The auxiliary mass flow approach

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Based on 1711.09572, 2107.01864 and 2201.11669

In collaboration with Yan-Qing Ma and Chen-Yu Wang

Precision calculations for future e^+e^- colliders: targets and tools

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Outline

High precision physics

➢ Multiloop scattering amplitudes

- most popular approach → talks by Vasily, Andreas, Long, Narayan
  - construct the amplitude → talk by Max

$$\mathcal{A} = \sum_j a_j I_j$$

  - reduce the scalar integrals to master integrals → talk by Tiziano

$$I_j = \sum_k b_{jk} \mathcal{I}_k$$

  - compute the master integrals → talks by Stefan, Vitalii, Janusz, Martijn

$$\mathcal{I}_k = \sum_{l=-2L} c_{kl} \epsilon^l$$

- other novel & promising approaches → talks by Valentin, Charalampos

High precision physics

➢ Master integrals calculation

- canonical differential equations [Kotikov, Phys. Lett. B, 1991][Henn, Phys. Rev. Lett., 2013]
- sector decomposition [Binoth and Heinrich, Nucl. Phys. B, 2000]
- Mellin-Barnes representation [Boos and Davydychev, Theor.Math.Phys., 1991][Smirnov, Phys. Lett. B, 1999]
- numerical (ordinary) differential equations [Czakon, Phys. Lett. B, 2008]
  - numerical solver [Hidding, Comput.Phys.Commun., 2021][Armadillo, Bonciani, et al, arXiv:2205.03345]
  - differential equations → IBP reduction [Chetyrkin and Tkachov, Nucl. Phys. B, 1981] [Laporta, Int. J. Mod. Phys. A, 2000]

$$\frac{\partial}{\partial x}\vec{\mathcal{I}}(x) = A(x)\vec{\mathcal{I}}(x)$$

  - boundary conditions → method of region [Beneke and Smirnov, Nucl. Phys. B, 1998], sector decomposition, auxiliary mass flow

$$\vec{\mathcal{I}}(x_0) \quad \text{or} \quad \vec{\mathcal{I}}(x) \overset{x\to x_0}{\sim} \cdots$$

Outline

Auxiliary mass flow

➢ Dimensionally regulated Feynman integrals

$$I(\vec{\nu}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0)^{\nu_K}}$$

- integrals with auxiliary mass parameter η

$$I_{\mathrm{aux}}(\vec{\nu};\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \eta)^{\nu_1} \cdots (\mathcal{D}_K - \eta)^{\nu_K}}$$

- obtain physical integrals through

$$I(\vec{\nu}) = \lim_{\eta \to \mathrm{i}0^-} I_{\mathrm{aux}}(\vec{\nu};\eta)$$

Auxiliary mass flow

➢ Expansion near $\eta = \infty$

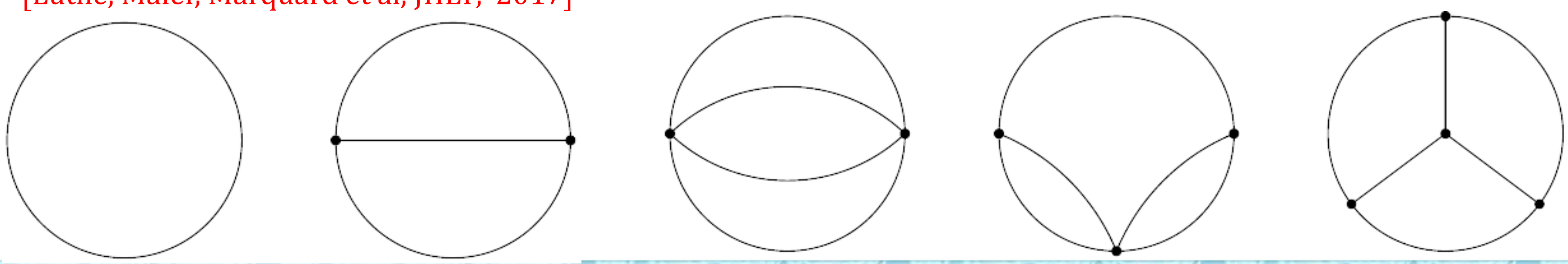
- method of region [Beneke and Smirnov, Nucl. Phys. B, 1998]
  - the only contributing region: $\ell_i^\mu \sim \sqrt{\eta}$

$$\frac{1}{((\ell+p)^2-m^2-\eta)^\nu} = \frac{1}{(\ell^2-\eta)^\nu}\sum_{i=0}^{\infty}\frac{(\nu)_i}{i!}\left(-\frac{2\ell\cdot p+p^2-m^2}{\ell^2-\eta}\right)^i$$

- Feynman parametric representation

$$\int \mathfrak{D}\vec{x}\frac{\mathcal{U}^{-D/2}}{(\mathcal{F}/\mathcal{U}+\eta)^{N_\nu-LD/2}} = \eta^{LD/2-N_\nu}\int \mathfrak{D}\vec{x}\,\mathcal{U}^{-D/2}\sum_{i=0}^{\infty}\frac{(N_\nu-LD/2)_i}{i!}\left(-\frac{\mathcal{F}}{\eta\,\mathcal{U}}\right)^i$$

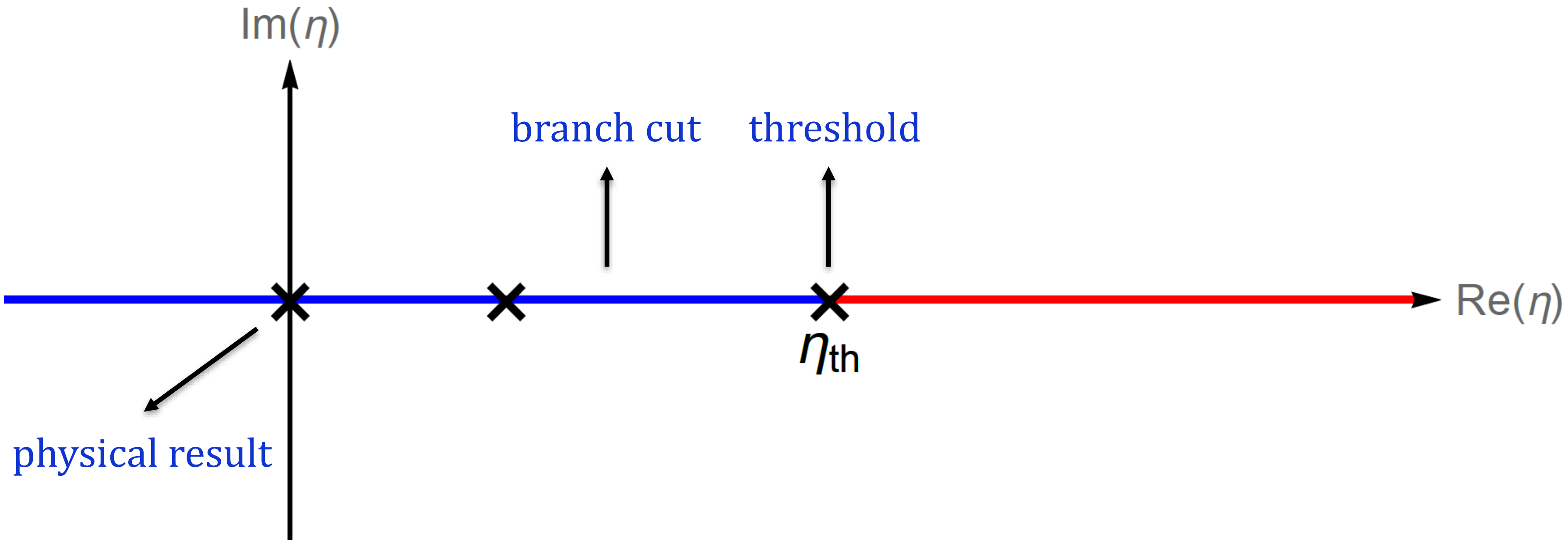
- fully massive vacuum integrals [Davydychev and Tausk, Nucl, Phys. B, 1993] [Broadhurst, Eur. Phys. J. C, 1999][Schroder and Vuorinen, JHEP, 2005] [Kniehl, Pikelner and Veretin, JHEP, 2017][Luthe, phdthesis, 2015] [Luthe, Maier, Marquard et al, JHEP, 2017]



Auxiliary mass flow

➢ $I(\eta)$ as an analytic function of η

- there should be a maximal threshold $\eta = \eta_{\text{th}}$ on the real axis
  - $I(\eta)$ is real-valued for $\eta > \eta_{\text{th}}$ and complex-valued for $\eta < \eta_{\text{th}}$
- branch cut can be chosen as the straight line connecting $\eta = -\infty$ and $\eta = \eta_{\text{th}}$ along the real axis, such that $I(\eta^*) = I^*(\eta)$

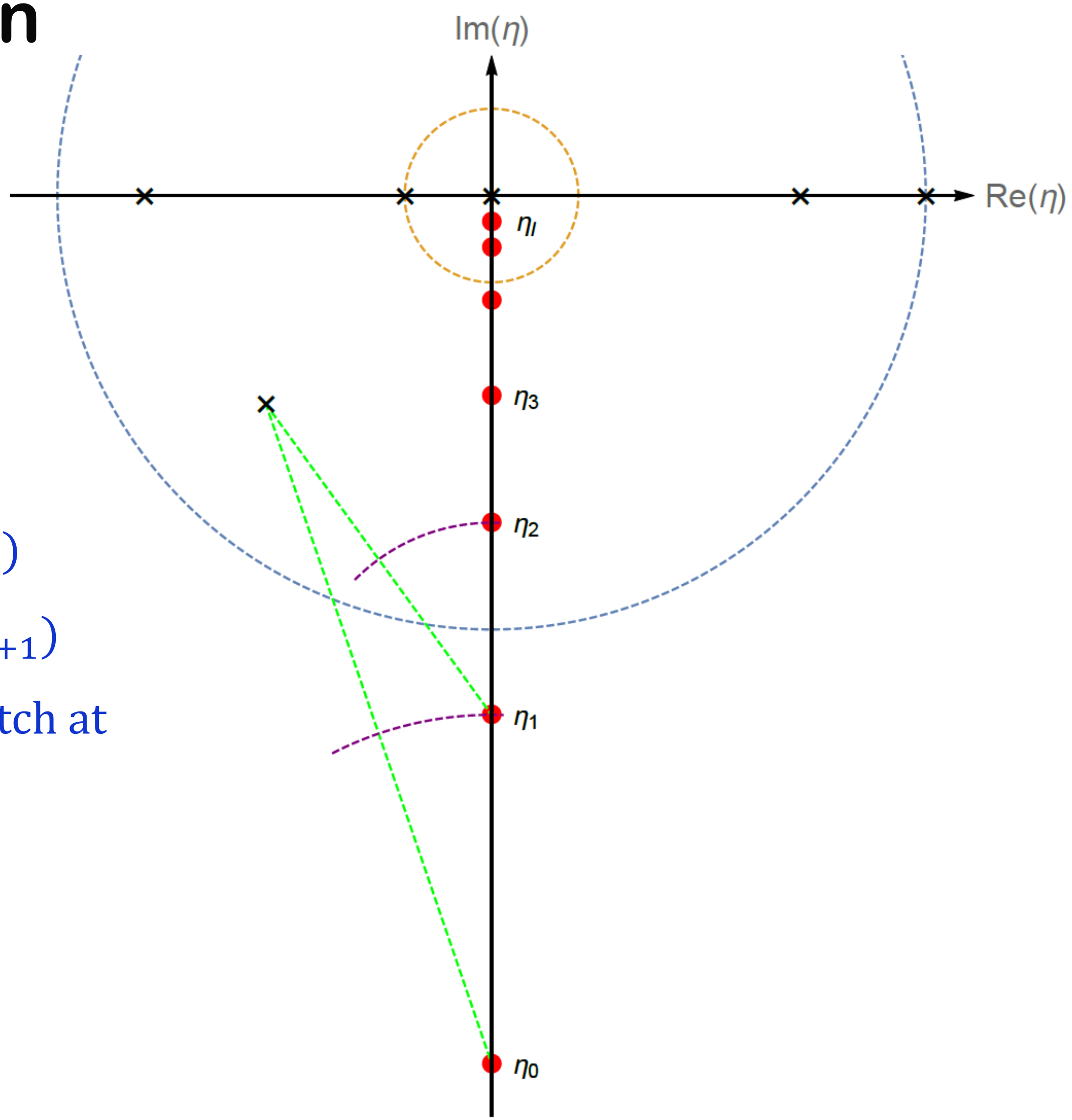


➢ Analytic continuation

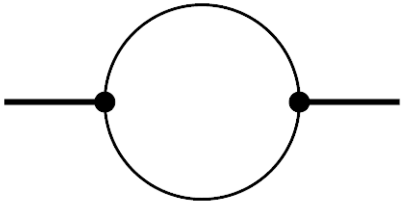
- differential equations

$$\frac{\partial}{\partial \eta}\vec{\mathcal{I}}_{\text{aux}}(\eta) = A(\eta)\vec{\mathcal{I}}_{\text{aux}}(\eta)$$

- boundary conditions at $\eta = \infty$
- define a path: $\{\eta_0, \eta_1, \ldots, \eta_l\}$
- expand at $\eta = \infty$ to estimate $I(\eta_0)$
- expand at $\eta = \eta_i$ to estimate $I(\eta_{i+1})$
- expand formally at $\eta = 0$ and match at $\eta = \eta_l$
    - η_0: outside the larger circle
    - η_l: inside the smaller circle
    - $|\eta_{i+1} - \eta_i| < r_i$



➢ A simple example: one-loop massless bubble



$$I(\nu_1,\nu_2)=\int\frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}}\frac{1}{(\ell^2)^{\nu_1}((\ell+p)^2)^{\nu_2}}$$

- master integral: $I(1,1)$

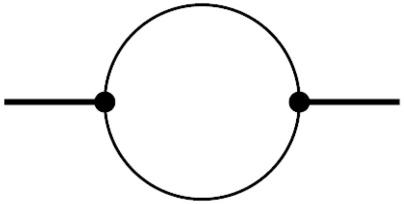
$$I(1,1)=(-p^2-i0)^{D/2-2}\times\frac{\Gamma\left(2-D/2\right)\Gamma\left(D/2-1\right)^2}{\Gamma(D-2)}$$

$$I(1,1)|_{p^2=1,D=4-2\epsilon}=\frac{1}{\epsilon}+(2-\gamma+i\pi)+O\left(\epsilon^1\right)$$

$$=\frac{1}{\epsilon}+(1.42278\ +3.14159i)+O\left(\epsilon^1\right)$$

Auxiliary mass flow

- insert auxiliary mass



$$I_{\text{aux}}(\nu_1,\nu_2;\eta) = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2-\eta)^{\nu_1}((\ell+p)^2-\eta)^{\nu_2}}$$

- master integrals: $\vec{I}_{\text{aux}}(\eta) = \{I_{\text{aux}}(1,0;\eta),\ I_{\text{aux}}(1,1;\eta)\}$
- construct differential equations using IBP reduction

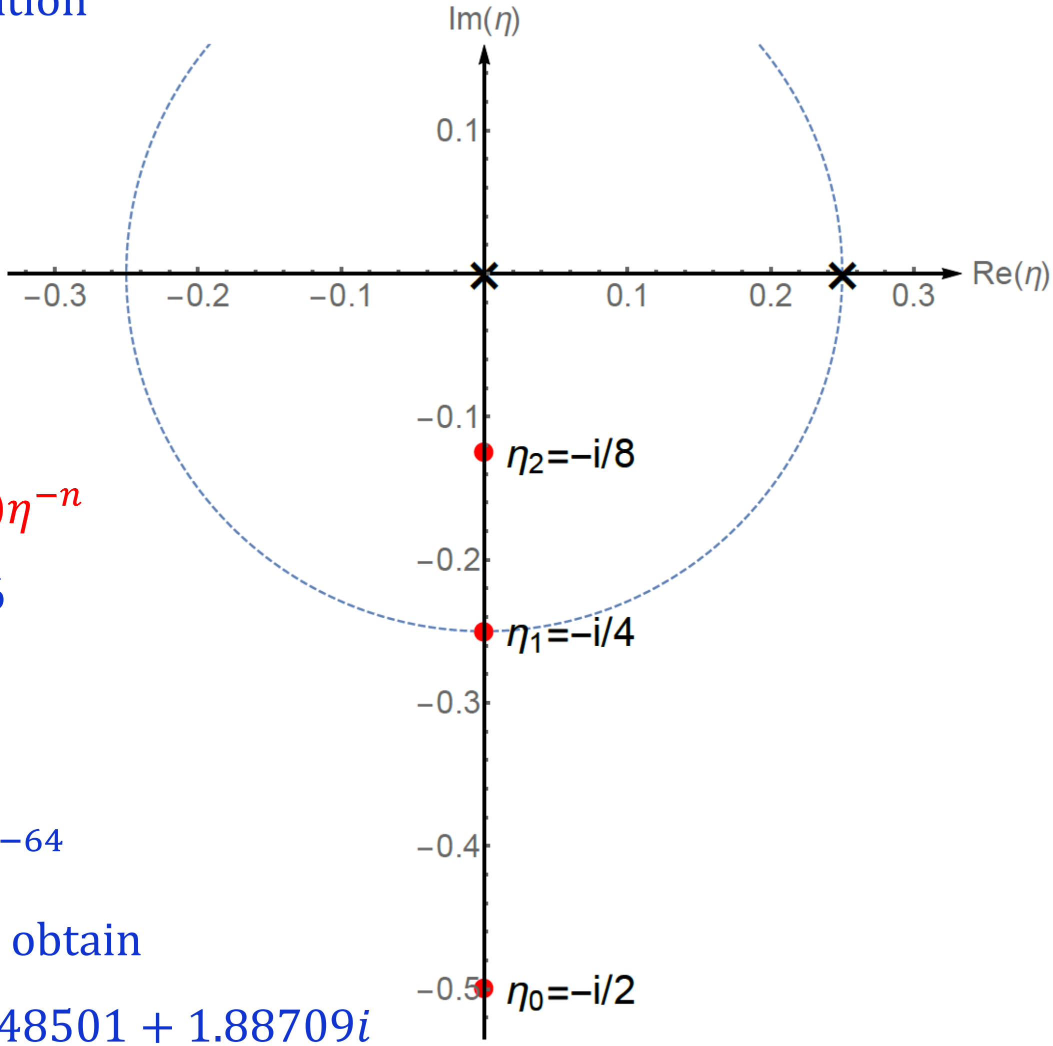
$$\frac{\partial}{\partial\eta}\vec{\mathcal{I}}_{\text{aux}}(\eta) = \begin{pmatrix} \frac{1-\epsilon}{\eta} & 0 \\ \frac{2(\epsilon-1)}{\eta(4\eta-1)} & -\frac{2(2\epsilon-1)}{4\eta-1} \end{pmatrix} \vec{\mathcal{I}}_{\text{aux}}(\eta)$$

- $\eta_{\text{th}} = 1/4$
- boundary conditions

$$I_{\text{aux}}(1,0;\eta) = \eta^{1-\epsilon} \times (-\Gamma(\epsilon-1))$$
$$I_{\text{aux}}(1,1;\eta) \sim \eta^{-\epsilon} \times (\Gamma(\epsilon) + \mathcal{O}(\eta^{-1}))$$

Auxiliary mass flow

- define a path for analytic continuation
  - singularities: {0, 1/4}
  - $R_{\mathrm{L}} = R_{\mathrm{S}} = 1/4$
  - $\{-i/2,\ -i/4,\ -i/8\}$



- expand near $\eta = \infty$
  - $I_{\mathrm{aux}}(1,1;\eta) = \eta^{-\epsilon} \sum_{n=0}^{\infty} a_n(\epsilon)\eta^{-n}$
    - $a_0(\epsilon) = \epsilon^{-1} - 0.577216$
    - $a_1(\epsilon) = 0.166667$
    - …
    - $a_{100}(\epsilon) = 5.49443 \times 10^{-64}$
  - estimate at $\eta = \eta_0 = -i/2$ to obtain

    $I_{\mathrm{aux}}(1,1;-i/2) = \epsilon^{-1} + 0.0548501 + 1.88709i$

Auxiliary mass flow

- expand near $\eta = \eta_0 = -i/2$
  - $I_{\text{aux}}(1,1;\eta) = \sum_{n=0}^{\infty} a_n(\epsilon)(\eta - \eta_0)^n$
    - $a_0(\epsilon) = \epsilon^{-1} + 0.0548501 + 1.88709i$
    - $a_1(\epsilon) = 0.5714 - 1.77538i$
    - …
    - $a_{100}(\epsilon) = -1.29958 \times 10^{24} + 1.28029 \times 10^{26}$
  - estimate at $\eta = \eta_1 = -i/4$ to obtain

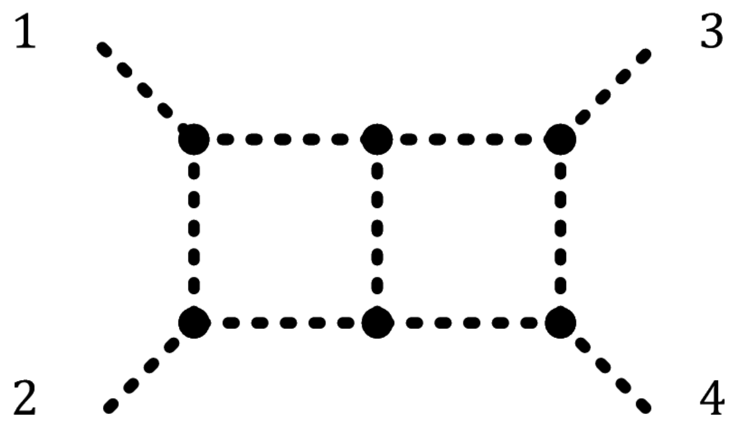
    $I_{\text{aux}}(1,1;-i/4) = \epsilon^{-1} + 0.609168 + 2.13174i$
- expand near $\eta = \eta_1 = -i/4$
  - $I_{\text{aux}}(1,1;\eta) = \sum_{n=0}^{\infty} a_n(\epsilon)(\eta - \eta_1)^n$
    - …
  - estimate at $\eta = -i/8$ to obtain

    $I_{\text{aux}}(1,1;-i/8) = \epsilon^{-1} + 0.994236 + 2.42639i$

Auxiliary mass flow

- expand near $\eta = 0$
  - $I_{\text{aux}}(1,1;\eta) = \sum_{n=0}^{\infty} a_n(\epsilon)\eta^n + \eta^{1-\epsilon} \sum_{n=0}^{\infty} b_n(\epsilon)\eta^n$
  - $b_n(\epsilon)$ can be totally determined by sub-topology
    - $b_0(\epsilon) = -2\Gamma(\epsilon - 1)$
    - $b_1(\epsilon) = 4\Gamma(\epsilon - 1)/(\epsilon - 2)$
    - $\ldots$
  - $a_n(\epsilon)$ cannot be totally determined but can be reduced to $a_0(\epsilon)$
    - $a_1(\epsilon) = 2(2\epsilon - 1)a_0(\epsilon)$
    - $a_2(\epsilon) = 2(2\epsilon - 1)(2\epsilon + 1)a_0(\epsilon)$
    - …
  - match at $\eta = \eta_2 = -i/8$ to obtain $a_0(\epsilon) = \epsilon^{-1} + 1.42278 + 3.14159i$
  - take the limit $\eta \to i0^-$
    - $\lim_{\eta \to i0^-} I_{\text{aux}}(1,1;\eta) = a_0(\epsilon) = \epsilon^{-1} + 1.42278 + 3.14159i$

➢ A two-loop example: massless double-box



- number of master integrals: 8 → 39

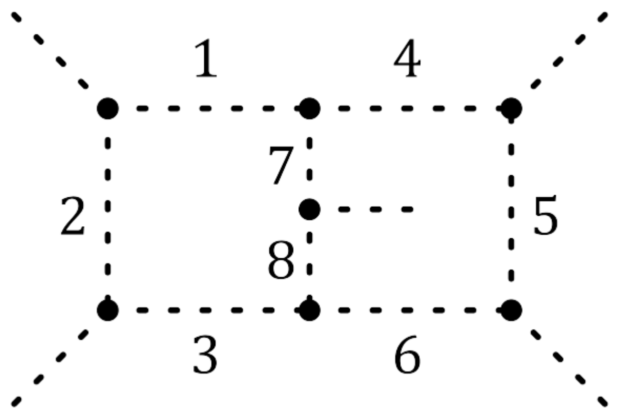
The growth of #MIs might result in prohibitive complexities for more complicated problems.

Iterative strategy.

Outline

Iterative strategy

➢ A simple observation



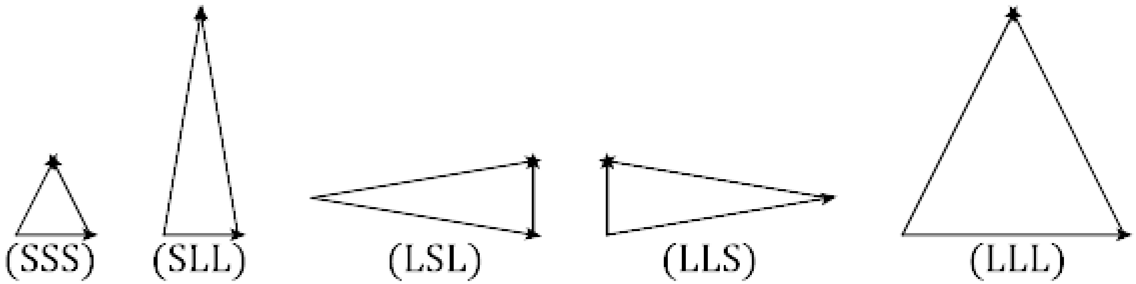
Mode	Propagators	Number of MIs
All	{1,2,3,4,5,6,7,8}	476
Loop	{4,5,6,7,8}	305
	{1,2,3,4,5,6}	319
Branch	{4,5,6}	233
	{7,8}	234
Propagator	{4}	178
	{5}	176
	{7}	220
Mass	$\cdots$	$\cdots$

- 108 master integrals
- inserting η to fewer propagators may reduce the number of master integrals

➢ Integration regions

- loop momentum of each branch can be either of $O(1)$ or $O(\sqrt{\eta})$
- regions for one-loop:

- regions for two-loop:



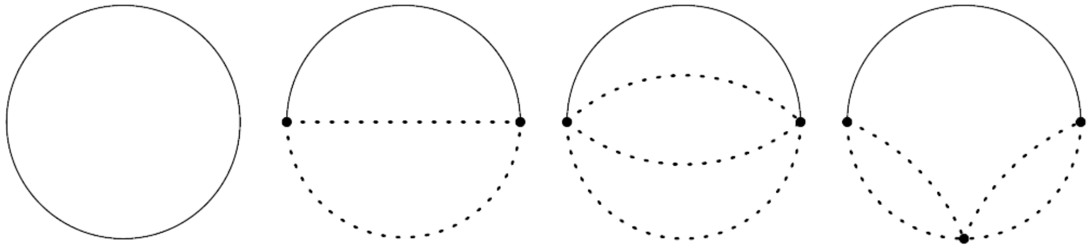
  - (LSS), (SLS), (SSL) excluded by momentum conservation
- $N_1 = 2, N_2 = 5, N_3 = 15, N_4 = 47, \ldots$

Iterative strategy

➢ Expansions

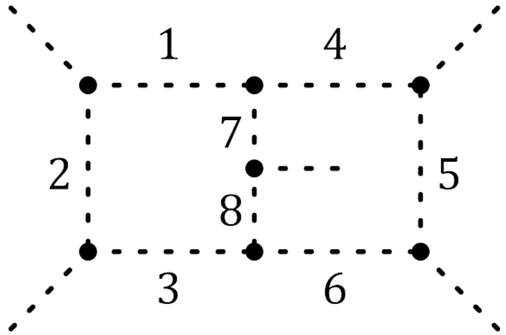
- (L ... L): $$\frac{1}{(\ell+p)^2-m^2-\kappa\eta} \sim \frac{1}{\ell^2-\kappa\eta}$$ ➡ vacuum
- (S ... S): $$\frac{1}{(\ell+p)^2-m^2-\eta} \sim \frac{1}{-\eta}$$ ➡ sub-family
- mixed: $$\frac{1}{(\ell_{\mathrm{L}}+\ell_{\mathrm{S}}+p)^2-m^2-\kappa\eta} \sim \frac{1}{\ell_{\mathrm{L}}^2-\kappa\eta}$$ ➡ factorized

Integrals can be solved iteratively.



- can be further transformed into p-integrals with fewer loops [Liu and Ma, arXiv: 2201.11637]

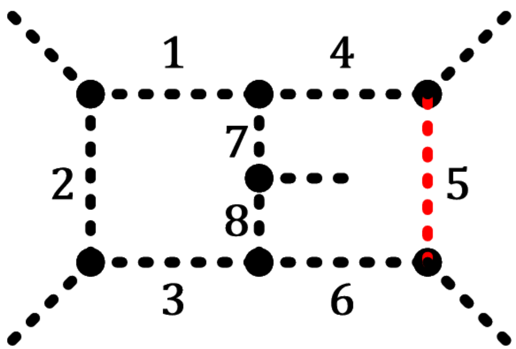
➢ Two-loop five-point massless double-pentagon



$\mathcal{D}_1 = \ell_1^2$, $\mathcal{D}_2 = (\ell_1 - p_1)^2$, $\mathcal{D}_3 = (\ell_1 - p_1 - p_2)^2$,
$\mathcal{D}_4 = \ell_2^2$, $\mathcal{D}_5 = (\ell_2 + p_5)^2$, $\mathcal{D}_6 = (\ell_2 + p_4 + p_5)^2$,
$\mathcal{D}_7 = (\ell_1 - \ell_2)^2$, $\mathcal{D}_8 = (\ell_1 - \ell_2 + p_3)^2$, $\mathcal{D}_9 = (\ell_1 + p_5)^2$,
$\mathcal{D}_{10} = (\ell_2 - p_1)^2$, $\mathcal{D}_{11} = (\ell_2 - p_1 - p_2)^2$,

$\vec{s} \equiv \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, $\qquad s_{ij} = (p_i + p_j)^2$

- introduce η to D_5

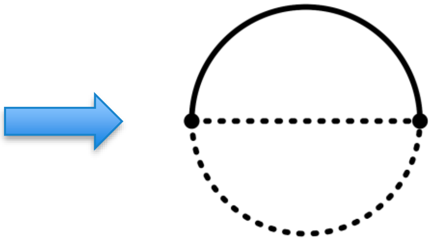


$$\frac{\mathcal{D}_9^{\nu_9}\mathcal{D}_{10}^{\nu_{10}}\mathcal{D}_{11}^{\nu_{11}}}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}(\mathcal{D}_5-\eta)^{\nu_5}\mathcal{D}_6^{\nu_6}\mathcal{D}_7^{\nu_7}\mathcal{D}_8^{\nu_8}}$$

Examples

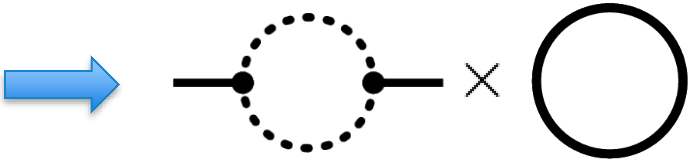
- all-large region (LLL): $\ell_1 \sim \sqrt{\eta},\ \ell_2 \sim \sqrt{\eta},\ \ell_1 - \ell_2 \sim \sqrt{\eta}$

$$\frac{1}{(\ell_1^2)^{\nu_1+\nu_2+\nu_3-\nu_9}(\ell_2^2)^{\nu_4+\nu_6-\nu_{10}-\nu_{11}}(\ell_2^2-\eta)^{\nu_5}(\ell_1-\ell_2)^{\nu_7+\nu_8}}$$



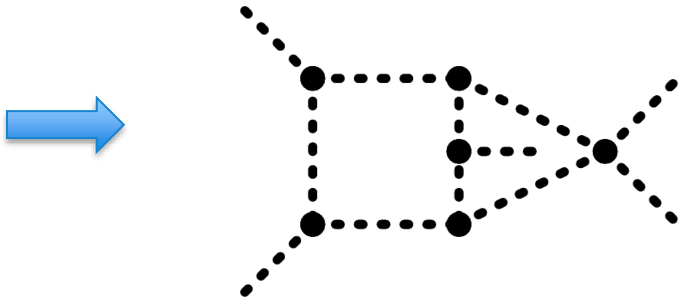
- mixed region (SLL): $\ell_1 \sim 1,\ \ell_2 \sim \sqrt{\eta},\ \ell_1 - \ell_2 \sim \sqrt{\eta}$

$$\frac{\mathcal{D}_9^{\nu_9}}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}(\ell_2^2)^{\nu_4+\nu_6+\nu_7+\nu_8-\nu_{10}-\nu_{11}}(\ell_2^2-\eta)^{\nu_5}}$$



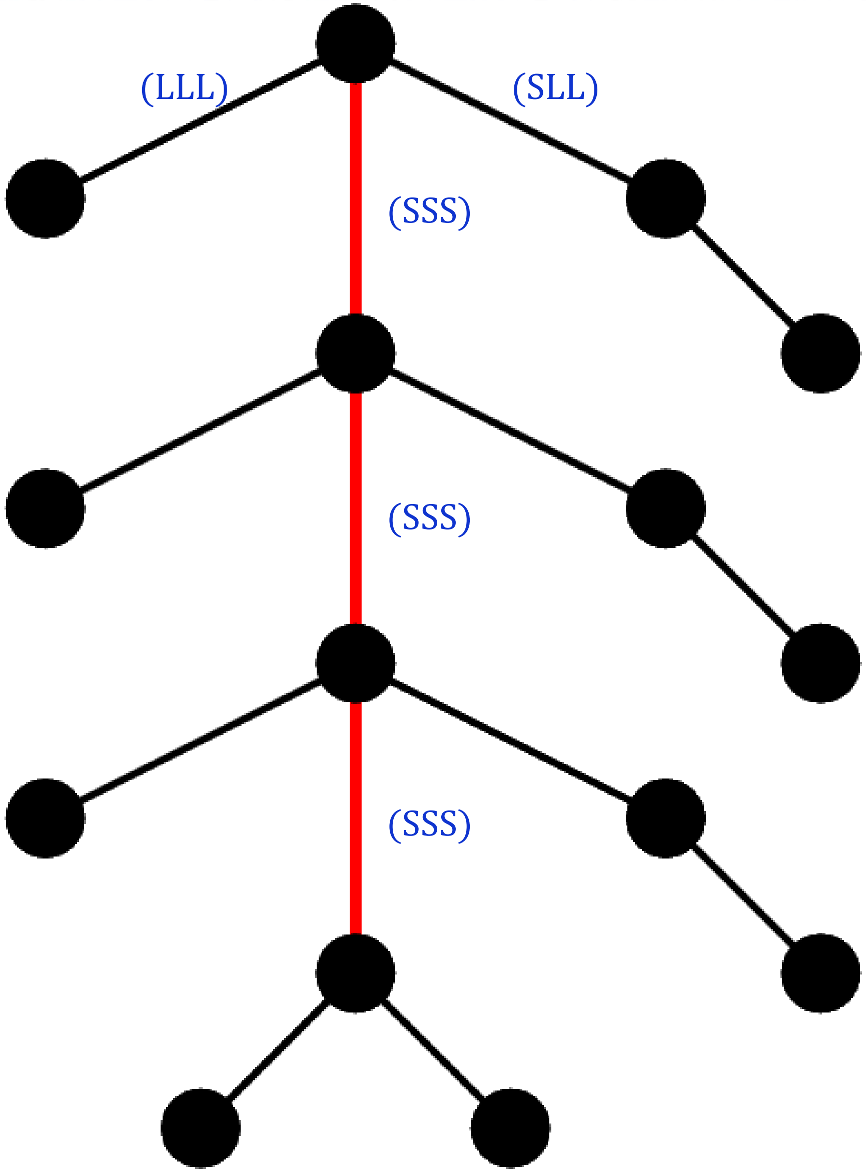
- all-small region (SSS): $\ell_1 \sim 1,\ \ell_2 \sim 1,\ \ell_1 - \ell_2 \sim 1$

$$\frac{\mathcal{D}_9^{\nu_9}\mathcal{D}_{10}^{\nu_{10}}\mathcal{D}_{11}^{\nu_{11}}}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_6^{\nu_6}\mathcal{D}_7^{\nu_7}\mathcal{D}_8^{\nu_8}}$$



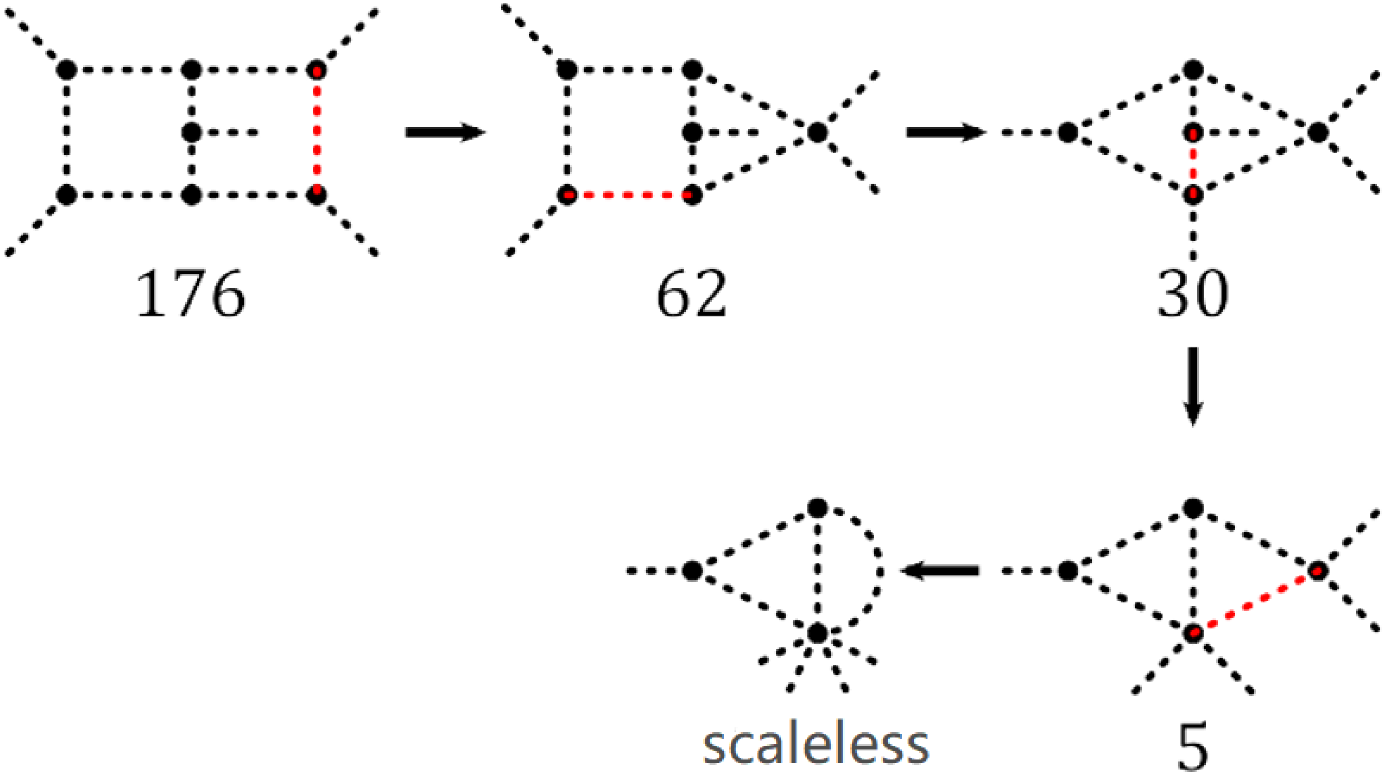
Examples

- repeat the above procedure and obtain a tree
- for each parent node
  - a system of differential equations
  - information of boundary integrals
  - a solver
- for each terminal node
  - results of master integrals
- master branch
  - all-small region iteration



Examples

- master branch of massless double-pentagon



- end up with scaleless integrals

Examples

- block-triangular systems [**XL** and Ma, Phys. Rev. D, 2019] [Guan, **XL** and Ma, Chin. Phys. C, 2020]
  - much smaller size
  - much better structure
  - 30~100 times faster on average for finite field computations

- differential equations at first step (176*176): block-triangular V.S. IBP (FiniteFlow+LiteRed) [Peraro, JHEP, 2019] [Lee, J. Phys. Conf. Ser., 2014]

	block-triangular	IBP
# relations	869	212847
t_{FFSample}	0.029s	3.93s

Examples

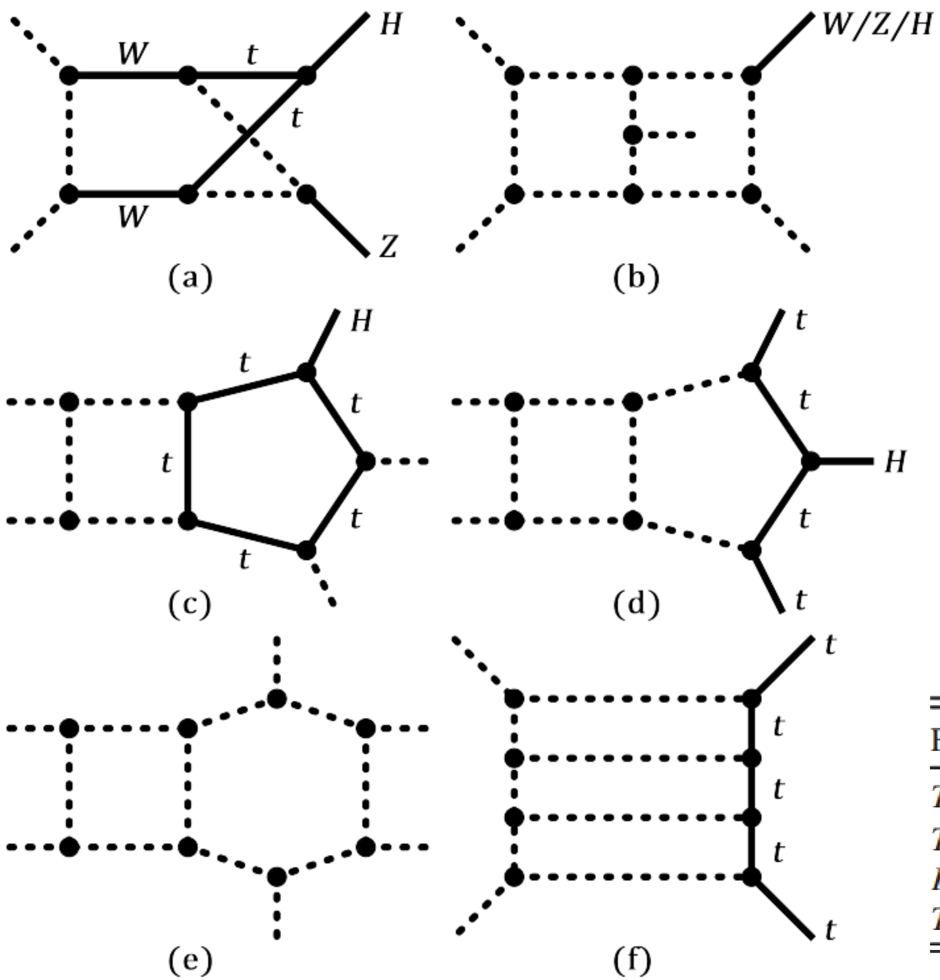
- $\vec{s}_0 = \{4, -\frac{113}{47}, \frac{281}{149}, \frac{349}{257}, -\frac{863}{541}\}$
- construction of the amflow-tree: 6 CPU hours
- 16-digit numerical solution: 7 CPU hours

$$
\begin{aligned}
&I(1,1,1,1,1,1,1,1,0,0,0)\\
&= -0.06943562517263776\epsilon^{-4} + (1.162256636711287 + 1.416359853446717i)\epsilon^{-3}\\
&\quad + (37.82474332116938 + 15.91912443581739i)\epsilon^{-2} + (86.2861798369034 + 166.8971535711277i)\epsilon^{-1}\\
&\quad - (4.143596557866 - 333.0996040071305i) - (531.834114822928 - 1583.724672502141i)\epsilon\\
&\quad - (2482.24025323261 - 2567.398291724192i)\epsilon^{2} - (8999.9036936711 - 19313.42643829926i)\epsilon^{3}\\
&\quad - (28906.95582696762 - 17366.82954322838i)\epsilon^{4}
\end{aligned}
$$

- checked against analytic solutions [Chicherin, Gehrmann, Henn et al, Phys. Rev. Lett., 2019] [Chicherin and Sotnikov, JHEP, 2020]

Examples

➢ Cutting-edge examples



- two-loop EW corrections to $H+Z$ production at e^+e^- colliders [Song, Freitas, JHEP, 2021]
- two-loop QCD corrections to $H/W/Z+2j, t\bar{t}H, 4j$ production at hadron colliders
- three-loop QCD correction to $t\bar{t}$ production at hadron colliders

Family	dp	a	b	c	d	e	f
T_{setup}	6	20	18	8	1	25	30
T_{solve}	7	11	15	6	3	15	42
P_1	95%	99%	96%	99%	98%	94%	93%
$T_{\vec{s}}$	2	916	64	1305	30	1801	63

Outline

I. Introduction

II. Auxiliary mass flow
  I. The method
  II. Iterative strategy

III. The package AMFlow
  I. **Basic usage**
  II. Applications to e+e- colliders phenomenology

IV. Summary and outlook

AMFlow

A

AMFlow

Project ID: 32748265

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52 Commits **1** Branch **2** Tags **1.5 MB** Project Storage **1** Release

A proof-of-concept implementation of auxiliary mass flow method.

- a Mathematica package for numerical computations of Feynman integrals using auxiliary mass flow
- available at https://gitlab.com/multiloop-pku/amflow
- current version: 1.1
- basic features
  - systematic: works for arbitrary integrals in principle
  - efficient: easy to reach high precision
  - user-friendly: press the button & wait for the results

AMFlow

- main package: AMFlow.m
  - provides functions to perform automatic computations
    - SolveIntegrals[targets, precision, epsorder]
- differential equation solver: diffeq_solver/DESolver.m
  - provides functions to solve differential equations numerically using series expansion
- interfaces to IBP reducers[Klappert, Lange, et al, Comput.Phys.Commun., 2021][Smirnov and Chuharev, Comput.Phys.Commun., 2020][Peraro, JHEP, 2019][Lee, J. Phys. Conf. Ser., 2014]
  - FiniteFlow+LiteRed: ibp_interface/FiniteFlow+LiteRed/interface.m & sup.m
  - Kira: ibp_interface/Kira/interface.m
  - Fire+LiteRed: ibp_interface/Fire+LiteRed/interface.m
  - BlockTriangular: in preparation
- examples/..

AMFlow

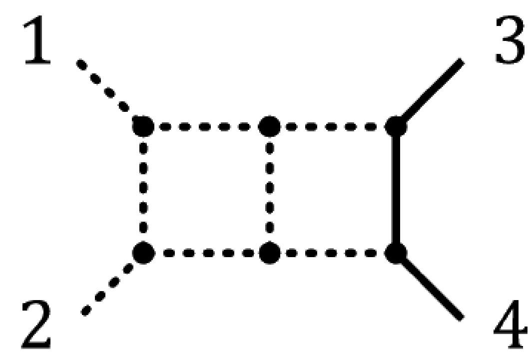
- examples/automatic_vs_manual

```
(*load the package*)
current = If[$FrontEnd===Null,$InputFileName,NotebookFileName[]]//DirectoryName;
Get[FileNameJoin[{current, "..", "..", "AMFlow.m"}]];

(*set ibp reducer, could be "FiniteFlow+LiteRed", "Kira" or "Fire+LiteRed"*)
SetReductionOptions["IBPReducer" -> "Kira"];

(*configuration of the integral family*)
AMFlowInfo["Family"] = tt;
AMFlowInfo["Loop"] = {l1, l2};
AMFlowInfo["Leg"] = {p1, p2, p3, p4};
AMFlowInfo["Conservation"] = {p4 -> -p1-p2-p3};
AMFlowInfo["Replacement"] = {p1^2 -> 0, p2^2 -> 0, p3^2 -> msq, p4^2 -> msq, (p1+p2)^2 -> s, (p1+p3)^2 -> t};
AMFlowInfo["Propagator"] = {l1^2, (l1+p1)^2, (l1+p1+p2)^2, l2^2, -msq+(l2+p3)^2, (l2+p3+p4)^2, (l1+l2)^2, (l1-p3)^2, (l2+p1)^2};
AMFlowInfo["Numeric"] = {s -> 30, t -> -10/3, msq -> 1};
AMFlowInfo["NThread"] = 4;

(*SolveIntegrals: computes given integrals with given precision goal up to given eps order*)
(*returned is a list of replacement rules like {j1 → v1, j2 → v2, ...}, where j1, j2, ... are integrals and v1, v2, ... are their results*)
target = {j[tt,1,1,1,1,1,1,1,-3,0], j[tt,1,1,1,1,1,1,1,-2,-1], j[tt,1,1,1,1,1,1,1,-1,-2], j[tt,1,1,1,1,1,1,1,0,-3]};
precision = 20;
epsorder = 4;
auto = SolveIntegrals[target, precision, epsorder];
```



- SetReductionOptions[“IBPReducer” -> “reducer”];
- AMFlowInfo[keyword] = object;
- SolveIntegrals[targets, precision, epsorder];

Outline

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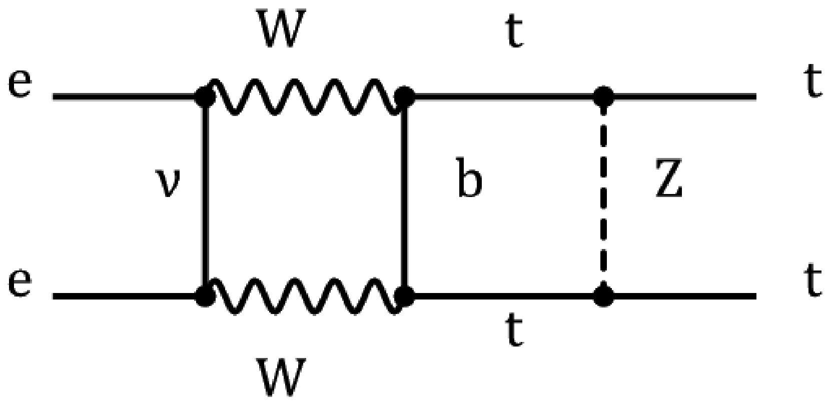
Applications

➢ $e^+e^- \to t\bar{t}$ @NNLO electroweak

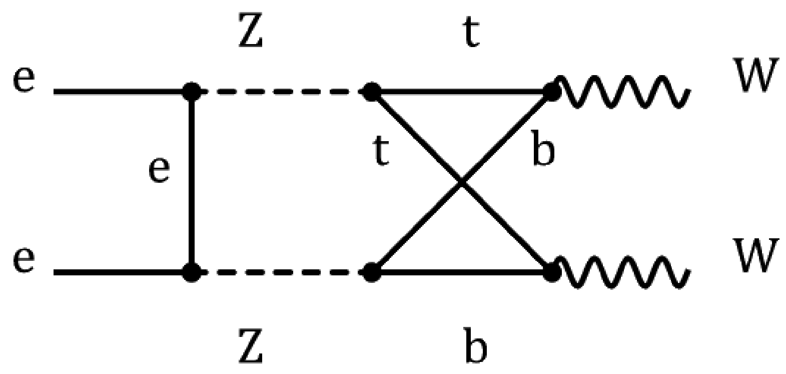


AMFlowInfo["Family"] = eett;
AMFlowInfo["Loop"] = {l1, l2};
AMFlowInfo["Leg"] = {p1, p2, p3, p4};
AMFlowInfo["Conservation"] = {p4 -> -p1-p2-p3};
AMFlowInfo["Replacement"] = {p1^2 -> 0, p2^2 -> 0, p3^2 -> mtsq, p4^2 -> mtsq, (p1+p2)^2 -> s, (p1+p3)^2 -> t};
AMFlowInfo["Propagator"] = {l1^2-mWsq, (l1+p1)^2, (l1+p1+p2)^2-mWsq, l2^2-mtsq, (l2+p3)^2-mZsq, (l2+p3+p4)^2-mtsq, (l1+l2)^2, (l1+p3)^2, (l2+p1)^2};
AMFlowInfo["Numeric"] = {s -> 10, t -> -11/3, mtsq -> 1, mWsq -> 35/162, mZsq -> 5/18};
AMFlowInfo["NThread"] = 20;

- master integrals: 84 → 84
- 20-digit results in physical region obtained in 0.4h (20 threads)

Applications

➢ $e^+e^- \rightarrow W^+W^-$@NNLO electroweak

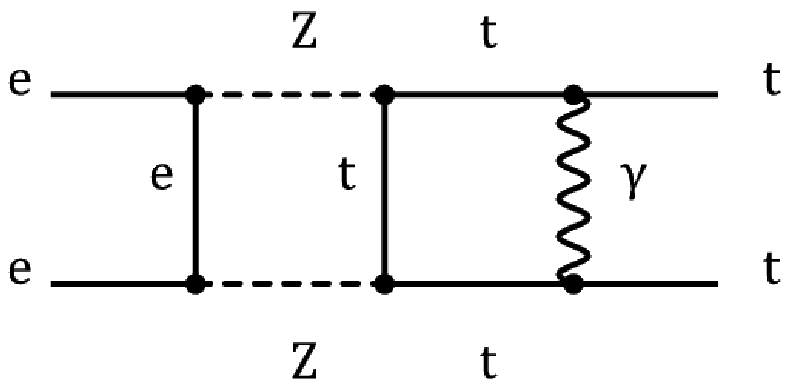



```
AMFlowInfo["Family"] = eeww;
AMFlowInfo["Loop"] = {l1, l2};
AMFlowInfo["Leg"] = {p1, p2, p3, p4};
AMFlowInfo["Conservation"] = {p4 -> -p1-p2-p3};
AMFlowInfo["Replacement"] = {p1^2 -> 0, p2^2 -> 0, p3^2 -> mWsq, p4^2 -> mWsq, (p1+p2)^2 -> s, (p1+p3)^2 -> t};
AMFlowInfo["Propagator"] = {l1^2-mZsq, (l1+p1)^2, (l1+p1+p2)^2-mZsq, l2^2-mtsq, (l2+p3)^2, (l1+l2-p4)^2, (l1+l2)^2-mtsq,
(l1+p3)^2, (l2+p1)^2};
AMFlowInfo["Numeric"] = {s -> 10, t -> -11/3, mtsq -> 162/35, mWsq -> 1, mZsq -> 9/7};
AMFlowInfo["NThread"] = 20;
```

- master integrals: 166 → 166
- 20-digit results in physical region obtained in 2.7h (20 threads)

Applications

➢ $e^+e^- \to t\bar{t}$ with complex Z mass




```
AMFlowInfo["Family"] = eettc;
AMFlowInfo["Loop"] = {l1, l2};
AMFlowInfo["Leg"] = {p1, p2, p3, p4};
AMFlowInfo["Conservation"] = {p4 -> -p1-p2-p3};
AMFlowInfo["Replacement"] = {p1^2 -> 0, p2^2 -> 0, p3^2 -> mtsq, p4^2 -> mtsq, (p1+p2)^2 -> s, (p1+p3)^2 -> t};
AMFlowInfo["Propagator"] = {l1^2-mZsq, (l1+p1)^2, (l1+p1+p2)^2-mZsq, l2^2-mtsq, (l2+p3)^2, (l2+p3+p4)^2-mtsq,
(l1+l2)^2-mtsq, (l1+p3)^2, (l2+p1)^2};
AMFlowInfo["Numeric"] = {s -> 10, t -> -11/3, mtsq -> 1, mZsq -> 5/18-I*55/7236};
AMFlowInfo["NThread"] = 20;
```

- master integrals: 69 → 69
- 20-digit results in physical region obtained in 0.3h (20 threads)

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Summary and Outlook

➢ What we have

- Auxiliary mass flow method fully automized the computation of boundary conditions for differential equations.
- AMFlow is the first public tool which can compute arbitrary Feynman loop integrals, at arbitrary kinematic point, to arbitrary precision.

➢ What we need

- Powerful reduction techniques are urgently needed to construct differential equations, both for η and for dynamical variables.
- A guide for choosing better master integrals in general cases is needed, which may strongly simplify the differential equations.