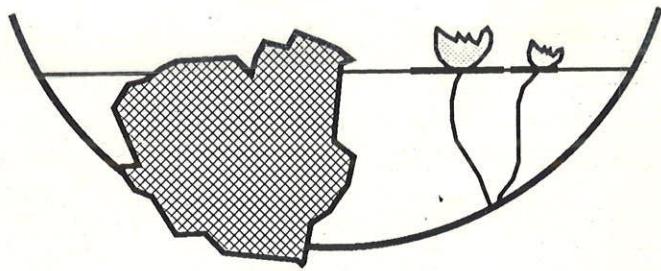


# PERCOLATION OR THE GAME OF STRUCTURE AND CHANCE

Ph. Blanchard

BiBoS: Bielefeld-Bonn Stochastics



Is the centre of the stone wetted  
by the liquid?

*The true logic  
of this world  
is the calculus  
of probability*

J.C. Maxwell

## Overview

- Happy Numbers
  - What are percolation phenomena?
  - Why to care?
    - Prototype Threshold Phenomena
    - Prototype Fractal
    - RCM: Connection with thermal critical phenomena
    - Thanks to conformal invariance of 2-dim critical percolation
- Probability represented now among Fields Medals: W. Werner (2006)  
S. Smirnov (2010)

- Examples and recent applications
  - Branching Process: Galton Watson
  - Classical Random Graphs: Erdős-Renyi. (1960)
  - Inhomogeneous Random Graphs : Cameo Principle (2004), BJR graphs (2007)
  - Generalized Epidemic Processes (2007)
  - Nemo-Project & Knowledge Transfer (2006-2009)
  - Corruption
  - Passive supporters of terrorism and phase transitions (2010)
  - On the Kertesz line: thermodynamic versus geometric criticality (2008)

## Interdisziplinäre Vorlesung

# PERKOLATION

Ph. Blanchard und H. Satz

### Thematik

Die Perkolationstheorie untersucht, wie aus einzelnen geometrischen Objekten größere zusammenhängende Gebilde entstehen; sie ist anwendbar auf eine Vielzahl von Fragen in den verschiedensten naturwissenschaftlichen Bereichen. Einige Beispiele von 'perkolatorischen' Problemen illustrieren dies:

- Wieviel Wasser muß man in ein Kaffeefilter gießen, bevor Kaffee ausfließt?
- Wie entwickelt sich die Struktur von Galaxien?
- Wie koppeln sich Einzelmoleküle zu Polymeren und zu amorpher Materie?
- Wie breiten sich Virus-Epidemien aus?
- Welche Baumdichte in Forsten ist optimal für die Bekämpfung von Waldbränden?
- Bei welcher Dichte verwandelt sich Kernmaterie in Quarkmaterie?

Die einfachste Illustration bietet wahrscheinlich das japanische Go-Spiel, in dem Perkolation Sieg bedeutet.

### Inhalt der Vorlesung

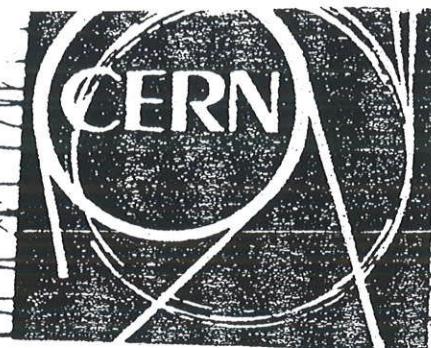
Es wird zunächst die Phänomenologie der Perkolation dargestellt und zur Definition der Grundbegriffe benutzt; danach werden einige Anwendungsbeispiele vorgeführt. Im folgenden wird die Perkolationstheorie für einfache Systeme entwickelt und die Struktur von Clustern untersucht; es werden Zusammenhänge zur Selbstähnlichkeit und fraktalen Systemen aufgedeckt, und es wird die Verbindung zwischen thermischem kritischen Verhalten (Phasenübergängen) und Perkolation behandelt.

### Voraussetzungen

Der Kurs ist interdisziplinär geplant, und daher sollten Grundkenntnisse von Mathematik und Naturwissenschaften (Physik, Chemie, Biologie) zum Verständnis genügen.

### Lehrbuch

D. Stauffer und A. Aharony: *Einführung in die Perkolationstheorie*.



WEDNESDAY 26 FEBRUARY 1992 at 2 p.m.

TH Conference Room

### THEORETICAL SEMINAR

#### PERCOLATION THEORY: SOME BASIC TECHNIQUES AND APPLICATIONS IN PHYSICS, EPIDEMIOLOGY AND IMMUNOLOGY

by Philippe BLANCHARD (Universität Bielefeld)

##### Abstract

The study of discrete random structures created by the removal at random of a fraction of the vertices or edges of a graph is the province of percolation theory. Percolation is one of the simplest systems which exhibits a "phase transition". Percolation theory is indeed considered primarily with the existence or non-existence of infinite connected clusters. Drinkers of Pernod are familiar with this type of phenomenon: in the process of adding the water drop by drop in a glass of Pernod, there arrives an instant at which the mixture becomes opaque.

Some basic techniques will be discussed: Increasing events, FKG-Inequality, Berg-Kesten Inequality, Russo's Formula, ..... As a model for a disordered medium, percolation theory can be used in many situations.

# Percolation Phenomena

## Basic Techniques and Applications

May 7 - 12, 2001

Center for Interdisciplinary Research (ZiF)

University of Bielefeld, Germany

Organizers: Ph. Blanchard, M. Röckner and H. Satz

**Invited Speakers include:**

- M. Aizenmann\* (Princeton)  
N. Armesto (Cordoba)  
K. Binder (Mainz)  
A. Bunde (Gießen)  
A. Coniglio (Napoli)  
J. Dias de Deus (Lisboa)  
A. Dress (Bielefeld)  
S. Fortunato (Bielefeld)  
D. Gandolfo (CPT Marseille)  
F. Götze (Bielefeld)  
P. Grassberger (NIC Jülich)  
R. van der Hofstad (TU Delft)  
D. Johnston (Heriot-Watt Univ. Edinburgh)  
J. Kertesz (TU Budapest)  
R. Kotecky\* (Praha)  
L. Laanait (ENS Rabat)  
C. Pajares (Santiago de Compostela)  
J. Seixas (Lisboa)  
M. Sirugue-Collin (CPT Marseille)  
G. Slade (UBC Vancouver)  
D. Stauffer (Köln)  
H. Wagner (LMU München)

\* to be confirmed



Thomas Kaminsky, 1997, Öl auf Leinwand, 180 x 170 cm

**Conference Secretary:**

M. Hoffmann  
ZiF, Wellenberg 1  
D-33615 Bielefeld  
Germany

Marina.Hoffmann@uni-bielefeld.de  
<http://www.uni-bielefeld.de/ZIF/>

Musée national suisse  
Château de Prangins



(1)

## HAPPY NUMBERS

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

### Eratosthenes sieve

$$\mathbb{N} \rightarrow \mathbb{P} = \{1, 3, 5, 7, \dots\}$$

prime numbers

### Happy sieve

1) Start with  $\mathbb{N}$

2) Eliminate 2 and each second number

1, 3, 5, 7,

3) Eliminate each third number

1, 3, 7, 13, ...

4) Eliminate each seventh number...

### Happy sieve

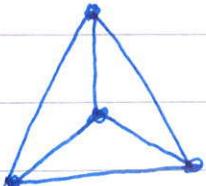
$$\mathbb{N} \rightarrow \mathbb{H} = \{h_n\}_{n \in \mathbb{N}}$$

$$h_{19} = 75 !$$

As said in China

"Ten thousand years of life for Helmut"

PERCOLATION = Random sieve on a graph



$$K_4 \rightarrow$$

Removal at random  
of bonds or sites  
of a graph  $G = (V, E)$

Like sudoku "Axiomatic" of percolation theory  
has few rules, all easy to remember.  
The art is entirely in the implementation.

## Main Questions

Is there macroscopic connectivity?

$x, y \in V \Rightarrow \exists$  open path  $\sigma_{x \rightarrow y}$

$C_0 = \{x \in V \mid \exists \text{ path } \sigma_{0 \rightarrow x}\}$

Cluster of 0

Is  $|C_0| = +\infty$

"Giant component" containing 0

## Main Result

Phase transition in the topological structure of random systems (lattices, graphs, population, ...)

- Phase-transition  $\Leftrightarrow$  0-1 Law
- Epidemiology  $\sim$  threshold
- $R_0$  reproduction number

# real life percolation

Philippe  
Blanchard, Tyll  
Krueger

Overview

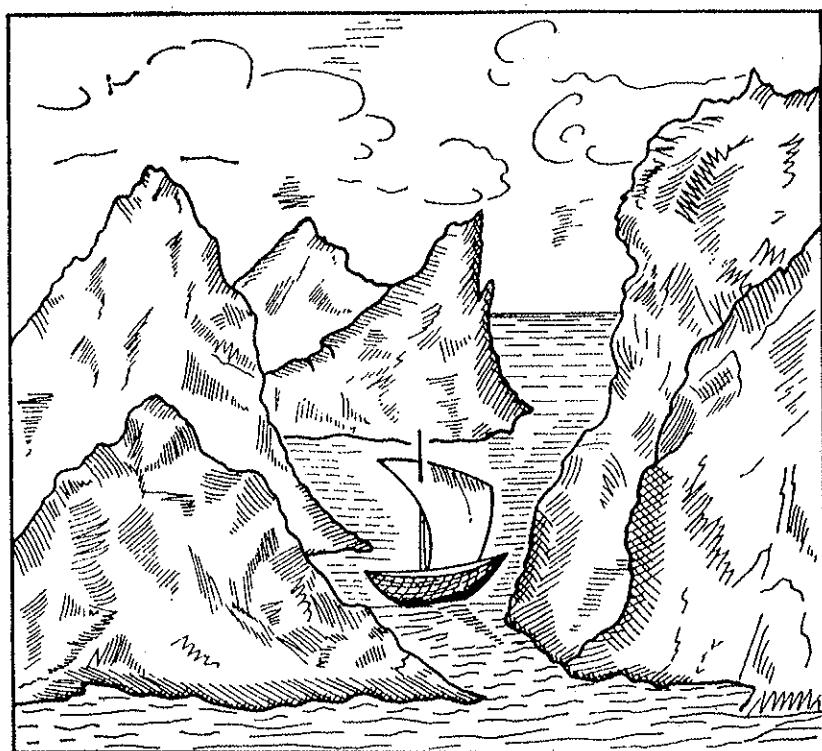
Percolation

Bootstrap  
percolation

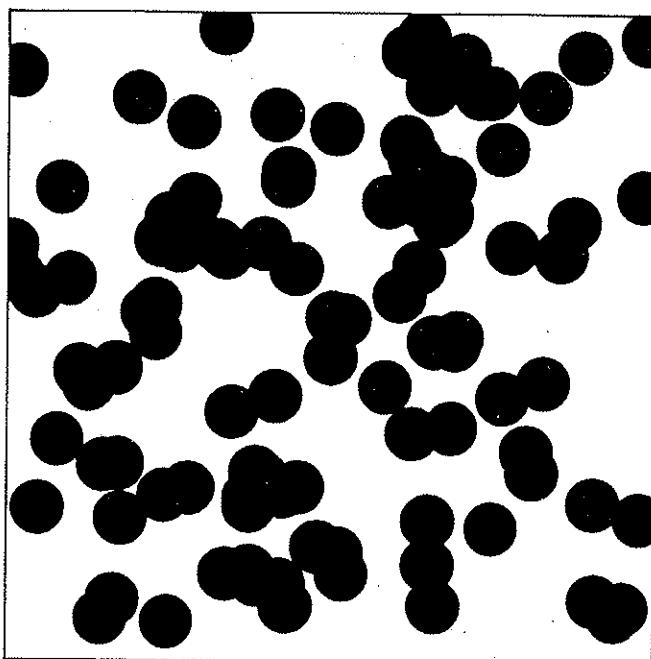
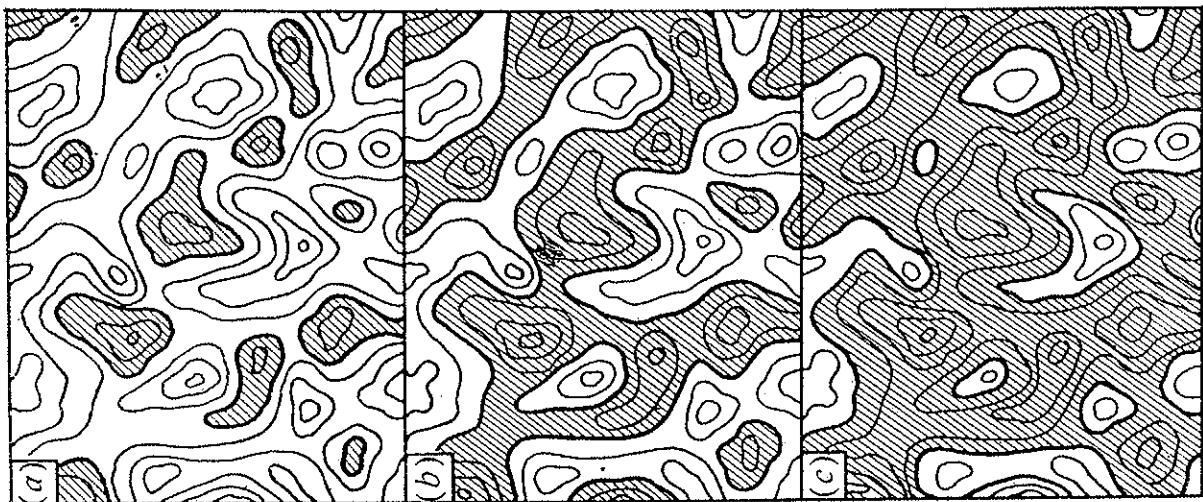
inhomogeneous  
random graph  
models

applications





Voyage during the Flood.



**Fig. 2.2.** Continuum percolation: Swiss cheese model

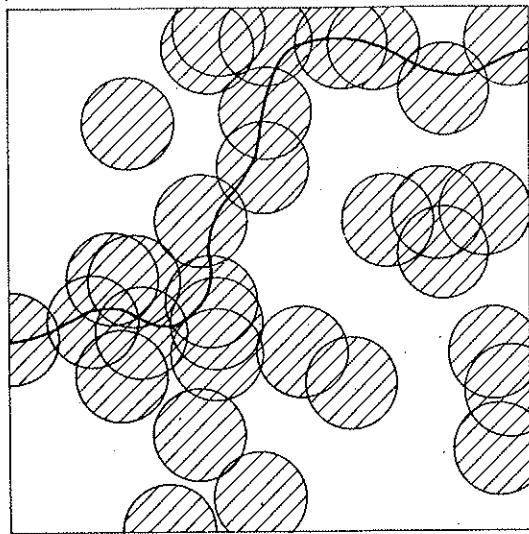


Figure 10.7. The thick line indicates a possible dry crossing of the lily pond. The centres of the circles are the points of a Poisson process. When the circles are sufficiently large, there exists a crossing of the box which is contained in the union of the circles.

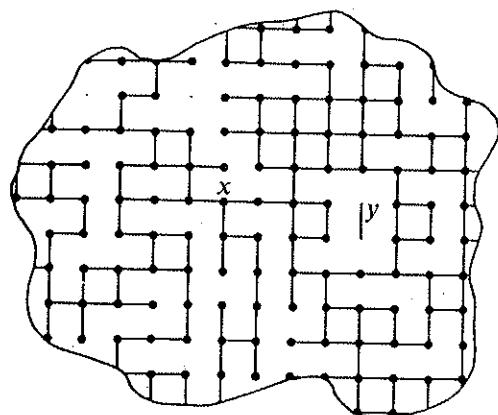
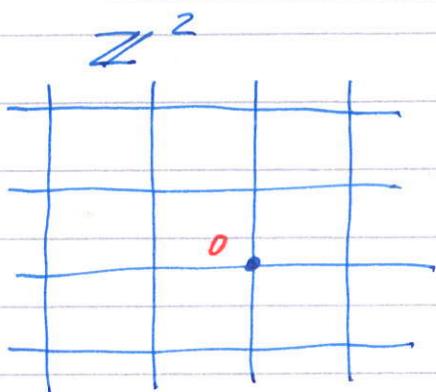


Figure 1.1. A sketch of the structure of a two-dimensional porous stone. The lines indicate the open edges; closed edges have been omitted. On immersion of the stone in water, vertex  $x$  will be wetted by the invasion of water, but vertex  $y$  will remain dry.

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Percolation : coherent theory of random spatial processes  $\sim$  "disordered medium"

"Standard" model



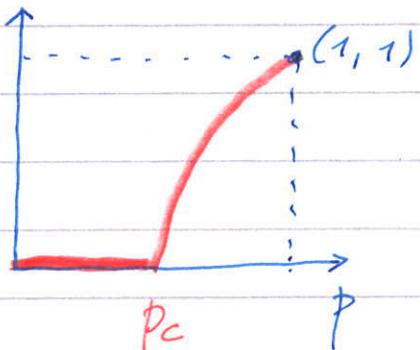
"edge  $\sim$  channel"  
"gas, flow, virus..."  
could flow

Edge  $\rightarrow$  open with probability  $p$   
 $0 \leq p \leq 1$   $\hookrightarrow$  closed with probability  $1-p$

$$\theta(p) = \text{Prob}(o \leftrightarrow \infty) = \text{Prob}(|C_0| = \infty)$$

$$p_c = \sup \{ p : \theta(p) = 0 \}$$

$$C_o = \{ x \mid \exists \text{ open path } \gamma_{o \rightarrow x} \}$$



Theorem (Kesten 1980)

$$p_c^b(Z^2) = \frac{1}{2}$$

What are transition phenomena?

### Why to care?

- Prototype threshold phenomena

Phase transitions, critical phenomena, critical exponents, scaling, universality, RG,...

### Overall picture

- infinite system
- smoothly varying parameter  $T, P$
- + macroscopic quantity "order parameter"  
 $\theta(T), \theta(p) \dots$
- $\theta$  may have singularities ( $T_c, p_c, \dots$ )  
 $\Rightarrow$  points of "phase transitions"

### Two contrasting approaches

- Physical approach (Ising model...)  
 Gibbs states  $\Rightarrow$  Singularities of the free energy
- Probabilistic approach

Asymptotic properties of appropriate random variables

- Branching process
- Random graphs
- Percolation (discrete, continuous, ...)
- RCM = Random Cluster Model
  - ~ generalisation of Ising & Potts models and percolation

## Behaviour near $p_c$

Bond percolation on  $\mathbb{Z}^d$

$$\theta(p) \sim (p - p_c)^\beta$$

$$\beta(2) = \frac{5}{36}$$

$$\beta(d) = 1 \quad d \geq 6$$

Renormalisation  $\Leftrightarrow$  looking to  $\mathbb{Z}^d$  from the wrong end of a telescope

Power increasing  $\Rightarrow$  supercritical process  
driven towards the extreme case  $p = 1$   
and a subcritical towards  $p = 0$

Critical case  $p = p_c \sim$  fixed point  
of the transformation.

Non rigorous but appealing theory!

(6)

## • Prototype fractal

$$\Theta(p) \sim (p - p_c)^{\beta}$$

$p < p_c$  Correlation length  $\xi < +\infty$

$$\xi \sim |p - p_c|^{-\nu}$$

$$\Theta(p) \sim \xi^{-\frac{\beta}{\nu}} \quad \text{density of } C_0$$

Hausdorff

$$D_H(C_0) = \begin{cases} d - \frac{\beta}{\nu} & d \leq 5 \\ 4 & d \geq 6 \end{cases}$$

Alles ist sich gleich, ein jeder Teil  
repräsentiert das Ganze

Lichtenberg

RCM : Connection with thermal critical phenomena

Fortuyn Kasteleyn 1969

Couiglio Klein 1980

Edwards Sokal 1988

Potts model  $\sim$  Ising with  $q$  states

Graph  $G = (V, E)$

$F \subseteq E$

$$\text{Probability } \phi_{p,q} = \frac{1}{Z} p^{|F|} (1-p)^{|E \setminus F|} q^{k(F)}$$

2 parameters  $\rightarrow q = 1, 2, 3, \dots$   
 $\rightarrow 0 \leq p \leq 1$

$k(F) = \#(\text{connected components of } (V, F))$

•  $q = 1$  bond percolation on  $G$

•  $q = 2, 3, \dots$  related to Potts model on  $G$

$$\theta(p, q) = \phi_{p,q} \quad (0 \in \text{giant component})$$

$$\exists p_c(q) \quad p_c(1) = p_c^b(G)$$

$p_c(q)$  expressed in terms of  $T_c(q)$

$$p_c(q) = 1 - e^{-\frac{J}{T_c(q)}}$$

Geometric representation of the Potts model

$$\pi(\sigma_i = \sigma_j) - \frac{1}{q} = \left(1 - \frac{1}{q}\right) \phi(i \leftrightarrow j)$$

2 point fn  
Potts

connectivity of  
RCM

$w_{E,S}(x, z)$   
 $\begin{array}{c} \uparrow \\ \text{spin} \end{array}$     $\begin{array}{c} \uparrow \\ \text{edges} \end{array}$

Edwards - Sokal

The set of geometric phase transitions is more general than the set of thermal phase transitions

## Conformal invariance and percolation on $\mathbb{Z}^2$

- Stimulating families of conjectures in geometric probability theory

Langlands, Polyakov, Cardy, Lawler  
Schramm ...

- ENS ~ 1968

Nicolas Bourbaki :

Probability useful for gambling but  
not proper work for a "serious"  
mathematician

| Probability represented now  
among Fields medals

ICM 2006 Madrid Wendelin WERNER  
ENS Paris XI

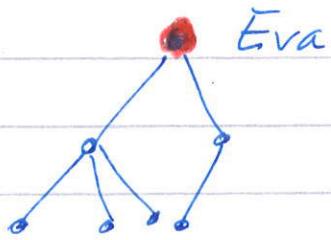
ICM 2010 Hyderabad Stas SMIRNOV  
Genève

(3)

## Examples and recent applications

### 1) Branching process : Galton - Watson

Francis Galton (1822-1911) Darwin's cousin



$Z$  children

Poisson process with mean  $c$

$$\text{Prob} [Z = k] = e^{-c} \frac{c^k}{k!}$$

Theorem 3 regimes for the total population  $G$

- Subcritical  $c < 1$   $G$  finite

$$E(G) = \sum_{k=0}^{\infty} c^k = \frac{1}{1-c}$$

- Critical  $c = 1$   $G$  finite

with probability 10 but  $E(G) = +\infty$

- Supercritical

$$\delta(c) = \text{Prob} [G = +\infty] = \alpha(c) > 0$$

$$\left[ e^{-c\alpha(c)} = 1 - \delta(c) \right]$$

2) Erdős-Renyi classical random graphs (1960)

Birth of the probabilistic method!

 $G(n, p)$  $n$  vertices

$$\text{Prob}(i \sim j) = \frac{c}{n} \quad c > 0$$

Bernoulli process  $\sim$  independent edge probabilities

$$\mathbb{E}(\#\text{edges}) = \frac{n(n-1)}{2} \cdot \frac{c}{n} = \frac{n-1}{2} c$$

 $\approx \mathbb{E}(\text{vertices})$  SPARSE

tree like

small clustering

$$\mathbb{E}(\#\text{triangles}) = \frac{n(n-1)(n-2)}{6} \cdot \frac{c^3}{n^3}$$

$$\text{Prob}[d(x) = k] = \text{Bin}\left(n-1, \frac{c}{n}\right) \approx \text{Poisson}_c$$

 $C_1$  the largest component  $\sim$  "giant" $C_2$  the second largestTheorem (Erdős-Renyi) 3 Refines- Subcritical  $c < 1$ 

$$|C_1| \sim \ln n$$

- Critical  $c = 1$ 

$$|C_1| \sim n^{2/3}$$

- Supercritical  $c > 1$ 

$$|C_1| \sim \delta(c)n$$

$$\boxed{e^{-c\delta(c)}} = 1 - \delta(c)$$

Remark 1 $C_1$  is unique. Not Jupiter and Saturn but more Jupiter and Ceres

$$\begin{cases} |C_1| > \delta n \\ |C_2| > \delta n \end{cases} \quad \text{Prob}(\textcircled{1} \textcircled{2}) \sim > 2\delta^2$$

High instability

Remark 2Connection between  $G(n, p)$  (tree like, degree Poisson distributed) and Galton-Watson

## Inhomogeneous Random Graphs

Motivation: Extend the Erdős - Renyi graphs to reproduce (partially) real-world networks

Barabasi - Albert preferential attachment (2000)

$\Rightarrow$  SCALE-FREE GRAPHS

$$\text{Prob} [d(\cdot) = k] \sim \frac{1}{k^\delta}$$

Power law

### SMALL WORLD PROPERTIES

$\text{diam } G \sim \ln n$

High clustering  $\sim$  many triangles

degree - degree correlations

Cameo - Graph B-Krämer J. Stat. Phys (2004)

Bollobas - Janson - Riordan BJR Random Structures

$w(i)$   $w(j)$  Random variables  
 $i$   $j$   $w(i)$  "properties" of  $i$   
 $\omega$  with values in  $S$   $\mu$ -distributed

$\mathcal{H}$  symmetric  $\mathcal{H}(\omega, \omega') = \mathcal{H}(\omega', \omega)$

$$\text{Prob}(i \sim j) = \frac{\mathcal{H}(w_i, w_j)}{n}$$

$\mathcal{H} = \text{const} \Rightarrow$  Erdős - Renyi:

$$\mathbb{E}(\# \text{edges}) = \frac{n}{2} \int \mathcal{H}(\omega, \omega') d\mu(\omega) d\mu(\omega')$$

$\Rightarrow$  Graphs are sparse

Cameo - Graphs

$$\omega = (w_1, w_2)$$

$$\begin{aligned} w_1 &\geq 0 \\ w_2 &\in [0, 1] \end{aligned}$$

$$S = \mathbb{R}^+ \times [0, 1]$$

$$w_2 = 0$$

Erdős - Renyi:

property Affinity for the property

## Local structure

- tree like

- Multitype Poisson process with intensity  
 $\lambda(\omega, z) dz$

-  $E(\# \text{ children of vertex } \text{ of type } \omega \text{ in } A) = \lambda_A(\omega)$

$$\lambda_A(\omega) = \int_A \lambda(\omega, z) dz$$

↓ Mean degree of  $\omega$ -type vertex

## The giant component $G_1$

$g(\omega) = \text{Prob} [\text{Poisson process starting with type } \omega \text{ survives}]$

$g(\omega)$  is solution of

$$f = 1 - e^{-Tf}$$

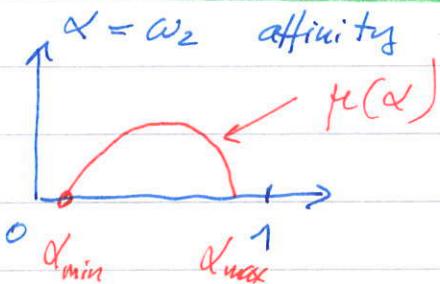
$$Tf(\omega) = \int_S \lambda(\omega, z) f(z) d\mu(z)$$

$$|C_1| \sim n \int g(\omega) d\mu(\omega)$$

Threshold

$$\|T\| = 1$$

## Extremists dictate the game to Cameo-graphs



$$\text{Prob}[d(\cdot) = k] \sim \frac{1}{k^\gamma}$$

$$\gamma = 1 + \frac{1}{\alpha_{\max}}$$

BKF

F = Santo Fortunato

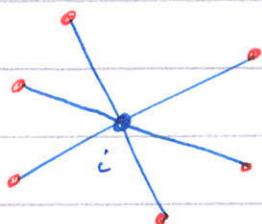
## Anomalous Epidemic Process on Scale Free graphs

$$\text{Prob} [ d(\cdot) ] \sim \frac{1}{k^\gamma}$$

$R_0$  reproduction number

$$R_0 = \mathbb{E} (\# \text{secondary infections by infected})$$

on graph



$$R_0 \sim \mathbb{E}(d^2) = \mu$$

$$2 < \gamma \leq 3 \Rightarrow \mu = +\infty$$

$$|G^{\text{inf}}| \sim e^{-\frac{1}{\mu} n}$$

$$R_0 = \beta \cdot \mu \cdot T$$

$\uparrow$  inf. probability       $\uparrow$  infections per node

$$\text{threshold } R_0 = 1$$

## Generalized Epidemic Process

BKK  $\geq 2006$

Social "contagion" ~ opinion, knowledge, innovation, corruption, terrorism...

Population  $\Leftrightarrow$  Graph  $G = (V, E)$

Infection  $\begin{cases} \xrightarrow{\quad} \text{local on the graph} \\ \xrightarrow{\quad} \text{with a threshold} \end{cases}$

$\begin{cases} \xrightarrow{\quad} \text{global (mean field)} \end{cases}$

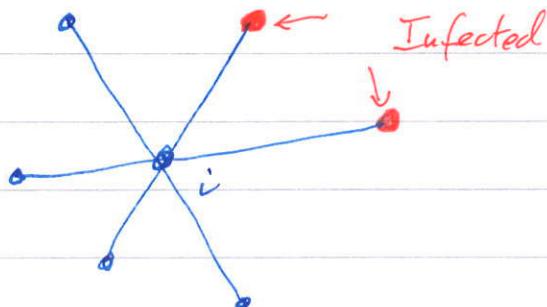
State space  $\Sigma = \{0 = \text{non infected}, 1 = \text{infected}, 2 = \dots\}$

$i \in V$  State  $X_i : V \rightarrow \Sigma$

$$\text{Prob} [X_i(t+1) = k' \mid X_i(t) = k]$$

$$= f_{\text{loc}}(X_{B_i(t)}) \oplus f_{\text{mean}}(X_G(t))$$

Threshold



$$|B_i(t)| = \chi(\text{infected neighbors}) < \Delta(i)$$

$\Rightarrow$  small infection rate  $\varepsilon$

$$|B_i(t)| = \chi(\text{infected neighbors}) \geq \Delta(i)$$

$\Rightarrow$  high infection rate  $\alpha$

### Properties

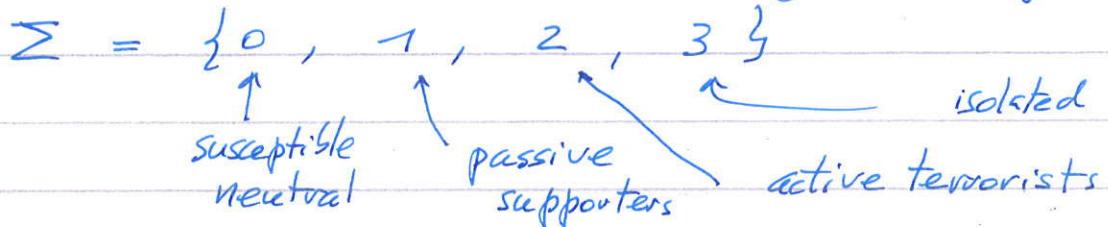
- $\varepsilon = 0 \quad \alpha = 1$  Bootstrap percolation
- Clustering accelerates GEP but slows down normal epidemics
- long transient dynamics
- degree-degree correlation important
- Time sharing communication index

## Passive Supporters of Terrorism and Phase transitions Only a model!

J. C. Camilo ... Common ecology quantifies  
human insurgency

Nature 426 / 17 December 2009

⇒ Coalition Military Fatalities in Afghanistan by Month



Graph  $G(n, p)$   $p = \frac{c}{n}$

$$c = 4$$

$$n = 200000$$

$$b_{t+1} = 1 - (1 - b_0) e^{-cbt} \sum_{k=0}^{\Delta-1} \frac{(cbt)^k}{k!}$$

$\downarrow$   
 $b_0^c$

Problem : Effect on "collateral damages" on the prevalence of passive supporters

- to capture/eliminate an active terrorist  $\Rightarrow m$  civilian victims

- relatives/friends of victims  $0 \rightarrow 1$

typically  $rm \sim 10 - 1000$

- Phase transitions in  $\#(\text{passive supporters})$

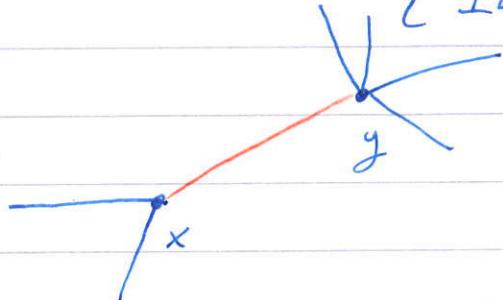
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FP1 to FP6  
 # (research projects) = 50590 # (organisations) = 49644

## EU - NEMO Project (2006 - 2009)

Network Models, Governance and R&D Project

with { ARC : Austrian Research Centers  
 Institute of Discrete Mathematics TU-Wien



Time sharing

$$t_{xy} = x \sim y$$

Communication index

$$c_{xy} = \min\left(\frac{1}{d(x)}, \frac{1}{d(y)}\right) = \frac{1}{\max(d(x), d(y))}$$

$$CI_x = \sum_{y \neq x} c_{xy}$$

$$CI = \frac{1}{m} \sum_x CI_x$$

} distributions  $\Rightarrow$  interesting characteristics of networks

For tree and some random graph spaces  $\Rightarrow$  analytical computations!

Combining inhomogeneous graph and communication index

Two different kernels  $\delta^*(\omega_x, \omega_y)$

- Multiplicative  $\sim \omega_x \cdot \omega_y$

- Additive  $\sim \omega_x + \omega_y$

Theorem

$$\|T_x\| \geq \|T_+\|$$

Multiplicative  $\sim \frac{1}{E(\omega)} \min(\omega_x, \omega_y)$

Additive  $\sim \frac{1 + \frac{\omega_y}{\omega_x}}{\frac{E(\omega)}{\omega_x} + 1} \quad \omega_y < \omega_x$

On the Kertesz Line : thermodynamic versus  
geometric criticality

H. Satz and BGLR

J. Phys. A. Math. Theor 41 (2008) 085001

Ising model + External field  $H$

No thermal phase transition

Percolation transition  $T_p(H)$

$T \leq T_p(H)$  percolation

$T > T_p(H)$  no percolation

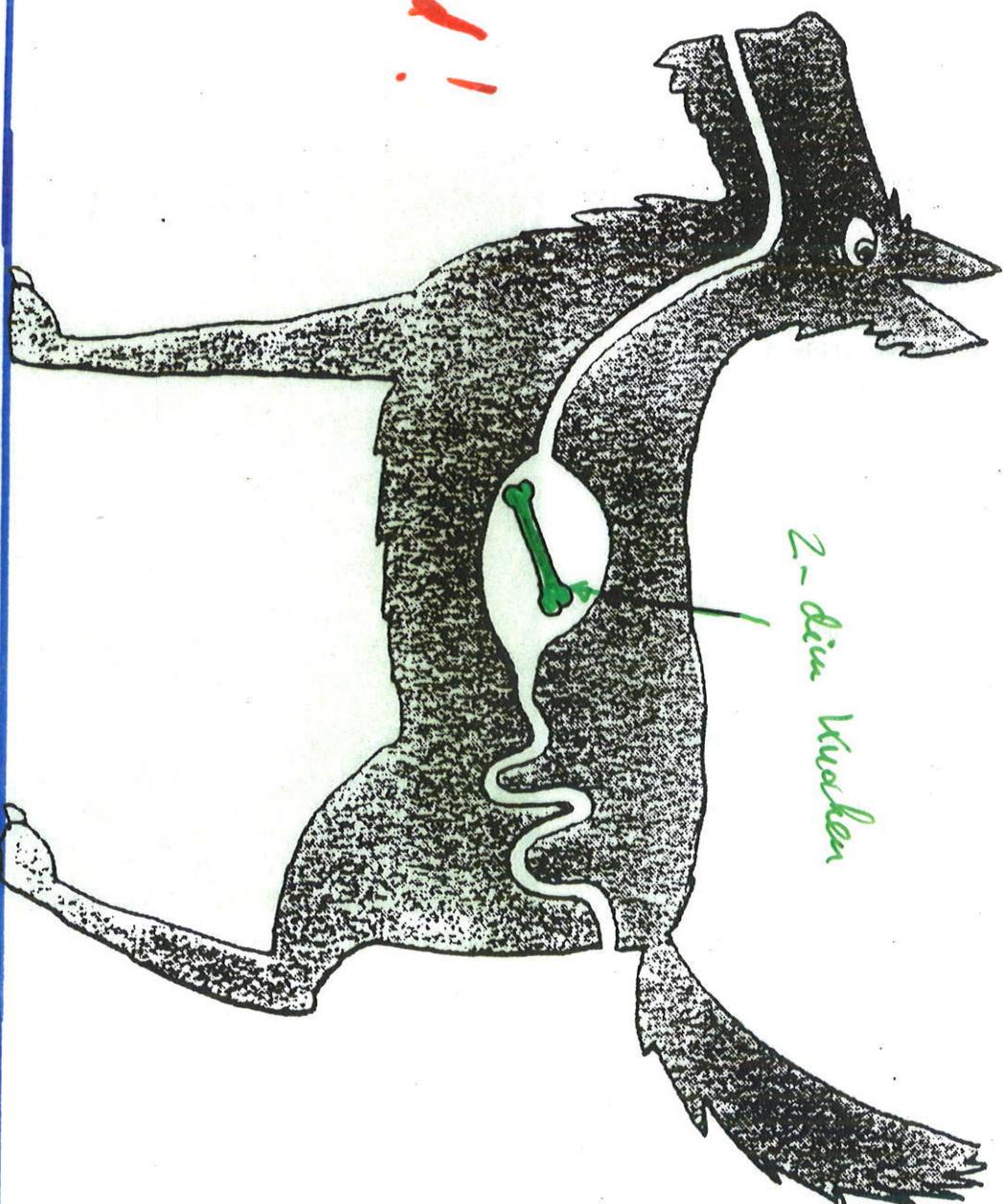
Potts Model on  $\mathbb{Z}^d$   $d \geq 3$

+ magnetic field  $q \geq 3$

Kertesz line both analytically and numerically

Warum IR 3?

2.-dim  
Hunde mit  
Verdauungskanal  
sind nicht  
verbunden!



↪ 1.-dim  
Erde

2.-dim Knochen