

Some ongoing Efforts for Evolving Spherical Einstein Field Equations on Hyperboloidal Slices

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Hyperboloidal Slices vs Cauchy Slices

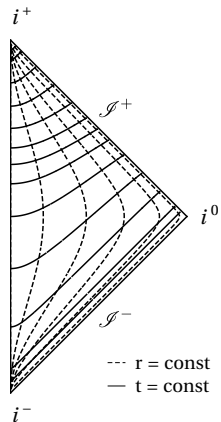


Figure: Hyperboloidal slices in Minkowski spacetime.

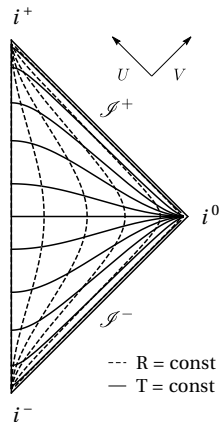


Figure: Cauchy Slices in Minkowski spacetime.

Motivation

- ▶ Why Hyperboloidal slices?
 - ▶ Need to include future null infinity, \mathcal{I}^+ , in the computational domain with a well-posed formalism.
- ▶ Why \mathcal{I}^+ ?
 - ▶ The only region where Gravitational Waves (GW) can be defined unambiguously.
 - ▶ Useful for studying phenomena of fundamental interest, like weak cosmic censorship conjecture, black hole formation, etc.
- ▶ Existing methods like Extrapolation from Cauchy evolution, CCE and CCM introduce errors due to their limitations.

Existing method 1: Extrapolation

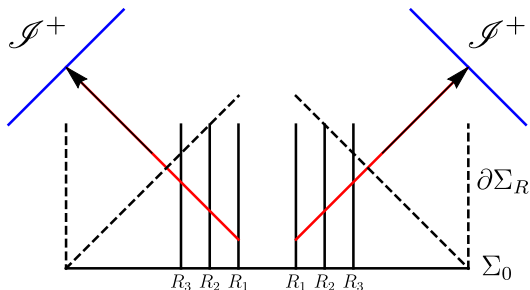


Figure: Extrapolation schematics.

Extrapolation Methods from Cauchy evolution:

- ▶ Boundary Errors, Extrapolation Errors.

Existing method 2: Cauchy Characteristic Extraction (CCE): (PRD 54, 6153)

- ▶ Data extracted at a timelike artificial boundary at a finite radius. Introduces gauge ambiguities.
- ▶ Unphysical boundary conditions at the boundary of the Cauchy domain. Limits time evolution.
- ▶ One-sided propagation of the signal.

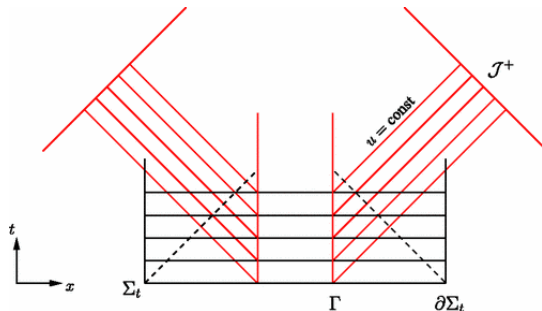


Figure: CCE schematics.

Source: C. Reisswig, N. T. Bishop, D. Pollney, and B. Szilgyi, Phys. Rev. Lett. 103, 221101 (2009).

Proposed method: Cauchy Characteristic Matching (CCM): (Winicour, 2009).

- ▶ Cauchy and Characteristic regions evolved at once. Data exchanged through boundary.
- ▶ 3D, in construction.
- ▶ Present formulations possibly ill-posed. (e.g. PRD 102, 064035).

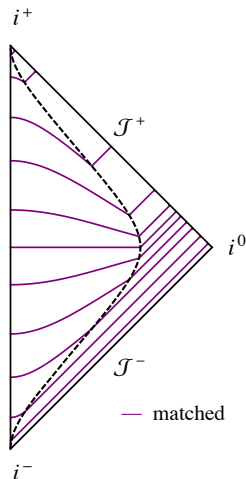


Figure: CCM schematics. Credit: Alex Vañó-Viñuales.

Hyperboloidal Slices^{1,2}

- ▶ Meet \mathcal{I}^+ , instead of i^0
 \Rightarrow signal can be directly collected at \mathcal{I}^+ .
- ▶ Overcomes all problem encountered in the existing methods.
- ▶ More gauge freedom than null slices.
- ▶ Suitable for dealing with initial-boundary value problems when initial data is not of compact support.

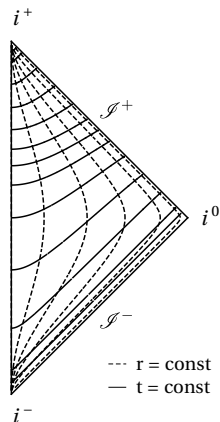


Figure: Hyperboloidal slices in Minkowski spacetime.

¹Mathematical front: Friedrich 1981-86, LeFloch et. al. 2014-19 etc.

²Numerical front: Hübner 1993, Frauendiener 1998-2006, Zenginoğlu 2005-11, Moncrief and Rinne 2009-18, Vañó-Viñuales 2015-18 etc.

Formulating EFEs on Hyperboloidal Slices

ToDo list:

- ▶ Goal 1: Derive a well-posed formulation of the Einstein Field Equations (EFEs) on hyperboloidal slices.
- ▶ Goal 2: Derive a discretization scheme preserving the well-posedness for the discrete Einstein Field Equation (dEFE) on hyperboloidal slices.

Our Approach:

- ▶ Goal 1a: Achieved by using the Dual-Foliation (DF) formalism³ together with the Generalized Harmonic Gauge (GHG)⁴.
- ▶ Goal 1b: Derive a suitable regularization scheme.
- ▶ Goal 2: Discretization scheme: Need to start from scratch, the Linear Wave Equation.

³(Hilditch 2015, 2016)

⁴Fourès-Bruhat 1952, Lindblad and Rodnianski 2003, Lindblom 2006, Gasperín and Hilditch 2018 etc.

Vacuum EFEs: Asymptotic Analysis (GBU Paper)⁵

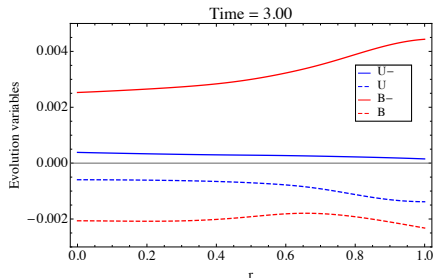
E. Gasperin, SG, D. Hilditch, A. Vañó-Viñuales, CQG, Vol. 37, No. 3

- ▶ EFEs in Harmonic Gauge (HG) give three classes of asymptotic equations.
- ▶ Good fields 'g', Bad fields 'b', Ugly fields 'u':

$$\square g = 0, \quad \square b = (\partial_T g)^2,$$

$$\square u = \frac{2}{R} \partial_T u.$$

- ▶ $g \sim 1/R$, $b \sim \ln R/R$,
 $u \sim 1/R^2$ towards \mathcal{I}^+ .



⁵See also: Gasperin and Hilditch, 2019, CQG 36, 195016.

Goal 2: Summation-by-Parts (SBP) scheme

SG, A. Vañó-Viñuales, D. Hilditch, S. Bose, PRD 103, 084045

- ▶ Derived for the class of equations

$$(\square - F)\psi = 0,$$

all in spherical symmetry,
 $F = F(R)$, $\psi = \psi(T, R)$.

- ▶ Assures stability, and hence convergence, by guaranteeing $\dot{\hat{E}} \leq 0$ on these slices, for all times.

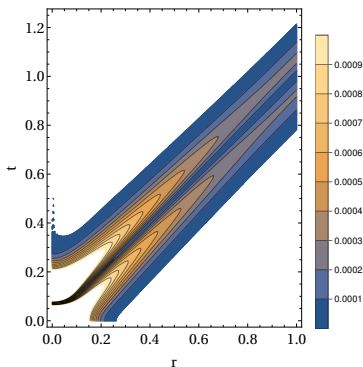


Figure: Propagation of a narrow pulse to \mathcal{I}^+ .

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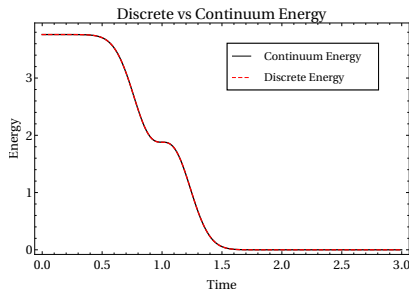


Figure: Discrete energy vs continuum energy, for the case $F = 0$.

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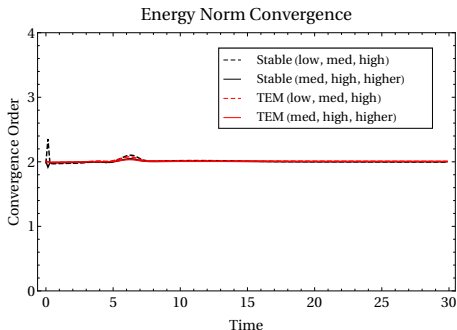


Figure: Convergence order in the energy norm.

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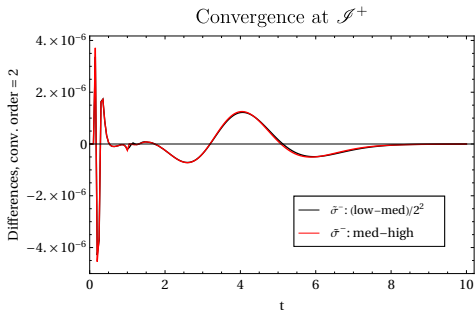


Figure: Convergence order at \mathcal{I}^+ .

Spherical GR: (Ongoing)

- ▶ Why Spherical symmetry?
 - ▶ To solve the problems involving spherical symmetry, like spherical collapse etc.
 - ▶ A stepping stone to full 3d.
- ▶ In spherical symmetry, the metric has the following general form

$$\mathbf{g} = \begin{pmatrix} g_{TT} & g_{TR} & 0 & 0 \\ g_{TR} & g_{RR} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ 0 & 0 & 0 & g_{\theta\theta} \sin^2 \theta \end{pmatrix}.$$

- ▶ Dynamical variables: g_{TT} , g_{TR} , g_{RR} and $g_{\theta\theta}$, all functions of (T, R) , cannot be classified as good, bad and ugly.

Spherical GR: (Ongoing)

- ▶ Converting into a GBU structure requires the following form

$$\mathbf{g} = \begin{pmatrix} \frac{2e^\delta C_+ C_-}{C_+ - C_-} & \frac{e^\delta (C_- + C_+)}{C_- - C_+} & 0 & 0 \\ \frac{e^\delta (C_- + C_+)}{C_- - C_+} & \frac{2e^\delta}{C_+ - C_-} & 0 & 0 \\ 0 & 0 & \dot{R}^2 & 0 \\ 0 & 0 & 0 & \dot{R}^2 \sin^2 \theta \end{pmatrix}.$$

- ▶ Dynamical variables: C_\pm , δ , \dot{R} , all functions of (T, R) .

Spherical GR:

For your eyes only!

- ▶ 'Causal structure' variables C_{\pm}^R :

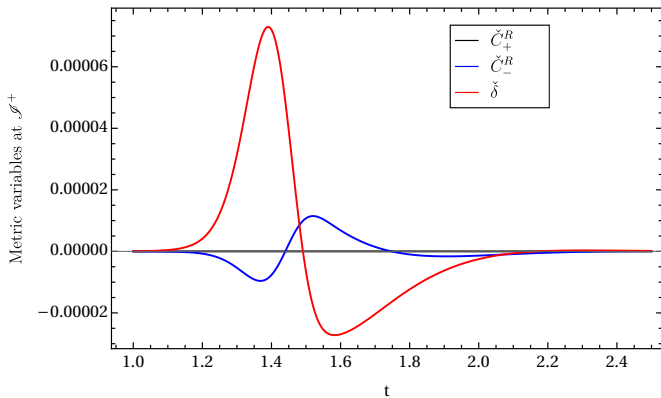
$$\nabla_{\xi} \left(\frac{\dot{R}^2}{e^{\delta\kappa}} \nabla_{\underline{\xi}} C_{+}^R \right) + \dot{R} \nabla_{\xi} H^{\eta} + \dot{R} (\nabla_{\xi} \dot{R}) \nabla_{\xi} C_{+}^R + 8\pi e^{-\delta} \dot{R}^2 T_{\xi\xi} = 0,$$

$$\nabla_{\underline{\xi}} \left(\frac{\dot{R}^2}{e^{\delta\kappa}} \nabla_{\xi} C_{-}^R \right) + \dot{R} \nabla_{\underline{\xi}} H^{\eta} + \dot{R} (\nabla_{\underline{\xi}} \dot{R}) \nabla_{\underline{\xi}} C_{-}^R + 8\pi e^{-\delta} \dot{R}^2 T_{\underline{\xi}\underline{\xi}} = 0.$$

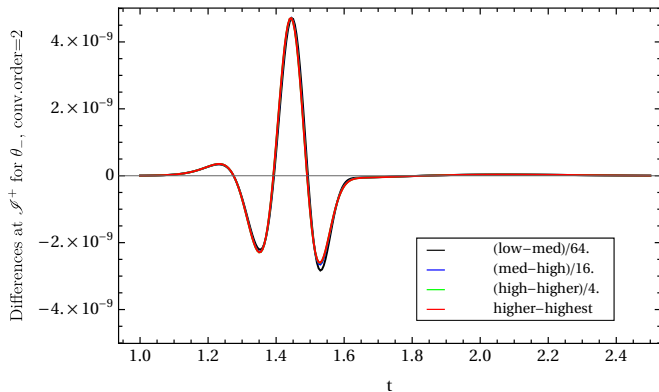
- ▶ 'Determinant' variables (e^{δ}, \dot{R}):

$$\begin{aligned} & \nabla_a \left(\frac{\xi^a}{e^{\delta\kappa}} \nabla_{\underline{\xi}} \delta \right) - \dot{R} \nabla_a \left(\frac{H^a}{\dot{R}} \right) + \frac{2}{\dot{R}^2 e^{\delta\kappa}} (\nabla_{\xi} \dot{R}) (\nabla_{\underline{\xi}} \dot{R}) \\ & + \frac{1}{4e^{\delta\kappa^3}} \left(\nabla_{\underline{\xi}} C_{-}^R \nabla_{\xi} C_{+}^R - \nabla_{\xi} C_{+}^R \nabla_{\underline{\xi}} C_{-}^R \right) + 16\pi T_T = 0, \\ & \nabla_a \left(\frac{\xi^a}{e^{\delta\kappa}} \nabla_{\underline{\xi}} \dot{R}^2 \right) + 2 = 0. \end{aligned}$$

Some preliminary results:



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Summary and Conclusions

- ▶ Continuum analysis:
 - ▶ Using asymptotic analysis, derived a regularization scheme on hyperboloidal slices for a 'model system' of equations that mimic the asymptotic form of the Einstein Field Equations (EFEs) in harmonic gauge (HG).
 - ▶ Propose that the EFEs in HG, and in generalized harmonic gauge (GHG) can be regularized similarly (work in progress).
- ▶ Discrete analysis:
 - ▶ Derived a 'good' discretization scheme on hyperboloidal slices that assures stability, and hence convergence, for a class of linear wave equations.
 - ▶ Propose that a similar scheme can improve the numerical results for the EFEs in GHG (future plan).
- ▶ Obtained some promising results for the spherical EFEs in GHG on hyperboloidal slices (stay tuned).

Thank You for your Attention!!
Questions? Comments?

References

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Dual-Foliation Formalism

- ▶ Consider two foliations of a spacetime $X^\mu = (T, X^i)$ and $x^\mu = (t, x^i)$, in spacelike hypersurfaces and relate the two geometries.

- ▶ Using a 3 + 1 split of the Jacobian $J^\mu{}_\mu = dX^\mu/dx^\mu$

$$\partial_T = (J^{-1})^\mu{}_{\underline{T}} \partial_\mu, \quad \partial_{\underline{i}} = (J^{-1})^\mu{}_{\underline{i}} \partial_\mu.$$

- ▶ Dynamical variables, or the tensor basis, remain the same. Only the coordinates are transformed.
- ▶ An equation for a state vector $\mathbf{u} = \{u_1, u_2, \dots, u_n\}$ of the form

$$\partial_T \mathbf{u} = \mathbf{A}^{\underline{p}} \partial_{\underline{p}} \mathbf{u} + \mathbf{A} \mathbf{S},$$

becomes

$$\partial_t \mathbf{u} = \mathbf{B}^{\underline{p}} \partial_{\underline{p}} \mathbf{u} + \mathbf{B} \mathbf{S}.$$

Nonlinear Change of Variables

- ▶ Define Operator

$$L^a = (\partial_T)^a + (\partial_R)^a, \quad \underline{L}^a = (\partial_T)^a - (\partial_R)^a, \quad s = \ln R.$$

- ▶ Good Fields:

$$G^+ = RL(Rg), \quad G^- = R\underline{L}g, \quad G = Rg.$$

- ▶ Bad Fields:

$$B^+ = RL(Rb + \frac{1}{8}s\eta), \quad B^- = R\underline{L}b + \frac{1}{8}s\underline{L}\eta, \quad B = Rb + \frac{1}{8}s\eta.$$

- ▶ Ugly Fields:

$$U^+ = RL(R^2u), \quad U^- = R^2\underline{L}u, \quad U = R^2u.$$

- ▶ $\partial_T \eta = (G^-)^2.$

Vacuum EFEs: Asymptotic Analysis (GBU Paper)

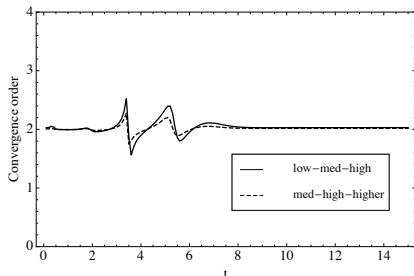
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