```

DiffExp: Transporting boundary conditions along $\langle\left|t \to -\frac{1.}{x}\right|\rangle$ from x = 0. to x = 1.

DiffExp: Preparing partial derivative matrices along current line..

DiffExp: Determining positions of singularities and branch-cuts.

DiffExp: Possible singularities along line at positions {0.}.

DiffExp: Analyzing integration segments.

DiffExp: Segments to integrate: 3.

DiffExp: Integrating segment: $\langle\left|t \to \frac{8.\ (-1. + 1.\ x)}{x}\right|\rangle$.

DiffExp: Integrated segment 1 out of 3 in 20.8565 seconds.

DiffExp: Evaluating at x = 0.0625

DiffExp: Current segment error estimate: 5.14483×10^{-31}

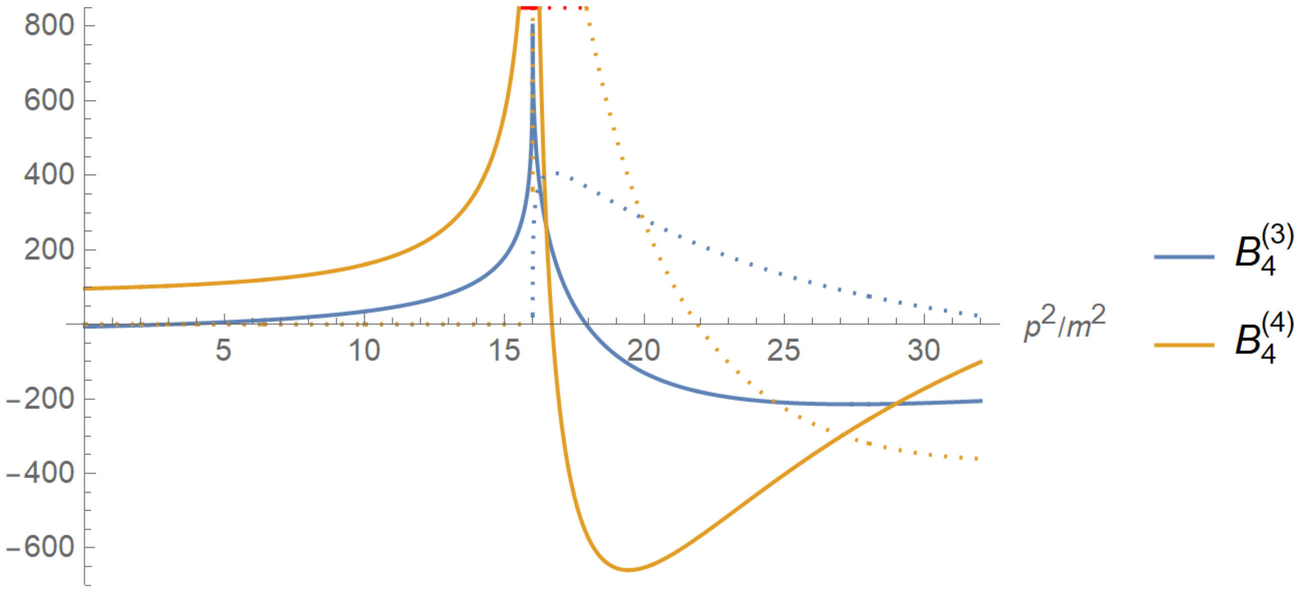
DiffExp: Total error estimate: 5.14483×10^{-31}

DiffExp: Integrating segment: $\langle|t \to$ -1. + 1. x

3-loop banana graph

- Lastly, we plot the result:

```
ResultsForPlotting = ToPiecewise[Transport2];
Quiet[ReImPlot[{ResultsForPlotting[[3, 4]][x], ResultsForPlotting[[3, 5]][x]}, {x, 0, 32},
  ClippingStyle → Red, PlotLegends → {"B4^(3)", "B4^(4)"}, AxesLabel → {"p^2/m^2"}, PlotRange → {-700, 850},
  MaxRecursion → 15, WorkingPrecision → 100]]
```



3-loop banana graph

- Computation time typically scales quadratically with expansion order:

Exp. order	Time (s)	Abs. error	Exp. order	Time (s)	Abs. error
155	310.	6.3×10^{-68}	85	91.9	8.3×10^{-35}
145	270.	3.5×10^{-63}	75	72.3	4.2×10^{-30}
135	236.	1.9×10^{-58}	65	55.9	2.1×10^{-25}
125	200.	1.0×10^{-53}	55	39.7	1.0×10^{-20}
115	170.	5.6×10^{-49}	45	27.6	4.7×10^{-16}
105	142.	3.0×10^{-44}	35	18.6	2.2×10^{-11}
95	116.	1.6×10^{-39}	25	11.7	1.4×10^{-6}

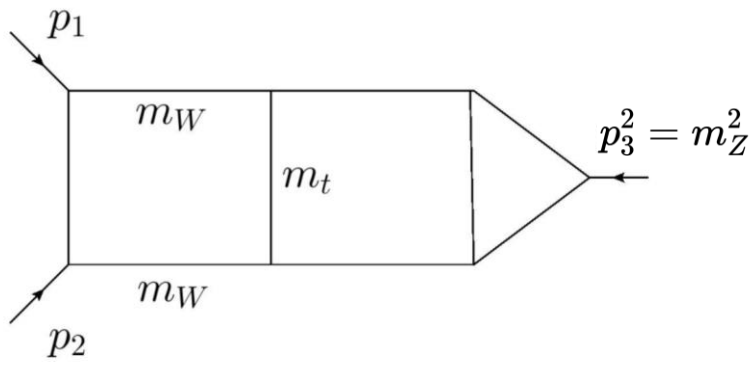
Table 1: The computation time that was needed to transport boundary conditions from $p^2/m^2 = -\infty$ to $p^2/m^2 = 32$, for various values of the expansion order. We used the options `ChopPrecision -> 225`, `DivisionOrder -> 3`, `RadiusOfConvergence -> 4`, `WorkingPrecision -> 400`, `UseMobius -> False`, `UsePade -> False`.

3-loop vertex topology, relevant for mixed-EW

- Let us consider the following 3-loop integral family:

Based on work with

[Ievgen Dubovyk, Ayres Freitas, Janusz Gluza, Krzysztof Grzanka, MH, Johann Usovitsch, 2201.02576]



$$
\begin{array}{llll}
D_1 = m_W^2 - k_3^2 & D_2 = -k_2^2 & D_3 = -k_1^2 & D_4 = -(k_1 - p_1 - p_2)^2 \\
D_5 = -(k_2 - p_1 - p_2)^2 & D_6 = m_W^2 - (k_3 - p_1 - p_2)^2 & D_7 = -(k_3 - p_1)^2 & D_8 = m_t^2 - (k_3 - k_2)^2 \\
D_9 = -(k_2 - k_1)^2 & N_{10} = -(k_1 - k_3)^2 & N_{11} = -(k_1 - p_2)^2 & N_{12} = -(k_2 - p_2)^2
\end{array}
$$

- The finite basis consists of 77 integrals in total. We choose 19 integrals in d = 4, 53 integrals in d = 6, and 5 integrals in d = 8. The differential equations are ~ 10 MB before expanding in ϵ.

3-Loop vertex topology, relevant for mixed-EW

- In the automatic approach that DiffExp uses, the differential matrix is required to be finite as $\epsilon \to 0$. For a general basis $\vec{f}$ this is not the case.

[Ievgen Dubovyk, Ayres Freitas, Janusz Gluza, Krzysztof Grzanka, MH, Johann Usovitsch, 2201.02576]

- Let's rescale each master integral by a power of ϵ:

$$\vec{g} = T\vec{f} \qquad T(\epsilon) = \mathrm{diag}\left(\epsilon^{t_1}, \ldots, \epsilon^{t_n}\right)$$

$$\frac{d}{dx}\vec{g}(x,\epsilon) = T(\epsilon)\hat{M}(x,\epsilon)T^{-1}(\epsilon)\vec{g}(x,\epsilon) \qquad \left(T\hat{M}T^{-1}\right)_{ij} = \epsilon^{t_i - t_j}M_{ij} \qquad M_{ij} = \sum_{n=-N_{ij}}^{\infty} M_{ij}^{(n)}\epsilon^n.$$

- Then we obtain the following linear system:

$$t_i - t_j - N_{ij} \geq 0, \qquad \text{for all } i, j = 1, \ldots, m$$

- We don't always find a solution for the above system, but empirically we do find a solution if we choose a finite basis of master integrals.

Basis integrals

$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{4,2,2,2,2,0,0,0,0,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=6-2\epsilon}_{3,2,2,2,2,1,0,0,0,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=6-2\epsilon}_{2,2,2,2,2,1,1,0,0,0,0,0}$	$\left(\frac{1}{\epsilon^4}\right) I^{d=6-2\epsilon}_{4,0,2,2,0,0,0,4,0,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{3,2,2,2,0,0,0,2,0,0,0,0}$
$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{3,0,2,2,2,0,0,2,0,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{3,0,2,2,1,0,0,3,0,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{2,0,2,2,2,0,0,3,0,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{0,2,2,2,2,0,0,4,0,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{3,0,2,2,0,1,0,4,0,0,0,0}$
$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,2,2,2,0,1,0,2,0,0,0,0}$	$I^{d=6-2\epsilon}_{2,2,2,2,1,1,0,1,0,0,0,0}$	$\left(\frac{1}{\epsilon^4}\right) I^{d=6-2\epsilon}_{0,2,2,2,0,0,2,3,0,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,0,2,2,2,0,1,2,0,0,0,0}$	$I^{d=6-2\epsilon}_{0,2,2,2,1,0,2,2,0,0,0,0}$
$I^{d=6-2\epsilon}_{0,2,2,2,1,0,1,3,0,0,0,0}$	$I^{d=6-2\epsilon}_{2,2,2,2,1,0,1,1,0,0,0,0}$	$I^{d=6-2\epsilon}_{2,1,2,2,2,0,1,1,0,0,0,0}$	$I^{d=6-2\epsilon}_{2,1,2,2,1,0,1,2,0,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,0,2,2,0,1,1,4,0,0,0,0}$
$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{1,2,2,2,0,1,1,2,0,0,0,0}$	$I^{d=6-2\epsilon}_{1,2,2,2,1,1,1,1,0,0,0,0}$	$\left(\frac{1}{\epsilon^4}\right) I^{d=8-2\epsilon}_{5,3,0,3,0,0,0,0,3,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=8-2\epsilon}_{4,3,0,3,0,1,0,0,3,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=8-2\epsilon}_{3,3,0,3,0,1,1,0,3,0,0,0}$
$\left(\frac{1}{\epsilon^5}\right) I^{d=6-2\epsilon}_{3,0,2,0,0,0,0,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^5}\right) I^{d=6-2\epsilon}_{4,0,2,0,0,0,0,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^5}\right) I^{d=6-2\epsilon}_{3,0,0,2,0,0,0,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^5}\right) I^{d=6-2\epsilon}_{4,0,0,2,0,0,0,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^5}\right) I^{d=6-2\epsilon}_{3,0,0,2,0,0,0,4,2,0,0,0}$
$\left(\frac{1}{\epsilon^5}\right) I^{d=6-2\epsilon}_{5,0,0,2,0,0,0,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^4}\right) I^{d=8-2\epsilon}_{0,3,0,3,0,0,0,5,3,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{3,2,0,2,0,0,0,1,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{3,0,2,0,2,0,0,1,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{3,0,2,0,1,0,0,2,2,0,0,0}$
$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,0,2,0,2,0,0,2,2,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{2,0,2,0,0,1,0,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{1,0,2,0,0,2,0,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,2,0,2,0,1,0,1,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,1,0,2,0,1,0,2,2,0,0,0}$
$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{1,0,1,1,0,1,0,2,1,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{2,0,1,1,0,1,0,2,1,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{1,0,1,1,0,1,0,3,1,0,0,0}$	$I^{d=4-2\epsilon}_{1,1,1,1,0,1,0,1,1,0,0,0}$	$\left(\frac{1}{\epsilon^5}\right) I^{d=8-2\epsilon}_{0,0,3,0,0,0,3,4,3,0,0,0}$
$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{2,0,0,2,0,0,1,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{0,2,0,2,0,0,2,2,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{0,2,0,2,0,0,1,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,2,0,2,0,0,1,1,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,1,0,2,0,0,1,2,2,0,0,0}$
$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{1,2,0,2,0,0,2,1,2,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{0,0,2,1,0,0,2,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{0,0,2,2,0,0,2,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{0,0,2,1,0,0,2,4,2,0,0,0}$	$I^{d=4-2\epsilon}_{1,0,1,1,0,0,1,2,1,0,0,0}$
$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{2,0,1,1,0,0,1,2,1,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{1,0,1,1,0,0,1,3,1,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{3,0,1,1,0,0,1,2,1,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{2,0,1,1,0,0,1,3,1,0,0,0}$	$I^{d=4-2\epsilon}_{0,1,1,1,0,0,1,2,1,0,0,0}$
$I^{d=4-2\epsilon}_{1,1,1,1,0,0,1,1,1,0,0,0}$	$I^{d=4-2\epsilon}_{2,1,1,1,0,0,1,1,1,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,0,2,0,2,0,1,1,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{2,0,2,0,1,0,1,2,2,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=6-2\epsilon}_{1,0,2,0,2,0,2,1,2,0,0,0}$
$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{1,0,2,0,2,0,1,2,2,0,0,0}$	$I^{d=4-2\epsilon}_{1,0,1,1,1,0,1,1,1,0,0,0}$	$I^{d=4-2\epsilon}_{2,0,1,1,1,0,1,1,1,0,0,0}$	$I^{d=4-2\epsilon}_{1,0,1,1,1,0,1,2,1,0,0,0}$	$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{1,0,2,0,0,1,1,3,2,0,0,0}$
$\left(\frac{1}{\epsilon^3}\right) I^{d=6-2\epsilon}_{2,0,2,0,0,1,1,3,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{1,2,0,2,0,1,1,1,2,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{1,1,0,2,0,1,1,2,2,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{1,0,1,1,0,1,1,2,1,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{2,0,1,1,0,1,1,2,1,0,0,0}$
$\left(\frac{1}{\epsilon}\right) I^{d=4-2\epsilon}_{1,0,1,1,0,1,1,3,1,0,0,0}$	$I^{d=4-2\epsilon}_{1,1,1,1,0,1,1,1,1,0,0,0}$	$\left(\frac{1}{\epsilon^2}\right) I^{d=6-2\epsilon}_{4,2,2,2,2,0,0,0,0,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=6-2\epsilon}_{3,2,2,2,2,1,0,0,0,0,0,0}$	$\left(\frac{1}{\epsilon}\right) I^{d=6-2\epsilon}_{2,2,2,2,2,1,1,0,0,0,0,0}$

Numerical boundary conditions using pySecDec

- In our basis choice, each $\mathrm{I}_{\nu_1,\dots,\nu_n}$ is a finite integral. The finite integral candidates were found using Reduze 2. This has the benefit of allowing fast(-ish) convergence of the integrals with pySecDec in the Euclidean region.

[Manteuffel, Studerus, 1201.4330]
[Manteuffel, Panzer, Schabinger, 1411.7392]

- We compute all basis integrals in the Euclidean region in the point $s = -2, m_W^2 = 4, m_t^2 = 16$, using the Qmc integrator configured with:

```
.ib.use_Qmc(minn=10**7, maxeval=10**9, transform='korobov3', epsabs=1e-12, cputhreads=16)
```

- The computation took between 1/2-1 day on a Ryzen Threadripper Pro 3955WX.
- We find for example: $\mathrm{I}_{1,1,1,1,0,1,1,1,1} = 0.133952666651743990 - 0.138991496465805000\,\epsilon + O(\epsilon^2)$
$$\pm\left(2.\times 10^{-10} + 7.\times 10^{-10}\,\epsilon\right)$$

3-Loop vertex topology, relevant for mixed-EW

[Ievgen Dubovyk, Ayres Freitas, Janusz Gluza, Krzysztof Grzanka, MH, Johann Usovitsch, 2201.02576]

- Another limitation of DiffExp is that $i\delta$-prescriptions for crossing thresholds are provided manually.
- We dealt with this by assuming all physical thresholds arise from unitarity cuts. We then add all prescriptions of the form:

$$s - \left(\sum_j m_j\right)^2 + i\delta$$

where s is the momentum along the cut, and the m_j are the masses of the cut propagators.

(For pseudo-thresholds these prescriptions are automatically ignored by DiffExp.)

3-loop vertex topology

- Using DiffExp we may transport from the Euclidean point to any other (real) point in phase-space.

- Transporting from $\left(s, m_W^2, m_t^2\right) = (-2,4,16)$ to $\left(s, m_W^2, m_t^2\right) = \left(1, \left(\frac{401925}{455938}\right)^2, \left(\frac{433000}{227969}\right)^2\right)$, we obtain:

$$I^{d=6-2\epsilon}_{1,1,0,2,0,1,1,2,2,0,0,0} = (0.125019 + 0.0127438\, i) - (0.334035 - 0.0731341\, i)\,\epsilon + (1.81433 + 0.208055\, i)\,\epsilon^2 - (6.08263 - 0.389921\, i)\,\epsilon^3 + O(\epsilon^4)$$
$$I^{d=4-2\epsilon}_{1,0,1,1,0,1,1,2,1,0,0,0} = (1.17171 + 1.03298\, i) - (3.13434 - 1.43328\, i)\,\epsilon + (5.9312 + 3.04346\, i)\,\epsilon^2 + O(\epsilon^3)$$
$$I^{d=4-2\epsilon}_{2,0,1,1,0,1,1,2,1,0,0,0} = (0.912403 + 0.837335\, i) - (1.66844 - 1.83869\, i)\,\epsilon + (2.25671 + 3.31779\, i)\,\epsilon^2 + O(\epsilon^3)$$
$$I^{d=4-2\epsilon}_{1,0,1,1,0,1,1,3,1,0,0,0} = (0.102616 + 0.123891\, i) - (0.137177 - 0.313638\, i)\,\epsilon - (0.0575107 - 0.560502\, i)\,\epsilon^2 + O(\epsilon^3)$$
$$I^{d=4-2\epsilon}_{1,1,1,1,0,1,1,1,1,0,0,0} = (1.30731 + 3.42323\, i) - (10.0551 - 8.533\, i)\,\epsilon + O(\epsilon^2)$$

- The computation involved 16 line segments and took 45 minutes on a single CPU core. The final precision reached has 8+ significant digits. We also obtain a piecewise function.

Feynman parameter integration through differential equations

Based on work with Johann Usovitsch

Feynman parametrization

- Consider a scalar Feynman integral:

$$I^{(d)}_{\nu_1\ldots\nu_n}(s_1,\ldots,s_N) = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) \prod_{j=1}^{n} \frac{1}{D_j^{\nu_j}}$$

Where:

$$d = d_{\text{int}} - 2\epsilon$$

$$D_i = -q_i^2 + m_i^2 - i\delta$$

- A formula by Feynman tells us that:

$$\frac{1}{D_1^{\nu_i}\ldots D_n^{\nu_n}} = \frac{\Gamma(\nu)}{\Gamma(\nu_1)\ldots\Gamma(\nu_n)} \int_0^1 d^n x \frac{x_1^{\nu_1-1}\ldots x_n^{\nu_n-1}\delta\left(1-\sum_{j=1}^n x_j\right)}{(x_1 D_1 + \ldots + x_n D_n)^\nu}$$

- This gives the well-known Feynman parametrization:

$$I^{(d)}_{\nu_1\ldots\nu_n} = \frac{\Gamma(\nu - ld/2)}{\prod_{j=1}^n \Gamma(\nu_j)} \int \left(\prod_{j=1}^{n} dx_j x_j^{\nu_j-1} \right) \frac{\mathcal{U}^{\nu-(l+1)d/2}}{\mathcal{F}^{\nu-ld/2}} \delta\left(1 - \sum_{j=1}^{n} x_j\right)$$

Direct integration & differential equations

See also:
[MH, Moriello, 1712.04441]
[Papadopoulos, Wever, 1910.06275]

- Alternatively, we may apply the formula recursively to two propagators:

$$D_{12} = x_1 D_1 + (1 - x_1) D_2$$
$$D_{123} = x_2 D_{12} + (1 - x_2) D_3$$
$$\ldots$$
$$D_{1\ldots n} = x_{n-1} D_{1\ldots,n-1} + (1 - x_{n-1}) D_n$$

- And we define a collection of integral families:

$$I^{(1,d)}_{\nu_1\ldots\nu_n} = I^{(d)}_{\nu_1\ldots\nu_n} = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) \prod_{j=1}^{n} D_j^{-\nu_j}$$

$$I^{(k,d)}_{\nu_1\ldots\nu_n} = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) D_{1\ldots k}^{-\nu_1} \prod_{j=k+1}^{n} D_j^{-\nu_{j-k+1}} \prod_{j=1}^{k-1} D_j^{-\nu_{j+n-k+1}} \quad \text{for } 1 < k < n-1$$

$$I^{(n,d)}_{\nu_1\ldots\nu_n} = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) D_{1\ldots n}^{-\nu_1} \prod_{j=1}^{n-1} D_j^{-\nu_{j+1}}$$

The orange propagators are included to provide a complete basis for IBP reductions, but do not play any further role in the discussion.

Direct integration & differential equations

- Example: $n = 4$ propagators

$$I^{(1,d)}_{\nu_1...\nu_n} = I^{(d)}_{\nu_1...\nu_n} = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) D_1^{-\nu_1} D_2^{-\nu_2} D_3^{-\nu_3} D_4^{-\nu_4}$$

$$I^{(2,d)}_{\nu_1...\nu_n} = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) D_{12}^{-\nu_1} D_3^{-\nu_2} D_4^{-\nu_3} D_1^{-\nu_4}$$

$$I^{(3,d)}_{\nu_1...\nu_n} = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) D_{123}^{-\nu_1} D_4^{-\nu_2} D_1^{-\nu_3} D_2^{-\nu_4}$$

$$I^{(4,d)}_{\nu_1...\nu_n} = \int \left(\prod_{j=1}^{l} \frac{d^d k_j}{i\pi^{\frac{d}{2}}} \right) D_{1234}^{-\nu_1} D_1^{-\nu_2} D_2^{-\nu_3} D_3^{-\nu_4}$$

Direct integration & differential equations

- Then we have:

$$I^{(k-1,d)}_{\nu_1...\nu_n} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 dx_k x_k^{\nu_1-1}(1-x_k)^{\nu_2-1} I^{(k,d)}_{\nu_1+\nu_2,\nu_3...\nu_n}$$

- Note that by iterating the recursion formula, we find:

$$I^{(d)}_{\nu_1...\nu_n} = \frac{\Gamma(\nu)}{\Gamma(\nu_1)\ldots\Gamma(\nu_n)} \left(\prod_{j=1}^{n-1} \int_0^1 dx_j x_j^{\mu_j-1}(1-x_j)^{\nu_{j+1}-1}\right) I^{(n,d)}_{\mu_n,0,...,0}$$

- The recursion ends at a generalized tadpole integral:

$$I^{(n,d)}_{\mu_n,0,...,0} = \int \left(\prod_{j=1}^{l} \frac{d^dk_j}{i\pi^{\frac{d}{2}}}\right) D^{-\nu}_{1...n} = \frac{\Gamma(\nu-ld/2)}{\Gamma(\nu)} \frac{\tilde{\mathcal{U}}^{\nu-(l+1)d/2}}{\tilde{\mathcal{F}}^{\nu-ld/2}}$$

$$x_1 \to \prod_{i=1}^{n-1} x_i \quad x_n \to (1-x_{n-1})$$

$$x_j \to (1-x_{j-1}) \prod_{i=j}^{n-1} x_i \quad \text{for } j=2,\ldots,n-1$$

- Where $\tilde{\mathcal{U}}$ and $\tilde{F}$ are rescaled versions of the standard Symanzik polynomials.

Direct integration & differential equations

- We may use this to obtain numerical results for the complete integral family:

  1. Set up a system of differential equations:

  $$\partial_{x_k}\vec{I}^{(k+1,d)} = M_{x_k}\vec{I}^{(k+1,d)}$$

  2. Transport boundary conditions to obtain a piecewise solution between $0 < x_k < 1$

  3. Integrate the expansions according to the recursion formula:

  $$I^{(k-1,d)}_{\nu_1...\nu_n} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_0^1 dx_k x_k^{\nu_1-1}(1-x_k)^{\nu_2-1} I^{(k,d)}_{\nu_1+\nu_2,\nu_3...\nu_n}$$

- The only input needed is:

$$I_\nu^{(n,d)} = \int\left(\prod_{j=1}^{l}\frac{d^dk_j}{i\pi^{\frac{d}{2}}}\right)D^{-\nu}_{1...n} = \frac{\Gamma(\nu-ld/2)}{\Gamma(\nu)}\frac{\tilde{\mathcal{U}}^{\nu-(l+1)d/2}}{\tilde{\mathcal{F}}^{\nu-ld/2}}$$

5-point 2-loop example:

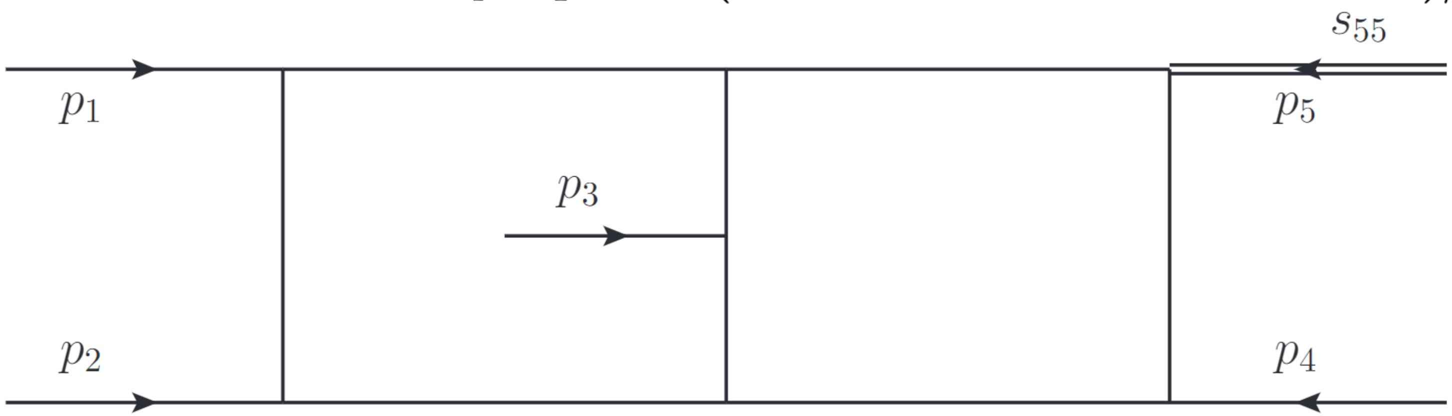
- We consider the following illustrative example:

$$I^{5\mathrm{p}} = \int \frac{dk_1^{4-2\varepsilon} dk_2^{4-2\varepsilon} ((k_2-p_1-p_2)^2)^{-a_9} ((k_1-p_1-p_2-p_3-p_4)^2)^{-a_{10}}}{((k_2-p_1-p_2-p_3-p_4)^2)^{a_1} ((k_2-p_1-p_2-p_3)^2)^{a_2} (k_2^2)^{a_3}} \times \frac{((k_2-p_1)^2)^{-a_{11}}}{((k_1-p_1-p_2)^2)^{a_4} ((k_1-p_1)^2)^{a_5} (k_1^2)^{a_6} ((k_1-k_2+p_3)^2)^{a_7} ((k_1-k_2)^2)^{a_8}},$$

- Where the kinematics is

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0,$$
$$p_1 \cdot p_2 = s_{12}/2, p_1 \cdot p_3 = s_{13}/2, p_1 \cdot p_4 = s_{14}/2, p_2 \cdot p_3 = s_{23}/2,$$
$$p_2 \cdot p_4 = -(s_{12}+s_{13}+s_{14}+s_{23}+s_{34}-s_{55})/2, p_3 \cdot p_4 = s_{34}/2$$

- Graphically, we have:



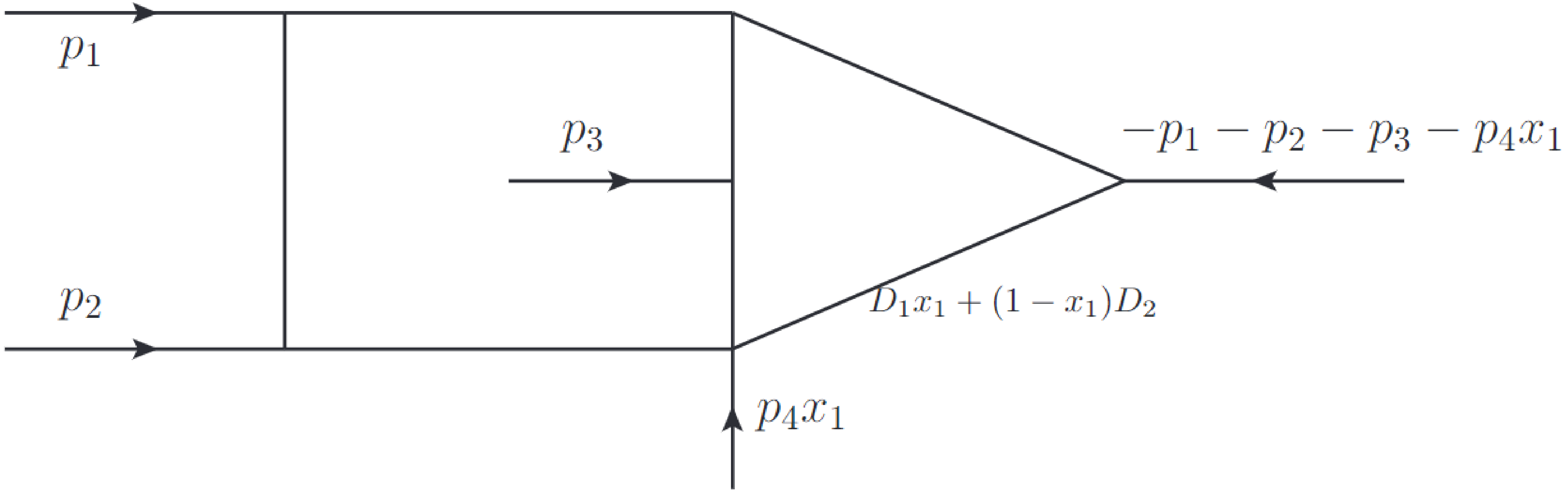
5-point 2-loop example:

- We combine our propagators in the following way:

Iterative Feynman trick			
j	input	output	Number of master integrals
1	—	uncombined	142
2	$\{D_1, D_2\}$	D_{12}	69
3	$\{D_4, D_5\}$	D_{45}	32
4	$\{D_7, D_8\}$	D_{78}	16
5	$\{D_{12}, D_3\}$	D_{123}	8
6	$\{D_{45}, D_6\}$	D_{456}	4
7	$\{D_{123}, D_{456}\}$	D_{123456}	2
8	$\{D_{123456}, D_{78}\}$	$D_{12345678}$	1

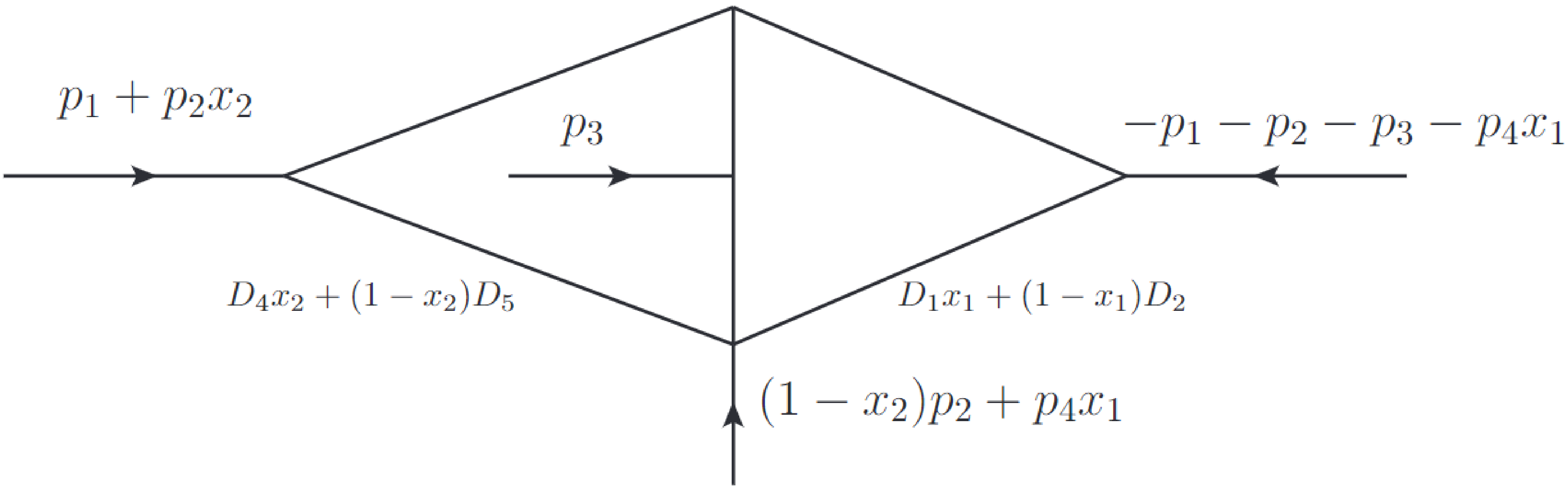
- The choices are motivated by first combining propagators which have the same internal momentum. This leads to simplifications of the graph.

5-point 2-loop example:



- Note that: $D_{12} = (k_2 \underbrace{- p_1 - p_2 - p_3 - p_4 x_1)^2}_{\tilde{Q}} - \underbrace{x_1(1 - x_1)(-p_4^2)}_{\widetilde{M}}$

5-point 2-loop example:



5-point 2-loop example:

We evaluate one of the most complicated master integrals at the numerical point

$$s_{14} = 3,\ s_{13} = -11/17,\ s_{23} = -13/17,\ s_{12} = -7/17,\ s_{34} = -7/13,\ s_{55} = -1 \text{ in } d = 4 - 2\varepsilon$$

$$
\begin{aligned}
\mathrm{I}^{5\mathrm{p}}[1,3,1,1,1,1,1,1,0,0,0] = \\
&\frac{1}{\varepsilon^{-4}}(-80991.44634941832815855134956686330134244459)+ \\
&\frac{1}{\varepsilon^{-3}}(-1176854.14050165085751620090895007182416011- \\
&\quad 303701.84533500293424001259182549353163494 29i)+ \\
&\frac{1}{\varepsilon^{-2}}(-13432835.84776929621856373949316048917976 74- \\
&\quad 4251651.64965980166114774272201533676580580i)+ \\
&\frac{1}{\varepsilon^{-1}}(-111346171.637045032880704355278590042329 21- \\
&\quad 32927342.39568833030002166578855680196817 6i)+ \\
&(-763045644.55613054420938678675134277317 42- \\
&\quad 183231121.40487741467886614905312052821 19i)+ \\
&+\varepsilon(-4428755434.16119754697555927652734791719- \\
&\quad 816059490.91219542938806845916619764871 9i)+ \\
&\varepsilon^2(-23085640630.25988952077799452653763919 9- \\
&\quad 3082908606.75512948115042154736426296 05i)+ \\
&\varepsilon^3(-110164352209.70924126524512566109439 38- \\
&\quad 10252510409.42185691550687766152353640i)+ \\
&\varepsilon^4(-497649560130.01520927919209863153192 0- \\
&\quad 30796992268.35160868705666559550754104i).
\end{aligned}
$$

Computational complexity (IBP):

- Combining two propagators leads to integral families with less master integrals than the deformations from auxiliary mass flow, and in turn faster IBP reductions:

Topology	No deformation	Combined propagators	AMFlow
topo7	31	19	31
topo7 with $m_1 = 0$, $m_2 = 0$	8	12	21
5p	142	69	191
5p with $s_{55} = 0$	108	69	174

- However, the DiffExp solver (based on Frobenius) is not as fast as the solver in AMFlow (local Fuchsian form + recursion for coefficients). We expect a combination of methods will be fruitful!

Conclusion

- Series expansion methods allow for obtaining high-precision numerical results for multiloop integrals with multiple scales.
- The Mathematica package DiffExp can be used for computing user-provided systems of differential equations.
- Analytic continuation past thresholds can be performed in an automatic way.
- Automated approaches exist for computing boundary conditions (i.e. AMFlow, or the here presented Feynman parameter approach.)

Thank you for listening!